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Second harmonic generation of propagating collective excitations in Bose–Einstein condensates*

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We consider a possible second harmonic generation (SHG) of propagating collective excitations in a two-component Bose–Einstein condensate (BEC) with repulsive atom–atom interactions. We show that the phase-matching condition for the SHG can be fulfilled if the wave vectors and frequencies of the excitations are chosen adequately from different dispersion branches. We solve the nonlinear amplitude equations for the SHG derived using a method of multiple-scales and provide SHG solutions similar to those obtained for a SHG in nonlinear optical media. A possible experimental realization of the SHG for the propagating collective modes in a cigar-shaped two-component BEC is also discussed.

Keywords: second harmonic generation, collective excitations, Bose–Einstein condensation
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1. Introduction

The remarkable experimental realization of Bose–Einstein condensation in weakly interacting atomic gases[1] has opened a new direction for the study on the nonlinear properties of matter waves.[2] The most spectacular experimental progress achieved recently concerns the demonstration of atomic four-wave mixing,[3] the discovery of superradiance,[4] the development of matter-wave amplification,[5,6] and the observation of dark and bright solitons as well as vortices in Bose–Einstein condensates (BECs).[7] At the same time, a large amount of theoretical study in this area has appeared,[8−14] and new phenomena such as atom holography through BEC,[15] coherent matter-wave amplification and superradiance in degenerate Fermi gases,[16] etc, have been predicted. These researches have enabled the extension of linear atom optics to a nonlinear regime, i.e. nonlinear atom optics,[17] very much like the laser led to the development of nonlinear optics in the 1960s.

Wave resonance plays an important role in nonlinear optics.[18] In the nonlinear atom optics based on BECs, even though many theoretical and experimental efforts on wave resonance exist,[3−6] the phenomena studied up to now belong to various processes of four-wave mixing of BEC matter waves. Because the interaction of the matter waves in BECs is described by a cubic nonlinearity, a second harmonic generation (SHG), which is a second-order process and thus requires a quadratic nonlinearity, is impossible. However, if we consider the excitations from the ground state of a condensate, the interaction between the excitations is of quadratic nonlinearity and hence a SHG is possible. Recently, nonlinear coupling and harmonic generation of collective modes created in a BEC have been considered theoretically (see Refs.[19–21] for details) and observed experimentally by Hechenblaikner *et al*[22] and Hodby *et al.*[23] In these interesting works the resonant interaction between two oscillating (or standing-wave) modes excited in BECs have been taken into account. For such low-frequency nonlinear oscillations, a resonant harmonic generation requires only a frequency-matching condition.

In the present work, we explore the possibility of a SHG based on the interaction of two propagating collective modes excited in a two-component BEC. A simple result for the limiting case in the absence of mass and trapping frequency differences has been given recently.[24] Here we provide a more general result to allow for different masses and trapping fre-

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quencies for the two BEC components. Note that different from standing-waves, the SHG for propagating waves requires a phase-matching condition (or called resonance condition, see Eq.(29) below) for wave vectors and frequencies of the fundamental wave and the second harmonic wave. For a single-component BEC the SHG is not possible for collective excitations because its excitation spectrum, which takes the form $\omega(q)=q(c^2+q^2/4)^{1/2}$ with the wave number of the excitation q and the sound speed of the system c,[13] can not satisfy the phase-matching condition. However, a two-component BEC displays an excitation spectrum with two branches and hence provides the possibility to fulfil the SHG phase-matching condition. Since the SHG is a process of energy up-conversion, at zero temperature such process can be well described by two coupled Gross–Pitaevskii (GP) equations. Using a method of multiple-scales we derive the nonlinearly coupled envelope equations describing the SHG and give their explicit solutions. We show that an experimental realization of such SHG may give information about the interaction between different components of the condensate.

The paper is organized as follows. Section 2 presents our model and gives the ground state solution of the system. In Section 3 we consider the linear excitations from the ground state. In Section 4 we analyse the phase-matching condition and derive the nonlinear amplitude equations for the SHG using a method of multiple scales. The SHG solutions for the amplitude equations similar to those obtained for a photon SHG in a nonlinear optical medium are also given in this section. The last section contains a discussion and summary of our results.

