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Non-Noether conserved quantity constructed by using form invariance for Birkhoffian system*

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Based on the invariance of Birkhoffian equations under the infinitesimal transformations of groups, the definition and the criterion of a form invariance for a Birkhoffian system are established. The condition under which the form invariance can lead to a non-Noether conserved quantity and the form of the conserved quantity are deduced by relying on the total time derivative along the trajectory of the equations, and two corollaries in special cases are presented. An example is finally given to illustrate the application of the results.

Keywords: Birkhoffian system, form invariance, non-Noether conserved quantity
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1. Introduction

The main methods to seek conserved quantities by using the symmetries of mechanical systems consist of the Noether method, the Lie method and the form invariance. A direct method in common use is to seek the Noether conserved quantity by using the Noether symmetry and the Noether theorem, and great progress is achieved in this method.[1−3] An indirect method to seek the Noether conserved quantity by using the condition that the Lie symmetry as well as the form invariance is a Noether symmetry has gained significant progress.[2−20] A direct method obtained by Hojman to seek the non-Noether conserved quantity by using the special Lie symmetry has made some progress too.[21−32]

The dynamics of a Birkhoffian system is an active branch in modern classical mechanics as well as one of the current developing directions in mathematical physics. It can be applied to quantum mechanics, statistical mechanics, biologic physics, mechanics of spaceflight and some fields of modern engineering.[33,34] In recent years, Birkhoffian equations and their transformation theory[33] have been studied, so also have the Noether theory,[35] the Poisson theory,[36] the Lie symmetry[37] and the form invariance[38,39] of Birkhoffian systems, and some important results have been obtained. The relation between the form invariance and the Lie symmetry for constrained Birkhoffian equations has been discussed in Ref.[40], and the Hojman conserved quantity of a Birkhoffian system has been studied in Ref.[41]. Research on the Birkhoffian system with unilateral constraints, the relativistic Birkhoffian system and the rotational relativistic Birkhoffian system has gained a few results.[42−45] In this paper, a kind of non-Noether conserved quantity deduced directly from the form invariance for a Birkhoffian system is presented by relying on the total time derivative along the trajectory of the equation for the system.

2. Definition and criterion of a form invariance

The Birkhoffian equations which show a $2n$-degree Birkhoffian system can be written in the form[33,34]

$$\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}=0$$

$$(\mu=1,2,\ldots,2n), \qquad (1)$$

where the function $B=B(t,\boldsymbol{a})$ is called a Birkhoffian, $R_\mu=R_\mu(t,\boldsymbol{a})$ are called Birkhoff's functions, t is time, and a^μ are variables. If the system (1) is

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non-singular, we have

$$\dot{a}^{\mu} = \sigma^{\mu}(t, \boldsymbol{a}) \qquad (\mu = 1, 2, \ldots, 2n). \tag{2}$$

Introducing the infinitesimal transformations

$$t^* = t + \varepsilon\xi_0(t, \boldsymbol{a}), \quad a^{\mu^*}(t^*) = a^{\mu}(t) + \varepsilon\xi_{\mu}(t, \boldsymbol{a}) \quad (\mu = 1, 2, \ldots, 2n), \tag{3}$$

where ε is an infinitesimal parameter, and ξ_0, ξ_{μ} are called infinitesimal generators, the functions B and R_{μ} respectively become B^* and R^*_{μ} under the transformations (3). Expanding B^* and R^*_{μ} into Taylor series at $(t, \boldsymbol{a})$, we obtain

$$\begin{aligned} B^* =& B(t^*, a^*) = B(t, \boldsymbol{a}) + \varepsilon X^{(0)}(B) + O(\varepsilon^2), \\ R^*_{\mu} =& R_{\mu}(t^*, a^*) = R_{\mu}(t, \boldsymbol{a}) + \varepsilon X^{(0)}(R_{\mu}) + O(\varepsilon^2) \\ & (\mu = 1, 2, \ldots, 2n), \end{aligned} \tag{4}$$

where

$$X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \xi_{\mu} \frac{\partial}{\partial a^{\mu}} \tag{5}$$

is the operator of the infinitesimal generators.

