Vol 13 No 12, December 2004
1009-1963/2004/13(12)/1977-05

Chinese Physics

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Filtering for linear systems with noise correlation and its application to singular systems*

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(Received 31 January 2004; revised manuscript received 13 May 2004)

In this paper, an optimal filter for a stochastic linear system with previous stage noise correlation is designed. Based on this result, together with the decomposition techniques of the stochastic singular linear system, the design of an optimal filter for a stochastic singular linear system is given.

Keywords: filter, stochastic system, singular system, noise correlation
PACC: 0250

1. Introduction

Since singular systems can be used to describe non-causal phenomena which arise naturally in robotics, physics, electric and chemical systems,[1] the analysis and design of singular systems has received great attention in the last few years (see, for example, Refs.[1–3] and references therein). It is shown that the solution of a singular system includes differential terms, so that singular systems are very sensitive to slight changes of the inputs. In practice, however, the systems are often affected by immeasurable disturbances. Therefore, the state estimations are very important. In fact, a few successes have been achieved in this field. In the deterministic case, the main achievements can be found in Refs.[4–6] and the references therein. As for the stochastic case, three basic methods, namely the least-squares method,[7] the maximum likelihood[8] and the minimum variance[9] approaches, have given rise to optimal estimators called generalized Kalman filters, which are algebraically equivalent. Recently, based on the modern time-series analysis method, a new time-domain Wiener filtering approach is presented,[10] in which the estimators can handle the optimal state filtering, smoothing and prediction problems in a unified framework.

In this paper, our purpose is to investigate the problem of filter for the stochastic singular systems in the other way. In Section 2, we investigate the optimal state estimation problem for previous stage noise correlation of linear systems by using the famous Projection Theorem[11] directly. In Section 3, we focus on the state estimation of linear singular systems. On the estimable assumption, the singular system is divided into two subsystems: one is non-casual and deterministic; the other is casual and stochastic, in which exists the one-step correlation between the inputs noise and the measurement noise. Applying the filter derived in Section 2, we obtain the filter for this stochastic subsystem. Then we can give the filter equations for the original system. Similar to the Kalman filter, the present filter is of simple form and compact structure. Besides, the filter of the stochastic subsystem is a reduced-order filter, so the computational burden is reduced.

2. Filtering for linear systems with previous stage noise correlation

In this section, our aim is to derive the Kalman filter equations which can account for time correlation

*Project supported by the State Key Development Programme for Basic Research of China (Grant No 2002CB312205), the Science Foundation of Jiangsu Education Bureau, and the Science Foundation from University of Science and Technology of Suzhou, China.

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in the disturbance inputs and measurement noise. We shall do this for the system for $k \geq 0$, defined as

$$x(k+1) = A(k)x(k) + G(k)\omega(k), \tag{1a}$$

$$y(k+1) = C(k+1)x(k+1) + \gamma(k+1). \tag{1b}$$

Here, $\{\omega(k)\}$ and $\{\gamma(k)\}$ are zero mean, Gaussian white processes with moments:

$$\mathrm{Cov}(\omega(k), \omega(j)) = Q(k)\delta(k,j),$$
$$\mathrm{Cov}(\gamma(k), \gamma(j)) = R(k)\delta(k,j).$$

Moreover, the inputs noise $\{\omega(k)\}$ and measurement noise $\{\gamma(k)\}$ are one-step correlative, that is,

$$\mathrm{Cov}(\omega(k), \gamma(j)) = S(k)\delta(k+1,j),$$

where $Q(k)$, $R(k)$ and $S(k)$ are positive definite, $\delta(k,j)$ is the Kronecker delta.

For convenience, we introduce the following symbols:

$$Y_k = \{y(0), y(1), \cdots, y(k)\},$$

$$\hat{x}(i|j) = E\{x(i)|Y_j\},$$

$$\hat{x}(k) = \hat{x}(k|k),$$

$$\tilde{x}(i|j) = x(i) - \hat{x}(i|j),$$

$$\tilde{x}(k) = x(k) - \hat{x}(k),$$

where $\hat{y}(i|j)$, $\hat{y}(k)$, $\tilde{y}(i|j)$ and $\tilde{y}(k)$are introduced similarly. For random variables ξ and ζ, V_ξ and $V_{\xi\zeta}$ denote the variance of ξ and the covariance of ξ and ζ respectively.

