Vol 13 No 12, December 2004
1009-1963/2004/13(12)/2003-05

Chinese Physics

© 2004 Chin. Phys. Soc.
and IOP Publishing Ltd

Exact invariants and adiabatic invariants of dynamical system of relative motion*

Chen Xiang-Wei(陈向炜), Wang Xin-Min(王新民), and Wang Ming-Quan(王明泉)

Department of Physics, Shangqiu Teachers College, Shangqiu 476000, China

(Received 25 February 2004; revised manuscript received 13 May 2004)

Based on the theory of symmetries and conserved quantities, the exact invariants and adiabatic invariants of a dynamical system of relative motion are studied. The perturbation to symmetries for the dynamical system of relative motion under small excitation is discussed. The concept of high-order adiabatic invariant is presented, and the form of exact invariants and adiabatic invariants as well as the conditions for their existence are given. Then the corresponding inverse problem is studied.

Keywords: dynamical system of relative motion, perturbation, exact invariant, adiabatic invariant
PACC: 0320, 0316

1. Introduction

More and more attention has been paid to the research of symmetries and conserved quantities of mechanical systems and many results have sprung up during recent years.[1−13] These results enriched the knowledge of the essence of mechanical systems. In 1917, Burgers first proposed adiabatic invariants which pointed to a special type of Hamiltonian system.[14] The adiabatic invariants mean that they are almost unchanged when the parameter varies very slowly and they play a very important role in the research into the quasi-integrability of mechanical systems. In fact, as the parameter varying very slowly is equivalent to the action of a small perturbation, we conduct our research by the latter model in this paper.

With the development of modern science and technology, the study on dynamics of complicated mechanical systems becomes more and more important.[15−17] The motion of these complicated systems includes the motion of the carrier and the motion of the carried carrier relative to the carrier. The motion of complicated systems can be studied in the inertial coordinate system, or in the mobile coordinate system closely linked to the carrier. The study on the dynamics of relative motion of a mechanical system by the method of analytical mechanics not only reaches unity in expression but also shows superiority for the complicated systems.

The study of adiabatic invariants has become a popular subject in mechanics,[18−21] atomic and molecular physics,[22] etc. In this paper, we discuss the adiabatic invariants of dynamical system of relative motion from the viewpoint of symmetries and conserved quantities. First, we construct the equations of the dynamical system of relative motion, and propose the concept of higher order adiabatic invariants. Second, we prove the conditions for the existence of the exact invariants and adiabatic invariants, and give their forms. Third, we study the inverse problem of the perturbation to symmetries of the system. Finally, we present an example to illustrate these results.

2. Dynamical equations of relative motion

Suppose that the system is composed of a carrier and a carried carrier. The velocity $\boldsymbol{v}_0$ of the base point O and the angular velocity $\boldsymbol{\omega}$ of the carrier are known functions of time. The carried carrier is composed of N particles, and the configuration of the system is determined by the n generalized coordinates $q_s (s = 1, \ldots, n)$. Suppose the motion of the carried

*Project supported by the National Natural Science Foundation of China (Grant No 10372053); the Natural Science Foundation of Henan Province, China (Grant No 0311010900), and the Foundation of Young Key Member of the Teachers in Institutions of Higher Learning of Henan Province, China.

carrier does not affect the original motion law of the carrier. The dynamical equations of relative motion of the holonomic mechanical system can be written in the form[17]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_r}{\partial \dot{q}_s}-\frac{\partial T_r}{\partial q_s}=Q_s-\frac{\partial}{\partial q_s}(V^0+V^\omega)+Q_s^{\dot{\omega}}+\Gamma_s,$$
$$(s=1,\ldots,n), \tag{1}$$

where T_r is the kinetic energy of relative motion of the system, Q_s the generalized forces, and V^0 is the uniform force field potential energy:

$$V^0=M(\overset{*}{\boldsymbol{v}}_0+\boldsymbol{\omega}\times\boldsymbol{v}_0)\cdot\boldsymbol{r}'_c, \tag{2}$$

where M is the total mass of the system, $\boldsymbol{r}'_c$ the radius vector of the mass centre in the reference system of motion, and V^ω the centrifugal force potential energy:

$$V^\omega=-\frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{\theta}^\circ\cdot\boldsymbol{\omega}, \tag{3}$$

where $\boldsymbol{\theta}^\circ$ is the inertia tensor in the point O of the system, $Q_s^{\dot{\omega}}$ the generalized inertial force of gyration:

$$Q_s^{\dot{\omega}}=-(\dot{\boldsymbol{\omega}}\times m_i\boldsymbol{r}'_i)\cdot\frac{\partial\boldsymbol{r}'_i}{\partial q_s}. \tag{4}$$

Γ_s is the generalized gyroscopic force

$$\Gamma_s=\gamma_{sk}\dot{q}_k,\quad \gamma_{sk}=2\boldsymbol{\omega}\cdot\left(m_i\frac{\partial\boldsymbol{r}'_i}{\partial q_s}\times\frac{\partial\boldsymbol{r}'_i}{\partial q_k}\right). \tag{5}$$

The generalized force Q_s is divided into the potential part Q'_s and the nonpotential part Q''_s, i.e.

$$Q_s=Q'_s+Q''_s,\quad Q'_s=-\frac{\partial V}{\partial q_s}. \tag{6}$$

Let

$$L_r=T_r-V-V^0-V^\omega, \tag{7}$$

then Eq.(1) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_r}{\partial\dot{q}_s}-\frac{\partial L_r}{\partial q_s}=Q''_s+Q_s^{\dot{\omega}}+\Gamma_s,\quad (s=1,\ldots,n). \tag{8}$$

Suppose that the system is not singular, i.e.

$$\det(h_{sk})=\det\left(\frac{\partial^2 L_r}{\partial\dot{q}_s\partial\dot{q}_k}\right)\neq 0, \tag{9}$$

then all the generalized accelerations can be solved from Eq.(8),

$$\ddot{q}_s=\alpha_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})\quad (s=1,\ldots,n). \tag{10}$$

3. Symmetries and exact invariants

Introduce the infinitesimal transformations of groups of time and generalized coordinates

$$t^*=t+\Delta t, q_s^*(t^*)=q_s(t)+\Delta q_s,$$
$$(s=1,\ldots,n), \tag{11}$$

and their expansions

$$t^*=t+\varepsilon\tau^0(t,\boldsymbol{q},\dot{\boldsymbol{q}}), q_s^*(t^*)=q_s(t)+\varepsilon\xi_s^0(t,\boldsymbol{q},\dot{\boldsymbol{q}}),$$
$$(s=1,\ldots,n), \tag{12}$$

where ε is an infinitesimal parameter, τ^0 and ξ_s^0 are infinitesimal generators. Introduce an infinitesimal generator vector

$$X^{(0)}=\tau^0\frac{\partial}{\partial t}+\xi_s^0\frac{\partial}{\partial q_s}, \tag{13}$$

and its first extension

$$X^{(1)}=X^{(0)}+(\dot{\xi}_s^0-\dot{q}_s\dot{\tau}^0)\frac{\partial}{\partial\dot{q}_s}. \tag{14}$$

The invariance of Eq.(10) under the infinitesimal transformations (12) leads to the following determining equations:

$$\ddot{\xi}_s^0-\dot{q}_s\ddot{\tau}^0-2\dot{\tau}^0\alpha_s=X^{(1)}(\alpha_s)\ ,\quad (s=1,\ldots,n). \tag{15}$$

Definition 1[17] If the generators τ^0 and ξ_s^0 satisfy the determining equation (15), then the corresponding transformations are called the Lie symmetrical transformations, and the corresponding symmetries are called the Lie symmetries.