Definition[38] Under the infinitesimal transformations (3), if the Birkhoffian equations (1) keep their form invariant, i.e.

$$\left(\frac{\partial R^*_{\nu}}{\partial a^{\mu}} - \frac{\partial R^*_{\mu}}{\partial a^{\nu}}\right)\dot{a}^{\nu} - \frac{\partial B^*}{\partial a^{\mu}} - \frac{\partial R^*_{\mu}}{\partial t} = 0 \quad (\mu = 1, 2, \ldots, 2n), \tag{6}$$

this kind of invariance is called a form invariance of the Birkhoffian system.

Substituting formulae (4) into Eqs.(6), neglecting the second and higher order terms of ε, and using Eqs.(1) and (2), we obtain

$$\begin{aligned} & \left[\frac{\partial X^{(0)}(R_{\nu})}{\partial a^{\mu}} - \frac{\partial X^{(0)}(R_{\mu})}{\partial a^{\nu}}\right]\sigma^{\nu} \\ & - \frac{\partial X^{(0)}(B)}{\partial a^{\mu}} - \frac{\partial X^{(0)}(R_{\mu})}{\partial t} = 0 \\ & (\mu = 1, 2, \ldots, 2n). \end{aligned} \tag{7}$$

Criterion If the infinitesimal generators ξ_0 and ξ_{μ} satisfy Eqs.(7), the Birkhoffian system (1) has a form invariance under the infinitesimal transformations (3).

Equations (7) are called the determining equations of the form invariance for the Birkhoffian system. It is obvious that the criterion in Ref.[38] is a special case of the criterion above.

3. Noether conserved quantity deduced from the form invariance

The condition, under which the form invariance of the Birkhoffian system can indirectly lead to a Noether conserved quantity by using the fact that the form invariance is also a Noether symmetry, and the form of the conserved quantity are given in the following proposition.

Proposition 1 If the infinitesimal generators ξ_0 and ξ_{μ} of the transformations (3) for the Birkhoffian system (1) satisfy Eqs.(7), and, besides, there exists a gauge function $G_{\rm N} = G_{\rm N}(t, \boldsymbol{a})$ satisfying the following structural equation:

$$X^{(0)}(R_{\mu})\dot{a}^{\mu} - X^{(0)}(B) + R_{\mu}\dot{\xi}_{\mu} - B\dot{\xi}_0 + \dot{G}_{\rm N} = 0, \tag{8}$$

there will be a Noether conserved quantity deduced from the form invariance for the system as follows:

$$I_{\rm N} = R_{\mu}\xi_{\mu} - B\xi_0 + G_{\rm N} = \text{const.} \tag{9}$$

4. Non-Noether conserved quantity deduced from the form invariance

The condition under which the form invariance can directly lead to a non-Noether conserved quantity and the form of the conserved quantity are given in the following proposition.

Proposition 2 If the infinitesimal generators ξ_0 and ξ_{μ} of the transformations (3) for the non-singular Birkhoffian system (2) satisfy Eqs.(7), and, besides, there exists a gauge function $G_{\rm F} = G_{\rm F}(t, \boldsymbol{a})$ satisfying the following generalized structural equation:

$$\begin{aligned} & X^{(0)}[X^{(0)}(R_{\mu})]\sigma^{\mu} \\ & - X^{(0)}[X^{(0)}(B)] + X^{(0)}(R_{\mu})\frac{\bar{\rm d}\xi_{\mu}}{{\rm d}t} \\ & - X^{(0)}(B)\frac{\bar{\rm d}\xi_0}{{\rm d}t} + \frac{\bar{\rm d}G_{\rm F}}{{\rm d}t} = 0, \end{aligned} \tag{10}$$

where

$$\frac{\bar{\rm d}}{{\rm d}t} = \frac{\partial}{\partial t} + \sigma^{\mu}\frac{\partial}{\partial a^{\mu}}, \tag{11}$$

the form invariance of the system will lead to a non-Noether conserved quantity as follows:

$$I_{\rm F} = X^{(0)}(R_{\mu})\xi_{\mu} - X^{(0)}(B)\xi_0 + G_{\rm F} = \text{const.} \tag{12}$$