Suppose that the initial state $x(0)$ is a Gaussian random variable with mean $\bar{x}(0)$ and covariance V_0; moreover, $x(0)$ is independent of both $\omega(k)$ and $\gamma(k)$, then $\{x(k)\}$, $\{y(k)\}$ are both Gaussian. Hence the minimum variance and linear minimum variance estimators coincide. Applying the Projection Theorem (see, for example, Theorems 5.3.4 and 5.3.5 in Ref.[11]), we have

$$\begin{aligned}
\hat{x}(k+1) =& E\{x(k+1)|Y_k, y(k+1)\} = E\{x(k+1)|Y_k, \tilde{y}(k+1|k)\} \\
=& E\{x(k+1)|Y_k\} + E\{x(k+1)|\tilde{y}(k+1|k)\} - E\{x(k+1)\} \\
=& \hat{x}(k+1|k) + K(k+1)(\tilde{y}(k+1|k) - E\{\tilde{y}(k+1|k)\}) \\
=& \hat{x}(k+1|k) + K(k+1)\tilde{y}(k+1|k),
\end{aligned}$$

where

$$\begin{aligned}
K(k+1) =& V_{x(k+1)\tilde{y}(k+1|k)} V^{-1}_{\tilde{y}(k+1|k)}, \\
\hat{x}(k+1|k) =& E\{A(k)x(k) + G(k)\omega(k)|Y_k\} \\
=& A(k)E\{x(k)|Y_k\} + G(k)E\{\omega(k)|Y_k\} = A(k)\hat{x}(k), \\
\hat{y}(k+1|k) =& E\{y(k+1)|Y_k\} = E\{C(k+1)x(k+1) + \gamma(k+1)|Y_k\} \\
=& C(k+1)\hat{x}(k+1|k) = C(k+1)A(k)\hat{x}(k), \\
\tilde{y}(k+1|k) =& y(k+1) - \hat{y}(k+1|k) = C(k+1)x(k+1) + \gamma(k+1) - C(k+1)\hat{x}(k+1|k) \\
=& C(k+1)\tilde{x}(k+1|k) + \gamma(k+1), \\
E[x(k+1)\gamma'(k+1)] =& E\{[A(k)x(k) + G(k)\omega(k)]\gamma'(k+1)\} \\
=& G(k)E\{\omega(k)\gamma'(k+1)\} = G(k)S(k), \\
V_{x(k+1)\tilde{y}(k+1|k)} =& E\{[x(k+1) - \bar{x}(k+1)]\tilde{y}'(k+1|k)\} \\
=& E\{x(k+1)[C(k+1)\tilde{x}(k+1|k) + \gamma(k+1)]'\} - \bar{x}(k+1)E\{\tilde{y}'(k+1|k)\} \\
=& E\{x(k+1)\tilde{x}'(k+1|k)\}C(k+1)' + E\{x(k+1)\gamma'(k+1)\} \\
=& E\{[\hat{x}(k+1|k) + \tilde{x}(k+1|k)]\tilde{x}'(k+1|k)\}C(k+1)' + E\{x(k+1)\gamma'(k+1)\} \\
=& E\{\tilde{x}(k+1|k)\tilde{x}'(k+1|k)\}C(k+1)' + E\{x(k+1)\gamma'(k+1)\} \\
=& V(k+1|k)C(k+1)' + G(k)S(k), \qquad (2) \\
V_{\tilde{y}(k+1|k)} =& E\{[C(k+1)\tilde{x}(k+1|k) + \gamma(k+1)][C(k+1)\tilde{x}(k+1|k) + \gamma(k+1)]'\} \\
=& C(k+1)V(k+1|k)C(k+1)' + C(k+1)E\{\tilde{x}(k+1|k)\gamma'(k+1)\} \\
& + E\{\gamma(k+1)\tilde{x}'(k+1|k)\}C(k+1)' + R(k+1).
\end{aligned}$$