Proposition 1 For the infinitesimal generators τ^0 and ξ_s^0 satisfying Eq.(15), if there exists a gauge function $G^0=G^0(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ satisfying the following structure equation:

$$L_r\dot{\tau}^0+X^{(1)}(L_r)+(Q''_s+Q_s^{\dot{\omega}}+\Gamma_s)$$
$$\times(\xi_s^0-\dot{q}_s\tau^0)+\dot{G}^0=0, \tag{16}$$

then the dynamical system of relative motion has the following exact invariant of Lie symmetries:

$$I_0=L_r\tau^0+\frac{\partial L_r}{\partial\dot{q}_s}(\xi_s^0-\dot{q}_s\tau^0)+G^0=\mathrm{const}. \tag{17}$$

Proof

$$\begin{aligned}\frac{\mathrm{d}I_0}{\mathrm{d}t}=&\dot{L}_r\tau^0+L_r\dot{\tau}^0+\frac{\partial L_r}{\partial\dot{q}_s}(\dot{\xi}_s^0-\ddot{q}_s\tau^0-\dot{q}_s\dot{\tau}^0)\\&+(\xi_s^0-\dot{q}_s\tau^0)\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_r}{\partial\dot{q}_s}-L_r\dot{\tau}^0-X^{(1)}(L_r)\\&-(Q''_s+Q_s^{\dot{\omega}}+\Gamma_s)(\xi_s^0-\dot{q}_s\tau^0)\\=&(\xi_s^0-\dot{q}_s\tau^0)\\&\times\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_r}{\partial\dot{q}_s}-\frac{\partial L_r}{\partial q_s}-Q''_s-Q_s^{\dot{\omega}}-\Gamma_s\right)=0.\end{aligned}$$

4. Perturbation to the symmetries and adiabatic invariants

Firstly we give the concept of high-order adiabatic invariants.

Definition 2 If $I_z(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \varepsilon)$ is a physical quantity including ε of which the highest power is z in a mechanical system, and its derivative with respect to time t is in direct proportion to ε^{z+1}, then I_z is called a zth-order adiabatic invariant of the mechanical system.

Suppose the dynamical system of relative motion corresponding to (8) is perturbed by small quantities εW_s, then the equations of motion of the system become

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_r}{\partial \dot{q}_s} - \frac{\partial L_r}{\partial q_s} = Q_s'' + Q_s^{\dot{\omega}} + \Gamma_s + \varepsilon W_s, \quad (s = 1, \ldots, n). \tag{18}$$

Due to the action of εW_s, the primary symmetries and invariants of the system may vary. Suppose the variation is a small perturbation based on the symmetrical transformation of the system without perturbation, and $\tau(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ and $\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ express the generators of time and space respectively after being perturbed, then

$$\begin{aligned} \tau &= \tau^0 + \varepsilon\tau^1 + \varepsilon^2\tau^2 + \cdots, \\ \xi_s &= \xi_s^0 + \varepsilon\xi_s^1 + \varepsilon^2\xi_s^2 + \cdots, \end{aligned} \tag{19}$$

and they satisfy

$$\begin{aligned} &L_r\dot{\tau} + X^{(1)'}(L_r) + (Q_s'' + Q_s^{\dot{\omega}} + \Gamma_s)(\xi_s - \dot{q}_s\tau) \\ &+ \varepsilon W_s(\xi_s + \dot{q}_s\tau) + \dot{G} = 0, \end{aligned} \tag{20}$$

where

$$\begin{aligned} X^{(1)'} =& X^{(0)'} + (\dot{\xi}_s - \dot{q}_s\dot{\tau})\frac{\partial}{\partial \dot{q}_s} \\ =& \tau\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} + (\dot{\xi}_s - \dot{q}_s\dot{\tau})\frac{\partial}{\partial \dot{q}_s}. \end{aligned}$$