Proof Taking the time derivative of Eq.(12) in terms of Eq.(11), and using Eq.(10), we have

$$\begin{aligned} \frac{\bar{\rm d}}{{\rm d}t}I_{\rm F} =& \left\{\left[\frac{\partial X^{(0)}(R_{\nu})}{\partial a^{\mu}} - \frac{\partial X^{(0)}(R_{\mu})}{\partial a^{\nu}}\right]\sigma^{\nu} \right. \\ & \left. - \frac{\partial X^{(0)}(B)}{\partial a^{\mu}} - \frac{\partial X^{(0)}(R_{\mu})}{\partial t}\right\}(\xi_0\sigma^{\mu} - \xi_{\mu}) \\ & - \xi_0\left[\frac{\partial X^{(0)}(R_{\nu})}{\partial a^{\mu}} - \frac{\partial X^{(0)}(R_{\mu})}{\partial a^{\nu}}\right]\sigma^{\nu}\sigma^{\mu}. \end{aligned} \tag{13}$$

According to Eqs.(7), the first term located on the right of formula (13) vanishes. The second term vanishes because of antisymmetry of the term in the bracket. Then we obtain

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I_{\mathrm{F}} = 0.$$

It is shown that the form invariance will lead to trivial conserved quantities under special cases in the following two corollaries.

Corollary 1 If the infinitesimal generators of the form invariance satisfy $\xi_\mu = \sigma^\mu \xi_0$, formula (12) will become a trivial conserved quantity $I_{\mathrm{F}} = 0$.

Proof According to Eq.(11) and $\xi_\mu = \sigma^\mu \xi_0$, we obtain the relation between two operators as follows:

$$X^{(0)} = \xi_0 \frac{\bar{\mathrm{d}}}{\mathrm{d}t}. \tag{14}$$

Then, we easily obtain $I_{\mathrm{F}} = 0$ from formulae (10), (12) and (14).

Corollary 2 If the infinitesimal generators of the form invariance satisfy $X^{(0)}(B) = 0$ as well as $X^{(0)}(R_\mu) = 0$, Proposition 2 will give the trivial conserved quantity.

5. An illustrative example

The differential equations of the Hojman–Urrutia case can be written in the form

$$\ddot{x} + \dot{y} = 0, \quad \ddot{y} + y = 0. \tag{15}$$

Let

$$a^1 = x, \quad a^2 = y, \quad a^3 = \dot{x}, \quad a^4 = \dot{y},$$

then we can rewrite Eqs.(15) as Birkhoffian equations (1) in which the Birkhoffian and the Birkhoff functions are given by

$$\begin{aligned} B &= (a^2 \sin t + a^4 \cos t)^2, \\ R_1 &= 0, \\ R_2 &= a^1 - a^4 - (a^2 + a^3)t + (a^2 \sin t + a^4 \cos t)\cos t, \\ R_3 &= a^1 - a^4 - (a^2 + a^3)t, \\ R_4 &= -(a^2 \sin t + a^4 \cos t)\sin t. \end{aligned} \tag{16}$$

Substituting the infinitesimal generators

$$\xi_0 = \xi_2 = \xi_4 = 0, \quad \xi_1 = t, \quad \text{and} \quad \xi_3 = 1 \tag{17}$$

into the determining equations (7), we can find that the symmetry has form invariance. The structural equation (8) gives

$$G_{\mathrm{N}} = 0,$$

and the Noether conserved quantity (9) indirectly deduced from the form invariance gives

$$I_{\mathrm{N}} = a^1 - a^4 - (a^2 + a^3)t = \text{const.}.$$

The generalized structural equation (10) gives

$$G_{\mathrm{F}} = 0,$$

and the corresponding conserved quantity from formula (12) is a trivial conserved quantity.

Taking the infinitesimal generators

$$\begin{aligned} &\xi_0 = \xi_2 = \xi_4 = 0, \\ &\xi_1 = (a^2 + a^3)t + 1, \\ &\xi_3 = a^2 + a^3, \end{aligned} \tag{18}$$

we can be sure that these are the generators of a form invariance according to Eqs.(7). No gauge function G_{N} has been found from Eq.(8), and this means that there is not any corresponding conserved quantity. However, the generalized structural equation (10) gives

$$G_{\mathrm{F}} = 0,$$

and formula (12) gives

$$I_{\mathrm{F}} = a^2 + a^3 = \text{const.}$$

This is a non-Noether conserved quantity deduced from the form invariance.

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