Note that

$$\begin{aligned}&E\{\tilde{x}(k+1|k)\gamma'(k+1)\}\\&=E\{[x(k+1)-\hat{x}(k+1|k)]\gamma'(k+1)\}\\&=E[x(k+1)\gamma'(k+1)]-E[\hat{x}(k+1|k)\gamma'(k+1)]\\&=E[x(k+1)\gamma'(k+1)]=G(k)S(k),\end{aligned}$$

and we have

$$\begin{aligned}V_{\tilde{y}(k+1|k)}=&C(k+1)V(k+1|k)C(k+1)'\\&+C(k+1)G(k)S(k)\\&+S(k)'G(k)'C(k+1)'+R(k+1).\end{aligned}\quad(3)$$

Since

$$\begin{aligned}\tilde{x}(k+1|k)=&x(k+1)-\hat{x}(k+1|k)\\=&A(k)\tilde{x}(k)+G(k)\omega(k)\end{aligned}$$

and

$$\begin{aligned}&[x(k+1)-\hat{x}(k+1|k)][x(k+1)-\hat{x}(k+1|k)]'\\=&A(k)\tilde{x}(k)\tilde{x}(k)'A(k)'+A(k)\tilde{x}(k)\omega(k)'G(k)'\\&+G(k)\omega(k)\tilde{x}(k)'A(k)'+G(k)\omega(k)\omega(k)'G(k)',\end{aligned}$$

we obtain the covariance estimate extrapolation

$$\begin{aligned}V(k+1|k)=&E\{\tilde{x}(k+1|k)\tilde{x}(k+1|k)'\}\\=&A(k)V(k)A(k)'+G(k)Q(k)G(k)'.\end{aligned}$$

Moreover,

$$\begin{aligned}\tilde{x}(k+1)=&x(k+1)-\hat{x}(k+1)\\=&x(k+1)-\hat{x}(k+1|k)-K(k+1)\\&\times[y(k+1)-C(k+1)\hat{x}(k+1|k)]\\=&\tilde{x}(k+1|k)-k(k+1)C(k+1)\\&\times\tilde{x}(k+1|k)-K(k+1)\gamma(k+1)\\=&[I-K(k+1)C(k+1)]\\&\times\tilde{x}(k+1|k)-K(k+1)\gamma(k+1),\end{aligned}$$

which implies the covariance estimate update

$$\begin{aligned}V(k+1)=&E\{\tilde{x}(k+1)\tilde{x}(k+1)'\}\\=&[I-K(k+1)C(k+1)]V(k+1|k)\\&\times[I-K(k+1)C(k+1)]'\\&-[I-K(k+1)C(k+1)]\\&\times G(k)S(k)K(k+1)'\\&-K(k+1)S(k)'G(k)'\\&\times[I-K(k+1)C(k+1)]'\\&+K(k+1)R(k+1)K(k+1)'.\end{aligned}\quad(4)$$

Substituting the equation $K(k+1)V_{\tilde{y}(k+1|k)}K(k+1)'-V_{x(k+1)\tilde{y}(k+1|k)}K(k+1)'=0$, together with Eqs.(2) and (3), into Eq.(4) provides another form for the covariance

$$\begin{aligned}V(k+1)=&[I-K(k+1)C(k+1)]V(k+1|k)\\&-K(k+1)S(k)'G(k)'.\end{aligned}$$

To sum up, the equations of the estimator are given by the state estimate extrapolation

$$\hat{x}(k+1|k)=A(k)\hat{x}(k)\quad(5)$$

and the covariance estimate extrapolation

$$V(k+1|k)=A(k)V(k)A(k)'+G(k)Q(k)G(k)'.\quad(6)$$

The recursive mean-value estimator provides the filter gain computation

$$\begin{aligned}K(k+1)=&[V(k+1|k)C(k+1)'+G(k)S(k)]\\&\times\{C(k+1)V(k+1|k)C(k+1)'\\&+C(k+1)G(k)S(k)\\&+S(k)'G(k)'C(k+1)'+R(k+1)\}^{-1},\end{aligned}\quad(7)$$

the state estimate update equation

$$\hat{x}(k+1)=\hat{x}(k+1|k)+K(k+1)\tilde{y}(k+1|k),\quad(8)$$

$$\tilde{y}(k+1|k)=y(k+1)-C(k+1)\hat{x}(k+1|k),\quad(9)$$

the covariance estimate update

$$\begin{aligned}V(k+1)=&[I-K(k+1)C(k+1)]V(k+1|k)\\&-K(k+1)S(k)'G(k)',\end{aligned}\quad(10)$$