G in Eq.(20) is a gauge function. Let

$$G = G^0 + \varepsilon G^1 + \varepsilon^2 G^2 + \cdots, \tag{21}$$

$$X^{(1)k} = \tau^k\frac{\partial}{\partial t} + \xi_s^k\frac{\partial}{\partial q_s} + (\dot{\xi}_s^k - \dot{q}_s\dot{\tau}^k)\frac{\partial}{\partial \dot{q}_s}. \tag{22}$$

Substituting Eqs.(19) and Eq.(21) into Eq.(20), we obtain

$$\begin{aligned} &L_r\dot{\tau}^k + X^{(1)k}(L_r) + (Q_s'' + Q_s^{\dot{\omega}} + \Gamma_s)(\xi_s^k - \dot{q}_s\tau^k) \\ &+ W_s(\xi_s^{k-1} + \dot{q}_s\tau^{k-1}) + \dot{G}^k = 0, \\ &(k = 0, 1, 2, \ldots, z), \end{aligned} \tag{23}$$

when $k = 0$, the condition $W_s = 0$ holds.

Proposition 2 For the dynamical systems of relative motion perturbed by small quantities εW_s, if the generators $\tau^k(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ and $\xi_s^k(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ under infinitesimal transformations satisfy Eq.(23), and $G^k(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is the gauge function, then

$$\begin{aligned} I_z = \varepsilon^k[&L_r\tau^k + \frac{\partial L_r}{\partial \dot{q}_s}(\xi_s^k - \dot{q}_s\tau^k) + G^k], \\ &(k = 0, 1, \ldots, z) \end{aligned} \tag{24}$$

is a zth-order adiabatic invariant of the mechanical system.

Proof

$$\begin{aligned} \frac{\mathrm{d}I_z}{\mathrm{d}t} =& \varepsilon^k\,[\,\dot{L}_r\tau^k + L_r\dot{\tau}^k + \frac{\partial L_r}{\partial \dot{q}_s}(\dot{\xi}_s^k - \ddot{q}_s\tau^k - \dot{q}_s\dot{\tau}^k) \\ &+ (\xi_s^k - \dot{q}_s\tau^k)\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_r}{\partial \dot{q}_s} - L_r\dot{\tau}^k - X^{(1)k}(L_r) \\ &- (Q_s'' + Q_s^{\dot{\omega}} + \Gamma_s)(\xi_s^k - \dot{q}_s\tau^k) \\ &- W_s(\xi_s^{k-1} - \dot{q}_s\dot{\tau}^{k-1})\,] \\ =& \varepsilon^k\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_r}{\partial \dot{q}_s} - \frac{\partial L_r}{\partial q_s} - Q_s'' - Q_s^{\dot{\omega}} - \Gamma_s\right)\right. \\ &\left.\times(\xi_s^k - \dot{q}_s\tau^k) - W_s(\xi_s^{k-1} - \dot{q}_s\dot{\tau}^{k-1})\right]. \end{aligned}$$

Using Eq.(18), we have

$$\frac{\mathrm{d}I_z}{\mathrm{d}t} = \varepsilon^k\,[\,\varepsilon W_s(\xi_s^k - \dot{q}_s\tau^k) - W_s(\xi_s^{k-1} - \dot{q}_s\dot{\tau}^{k-1})\,].$$

Expanding the above formula and making summation, we have

$$\frac{\mathrm{d}I_z}{\mathrm{d}t} = \varepsilon^{z+1}W_s(\xi_s^z - \dot{q}_s\tau^z). \tag{25}$$

This shows that $\mathrm{d}I_z/\mathrm{d}t$ is in direct proportion to ε^{z+1}, so I_z is a zth-order adiabatic invariant of the mechanical system.