and the initial conditions

$$\hat{x}(0)=\bar{x}(0),\quad V(0)=V_0.\quad(11)$$

3. Estimation of linear singular systems

Consider the following discrete-time singular stochastic system:

$$E(k+1)x(k+1)=A(k)x(k)+G(k)\omega(k),\quad(12a)$$

$$y(k+1)=C(k+1)x(k+1)+\gamma(k+1),\quad(12b)$$

where $E(k)$, $A(k)\in R^{m\times n}$, $x(k)\in R^n$, $k=0,1,2,\cdots$; $\{\omega(k)\}$ and $\{\gamma(k)\}$ are zero mean, Gaussian white processes with moments

$$\begin{aligned}&E(\omega(k)\omega(j)')=Q(k)\delta(k,j),\\&E(\gamma(k)\gamma(j)')=R(k)\delta(k,j),\\&E(\omega(k)\gamma(j)')=0.\end{aligned}$$

$Q(k)$ and $R(k)$ are positive definite. We suppose that the initial state $x(0)$ is a Gaussian random variable with mean $\bar{x}(0)$ and covariance V_0, moreover, $x(0)$ is independent of both $\omega(k)$ and $\gamma(k)$.

Assumption 1: The system (12) is estimable, i.e.

$$\text{rank}\begin{bmatrix} E(k) \\ C(k) \end{bmatrix} = n. \tag{13}$$

Then there exist two nonsingular matrices $T_1(k)$ and $T_2(k)$, such that

$$T_1(k)E(k) + T_2(k)C(k) = I. \tag{14}$$

Therefore, the system (12) can be rewritten as

$$\begin{aligned} x(k+1) =& T_1(k+1)A(k)x(k) + T_1(k+1) \\ & \times G(k)\omega(k) + T_2(k+1)y(k+1) \\ & - T_2(k+1)\gamma(k+1) \\ =& T_1(k+1)A(k)x(k) \\ & + T_2(k+1)y(k+1) + D(k)\xi(k), \quad (15a) \\ y(k+1) =& C(k+1)x(k+1) + \gamma(k+1), \quad (15b) \end{aligned}$$

where

$$D(k) = [T_1(k+1)G(k) - T_2(k+1)],$$

$$\xi(k) = \begin{bmatrix} \omega(k) \\ \gamma(k+1) \end{bmatrix}.$$

Obviously, the regular system described in Eq.(15) is not strictly casual in that the linear recursive state equation needs $y(k+1)$ to obtain $x(k+1)$. However, it can be split into two subsystems as follows:

$$\begin{aligned} x_1(k+1) &= T_1(k+1)A(k)x_1(k) + D(k)\xi(k), \quad (16a) \\ y_1(k+1) &= C(k+1)x_1(k+1) + \gamma(k+1), \quad (16b) \\ x_1(0) &= x(0) - \bar{x}(0), \quad (16c) \\ x_2(k+1) &= T_1(k+1)A(k)x_2(k) \\ & \quad + T_2(k+1)y(k+1), \quad (17a) \\ y_2(k+1) &= C(k+1)x_2(k+1), \quad (17b) \\ x_2(0) &= \bar{x}(0), \quad (17c) \end{aligned}$$

where $x(k) = x_1(k) + x_2(k)$, $y(k+1) = y_1(k+1) + y_2(k+1)$.

We can see that Eq.(17) is a non-casual deterministic system, while Eq.(16) is a casual stochastic system.

Note that

$$E(\xi(k)\xi(j)') = V_\xi(k)\delta(k,j),$$
$$E(\xi(k)\gamma(j)') = V_{\xi\gamma}(k)\delta(k+1,j),$$

where

$$V_\xi(k) = \begin{bmatrix} Q(k) & 0 \\ 0 & R(k+1) \end{bmatrix},$$

$$V_{\xi\gamma}(k) = \begin{bmatrix} 0 \\ R(k+1) \end{bmatrix},$$

we can also see that there is a one-step correlation between state and output noise of system (16). Applying the proposed filter described by Eqs.(5)–(11) in Section 2, the optimal estimate for Eq.(16) is given as follows:

variance algorithm

$$V_{\tilde{x}_1}(k+1|k) = T_1(k+1)A(k)V_{\tilde{x}_1}(k)A(k)'T_1(k+1)' + D(k)V_\xi(k)D(k)',$$