5. Inverse problem

Suppose that the system has a first-order adiabatic invariant as follows:

$$I_1 = \lambda_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon\lambda_1(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{26}$$

Differentiating this with respect to t, we have

$$\begin{aligned} \frac{\mathrm{d}I_1}{\mathrm{d}t} =& \frac{\partial\lambda_0}{\partial t} + \left(\frac{\partial\lambda_0}{\partial q_s}\dot{q}_s + \frac{\partial\lambda_0}{\partial \dot{q}_s}\ddot{q}_s\right) \\ &+ \varepsilon\left[\frac{\partial\lambda_1}{\partial t} + \left(\frac{\partial\lambda_1}{\partial q_s}\dot{q}_s + \frac{\partial\lambda_1}{\partial \dot{q}_s}\ddot{q}_s\right)\right]. \end{aligned} \tag{27}$$

Multiplying Eq.(18) by $\xi_s - \dot{q}_s\tau$, and taking the summation for s, we have

$$\begin{aligned} &(\xi_s - \dot{q}_s\tau) \\ &\times\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_r}{\partial \dot{q}_s} - \frac{\partial L_r}{\partial q_s} - Q_s'' - Q_s^{\dot{\omega}} - \Gamma_s - \varepsilon W_s\right) = 0. \end{aligned} \tag{28}$$

According to Eqs.(25), and adding Eq.(27) and Eq.(28), we have

$$\begin{aligned}&\frac{\partial\lambda_0}{\partial t}+\left(\frac{\partial\lambda_0}{\partial q_s}\dot{q}_s+\frac{\partial\lambda_0}{\partial\dot{q}_s}\ddot{q}_s\right)\\&+\varepsilon\left[\frac{\partial\lambda_1}{\partial t}+\left(\frac{\partial\lambda_1}{\partial q_s}\dot{q}_s+\frac{\partial\lambda_1}{\partial\dot{q}_s}\ddot{q}_s\right)\right]+(\xi_s-\dot{q}_s\tau)\\&\times\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_r}{\partial\dot{q}_s}-\frac{\partial L_r}{\partial q_s}-Q_s''-Q_s^{\dot{\omega}}-\Gamma_s-\varepsilon W_s\right)\\=&\varepsilon^2W_s(\xi_s^1-\dot{q}_s\tau^1),\end{aligned}\tag{29}$$

where the expressions of τ and ξ_s are Eqs.(19).

Firstly, we seek the generators τ^0 and ξ_s^0 of the infinitesimal transformations without perturbation. Separating the terms not containing ε in Eq.(29), then separating the terms containing $\ddot{q}_k$ and taking their coefficients as zero, we have

$$\frac{\partial\lambda_0}{\partial\dot{q}_s}-\frac{\partial^2L_r}{\partial\dot{q}_s\partial\dot{q}_k}(\xi_k^0-\dot{q}_k\tau^0)=0\quad(k=1,\ldots,n).\tag{30}$$

Letting $\dfrac{\partial^2L_r}{\partial\dot{q}_s\partial\dot{q}_k}=H_{sk}$, $h_{sk}=(H_{sk})^{-1}$. Here H_{sk} is called the Hess matrix. From the above equation we obtain

$$\xi_k^0-\dot{q}_k\tau^0=h_{sk}\frac{\partial\lambda_0}{\partial\dot{q}_s}.\tag{31}$$

Further let

$$\lambda_0=L_r\tau^0+\frac{\partial L_r}{\partial\dot{q}_k}(\xi_k^0-\dot{q}_k\tau^0)+G^0,\tag{32}$$

from Eqs.(31) and (32), we can obtain the items without perturbation:

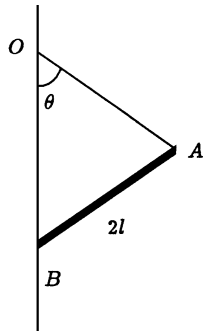
$$\begin{aligned}\tau^0&=L_r^{-1}(\lambda_0-\frac{\partial L_r}{\partial\dot{q}_k}h_{sk}\frac{\partial\lambda_0}{\partial\dot{q}_s}-G^0)\ ,\\\xi_k^0&=h_{sk}\frac{\partial\lambda_0}{\partial\dot{q}_s}+\dot{q}_kL_r^{-1}\left(\lambda_0-\frac{\partial L_r}{\partial\dot{q}_k}h_{sk}\frac{\partial\lambda_0}{\partial\dot{q}_s}-G^0\right).\end{aligned}\tag{33}$$