gain equation

$$\begin{aligned} K(k+1) =& [V_{\hat{x}_1}(k+1|k)C(k+1)' + D(k)V_{\xi\gamma}(k)] \\ & \times \{C(k+1)V_{\hat{x}_1}(k+1|k)C(k+1)' + C(k+1)D(k)V_{\xi\gamma}(k) + V_{\xi\gamma}(k)'D(k)'C(k+1)' + R(k+1)\}^{-1}, \end{aligned}$$

filter algorithm

$$\hat{x}_1(k+1) = T_1(k+1)A(k)\hat{x}_1(k) + K(k+1)[y_1(k+1) - C(k+1)T_1(k+1)A(k)\hat{x}_1(k)],$$

variance estimate update

$$V_{\tilde{x}_1}(k+1) = [I - K(k+1)C(k+1)]V_{\tilde{x}_1}(k+1|k) - K(k+1)V_{\xi\gamma}(k)'D(k)',$$

initial conditions

$$\hat{x}_1(0) = 0, \quad V_{\hat{x}_1}(0) = V_{\hat{x}_1(0)}.$$

Recall that Eq.(17) is a deterministic system, we have

$$\begin{aligned}
\hat{x}(k+1) =& \hat{x}_1(k+1) + x_2(k+1) \\
=& T_1(k+1)A(k)\hat{x}_1(k) + K(k+1)[y_1(k+1) - C(k+1)T_1(k+1)A(k)\hat{x}_1(k)] \\
&+ T_1(k+1)A(k)x_2(k) + T_2(k+1)y(k+1) \\
=& T_1(k+1)A(k)\hat{x}(k) + K(k+1)[y(k+1) - C(k+1)x_2(k+1) \\
&- C(k+1)T_1(k+1)A(k)\hat{x}_1(k)] + T_2(k+1)y(k+1) \\
=& T_1(k+1)A(k)\hat{x}(k) + K(k+1)[y(k+1) - C(k+1)T_1(k+1)A(k)\hat{x}(k) \\
&- C(k+1)T_2(k+1)y(k+1)] + T_2(k+1)y(k+1) \\
=& [I - K(k+1)C(k+1)]T_1(k+1)A(k)\hat{x}(k) \\
&+ [K(k+1) - K(k+1)C(k+1)T_2(k+1) + T_2(k+1)]y(k+1), \\
V_{\tilde{x}}(k+1|k) =& V_{\tilde{x}_1}(k+1|k), \\
V_{\tilde{x}}(k+1) =& V_{\tilde{x}_1}(k+1).
\end{aligned}$$

Additionally,

$$D(k)V_{\xi\gamma}(k) = [T_1(k+1)G(k) - T_2(k+1)]\begin{bmatrix} 0 \\ R(k+1) \end{bmatrix} = -T_2(k+1)R(k+1),$$

$$K(k+1)V_{\xi\gamma}(k)'D(k)' = -K(k+1)R(k+1)T_2(k+1)'.$$

These considerations justify the following filter for the singular system (12):

variance algorithm

$$V(k+1|k) = T_1(k+1)A(k)V(k)A(k)'T_1(k+1)' + D(k)V_\xi(k)D(k)',$$

gain equation

$$\begin{aligned}
K(k+1) =& [V(k+1|k)C(k+1)' - T_2(k+1)R(k+1)] \\
&\times \{C(k+1)V(k+1|k)C(k+1)' - C(k+1)T_2(k+1)R(k+1) \\
&- R(k+1)T_2(k+1)'C(k+1)' + R(k+1)\}^{-1},
\end{aligned}$$

filter algorithm

$$\begin{aligned}
\hat{x}(k+1) =& [I - K(k+1)C(k+1)]T_1(k+1)A(k)\hat{x}(k) \\
&+ [K(k+1) - K(k+1)C(k+1)T_2(k+1) + T_2(k+1)]y(k+1),
\end{aligned}$$

variance estimate update

$$V(k+1) = [I - K(k+1)C(k+1)]V(k+1|k) + K(k+1)R(k+1)T_2(k+1)',$$

Initial conditions

$$\hat{x}(0) = \bar{x}(0), V(0) = V_0.$$

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