Further we can obtain the results of perturbation items of generator functions under small disturbance:

$$\begin{aligned}\tau^1&=L_r^{-1}(\lambda_1-\frac{\partial L_r}{\partial\dot{q}_k}h_{sk}\frac{\partial\lambda_1}{\partial\dot{q}_s}-G^1)\ ,\\\xi_k^1&=h_{sk}\frac{\partial\lambda_1}{\partial\dot{q}_s}+\dot{q}_kL_r^{-1}(\lambda_1-\frac{\partial L_r}{\partial\dot{q}_k}h_{sk}\frac{\partial\lambda_1}{\partial\dot{q}_s}-G^1).\end{aligned}\tag{34}$$

6. An illustrative example

A uniform-mass bar AB uniformly rotates around an axis with angular velocity ω, the length of the bar is $2l$, one end B of the bar may move along the vertical axis, and the other end A is connected to a point O of the axis by a rope of length $2l$, as shown in the right graph. Try to study the exact invariants and adiabatic invariants of the system.



From the known conditions we can obtain

$$\begin{cases}T_r=\dfrac{1}{6}ml^2\dot{q}^2+\dfrac{1}{2}m(9l^2\dot{q}^2\sin^2q+l^2\dot{q}^2\cos^2q),\\V=-3mgl\cos q,\ V^{\omega}=-\dfrac{2}{3}ml^2\omega^2\sin^2q,\\Q''=V^0=\Gamma=Q^{\dot{\omega}}=0\ ,\end{cases}\tag{35}$$

where m and g are the mass of the bar and acceleration of gravity.

Firstly, we seek the exact invariant. Equations (8) give the dynamical equations of relative motion of the system as follows:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{3}ml^2\dot{q}+ml^2\dot{q}(9\sin^2q+\cos^2q)\right]\\&-8ml^2\dot{q}^2\sin q\cos q\\&+3mgl\sin q-\frac{4}{3}ml^2\omega^2\sin q\cos q=0\ ,\end{aligned}\tag{36}$$

thus we obtain

$$\ddot{q}=\frac{\sin q}{\dfrac{4}{3}+8\sin^2q}\left(\frac{4}{3}\omega^2\cos q-8\dot{q}^2\cos q-\frac{3g}{l}\right)=\alpha.\tag{37}$$

The determining equation (15) gives

$$\ddot{\xi}^0-\dot{q}\ddot{\tau}^0-2\dot{\tau}^0\alpha=X^{(1)}(\alpha).\tag{38}$$

Equation (38) has solutions

$$\tau^0=-1,\quad\xi^0=0.\tag{39}$$

The structure equation (16) gives

$$\begin{aligned}&L_r\dot{\tau}^0+\xi^0(8ml^2\dot{q}^2\sin q\cos q-3mgl\sin q\\&+\frac{4}{3}ml^2\omega^2\sin q\cos q)+(\dot{\xi}^0-\dot{q}\dot{\tau}^0)\\&\times\left[\frac{1}{3}ml^2\dot{q}+m(9l^2\dot{q}\sin^2q+l^2\dot{q}\cos^2q)\right]+\dot{G}^0=0.\end{aligned}\tag{40}$$

It has a solution

$$G^0=0.\tag{41}$$

So we can obtain the following exact invariant by Proposition 1:

$$I_0 = \frac{1}{6}ml^2\dot{q}^2 + \frac{1}{2}ml^2\dot{q}^2(9\sin^2 q + \cos^2 q) - 3mgl\sin q - \frac{2}{3}ml^2\omega^2\sin^2 q = \text{const.} \quad (42)$$

Next we study the adiabatic invariants of the system. Suppose the system is perturbed by the small quantity

$$\varepsilon W = \varepsilon q. \quad (43)$$

The structure equation (23) gives

$$L_r\dot{\tau}^1 + \xi^1(8ml^2\dot{q}^2\sin q\cos q - 3mgl\sin q + \frac{4}{3}ml^2\omega^2\sin q\cos q) + (\dot{\xi}^1 - \dot{q}\dot{\tau}^1) \times \left[\frac{1}{3}ml^2\dot{q} + m(9l^2\dot{q}\sin^2 q + l^2\dot{q}\cos^2 q)\right] + q(\dot{\xi}^0 - \dot{q}\tau^0) + \dot{G}^1 = 0. \quad (44)$$

It has solutions as

$$\tau^1 = -1, \quad \xi^1 = 0, \quad G^1 = \frac{1}{2}q^2. \quad (45)$$

So we can obtain the following first-order adiabatic invariant by proposition 2:

$$I_1 = \frac{1}{6}ml^2\dot{q}^2 + \frac{1}{2}ml^2\dot{q}^2(9\sin^2 q + \cos^2 q) - 3mgl\sin q - \frac{2}{3}ml^2\omega^2\sin^2 q + \varepsilon\left[\frac{1}{6}ml^2\dot{q}^2 + \frac{1}{2}ml^2\dot{q}^2(9\sin^2 q + \cos^2 q) - 3mgl\sin q - \frac{2}{3}ml^2\omega^2\sin^2 q + \frac{1}{2}q^2\right]. \quad (46)$$

Moreover, we can obtain further higher order adiabatic invariants.

References

[1] Vujanovic B D 1995 *Int. J. Non-Linear Mech.* **30** 783
[2] Lutzky M 1979 *J. Phys. A: Math. Gen.* **12** 973
[3] Lakshmanan M and Senthil V M 1992 *J. Phys. A: Math. Gen.* **25** 1259
[4] Mei F X, Zhang Y F and Shang M 1999 *Mech. Res. Commun.* **26** 7
[5] Guo Y X 2001 *Chin. Phys.* **10** 181
[6] Zhang R C, Chen X W and Mei F X 2001 *Chin. Phys.* **10** 12
[7] Mei F X 2001 *Chin. Phys.* **10** 177
[8] Qiao Y F 2001 *Acta Phys. Sin.* **50** 811 (in Chinese)
[9] Luo S K 2003 *Chin. Phys.* **12** 841
[10] Luo S K and Cai J L 2003 *Chin. Phys.* **12** 357
[11] Fu J L and Chen L Q 2003 *Phys. Lett.* A **317** 255
[12] Fu J L and Chen L Q 2004 *Mech. Res. Commun.* **31** 9
[13] Fu J L, Chen L Q and Yang X D 2004 *Chin. Phys.* **13** 287
[14] Burgers J M 1917 *Ann. Phys., Lpz.* **52** 195
[15] Chen LQ 1992 *Mech. Eng.* **14** 63 (in Chinese)
[16] Chen LQ 1993 *J. Anshan Gang Tie Inst.* **16** 8 (in Chinese)
[17] Mei F X 1999 *The Application of Lie Groups and Lie Algebra to Constrained Mechanical Systems* (Beijing: Science Press) (in Chinese)
[18] Hu J H 1999 *SIAM J. Appl. Math.* **59** 322
[19] Zhao Y Y and Mei F X 1996 *Acta Mech. Sin.* **28** 207 (in Chinese)
[20] Chen X W and Mei F X 2000 *Chin. Phys.* **9** 721
[21] Chen X W and Li Y M 2003 *Chin. Phys.* **12** 936
[22] Ostrovsky V N and Prudov N V 1995 *J. Phys. B: At. Mol. Opt. Phys.* **20** 4435