2. Model and ground state solution

2.1. The model

We consider a two-component BEC which is a binary mixture of alkali condensates. Such mixture may consist of different alkalis such as ^{87}Rb and ^{23}Na, or different isotopes such as ^{87}Rb and ^{85}Rb, or even different hyperfine spin states of the same alkali such as the $|F=2,m_F=2\rangle$ and $|F=1,m_F=1\rangle$ states of ^{87}Rb. Denoting $\Psi_j(\boldsymbol{r},t)$ as the order parameter of species j with particle number $N_j=\int \mathrm{d}\boldsymbol{r}|\Psi_j|^2$ $(j=1,2)$, the equations of motion controlling Ψ_j are

$$\mathrm{i}\hbar\frac{\partial\Psi_1}{\partial t}=\Big[-\frac{\hbar^2}{2m_1}\nabla^2+V_1(\boldsymbol{r}) + g_{11}|\Psi_1|^2+g_{12}|\Psi_2|^2\Big]\Psi_1, \tag{1}$$

$$\mathrm{i}\hbar\frac{\partial\Psi_2}{\partial t}=\Big[-\frac{\hbar^2}{2m_2}\nabla^2+V_2(\boldsymbol{r}) + g_{21}|\Psi_1|^2+g_{22}|\Psi_2|^2\Big]\Psi_2, \tag{2}$$

where m_j and $V_j(\boldsymbol{r})$ are respectively the atomic mass and external trapping potential for the species j, $g_{jl}=2\pi\hbar^2a_{jl}/m_{jl}$ is the interaction parameter with a_{jl} $(j,l=1,2)$ being the s-wave scattering length between the species j and the species l ($a_{jl}>0$ for repulsive interaction) and $m_{jl}=m_jm_l/(m_j+m_l)$ being the reduced mass. We consider an anisotropic harmonic trap for which the trapping potentials take the form

$$V_j(\boldsymbol{r})=\frac{m_j}{2}[\omega_{jx}^2x^2+\omega_{j\perp}^2(y^2+z^2)], \tag{3}$$

where ω_{jx} and $\omega_{j\perp}$ are the trap frequencies of the species j in the axial (x) and the transverse (y and z) directions, respectively.

Expressing the order parameters in terms of their modulus and phases, i.e. $\Psi_j=\sqrt{n_j}\exp(\mathrm{i}\phi_j)$, we obtain a set of coupled nonlinear equations for n_j and ϕ_j $(j=1,2)$, which have the dimensionless form

$$\frac{\partial n_1}{\partial t}+\nabla_\perp\cdot(n_1\nabla_\perp\phi_1)+\varepsilon\frac{\partial}{\partial x}\left(n_1\frac{\partial\phi_1}{\partial x}\right)=0, \tag{4}$$

$$\frac{\partial n_2}{\partial t}+\frac{1}{\gamma_m}\left[\nabla_\perp\cdot(n_2\nabla_\perp\phi_2)+\varepsilon\frac{\partial}{\partial x}\left(n_2\frac{\partial\phi_2}{\partial x}\right)\right]=0, \tag{5}$$

$$\left[\frac{\partial\phi_1}{\partial t}-\frac{1}{2}\nabla_\perp^2+\frac{1}{2}(y^2+z^2)+\frac{1}{2}(\nabla_\perp\phi_1)^2\right]\sqrt{n_1} + \left[-\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{1}{2}\varepsilon^{-2}\delta^2x^2 + \frac{1}{2}\left(\frac{\partial\phi_1}{\partial x}\right)^2+G_{11}n_1+G_{12}n_2\right]\sqrt{n_1}=0, \tag{6}$$

$$\left[\frac{\partial\phi_2}{\partial t}-\frac{1}{2\gamma_m}\nabla_\perp^2+\frac{1}{2}\gamma_m\gamma_\omega^2(y^2+z^2) + \frac{1}{2\gamma_m}(\nabla_\perp\phi_2)^2\right]\sqrt{n_2} + \varepsilon\left[-\frac{1}{2\gamma_m}\frac{\partial^2}{\partial x^2}+\frac{1}{2}\gamma_m\gamma_x^2\varepsilon^{-2}\delta^2x^2 + \frac{1}{2\gamma_m}\left(\frac{\partial\phi_2}{\partial x}\right)^2+G_{21}n_1+G_{22}n_2\right]\sqrt{n_2}=0, \tag{7}$$

where $\gamma_m=m_2/m_1$, $\gamma_\omega=\omega_{2\perp}/\omega_{1\perp}$, $\gamma_x=\omega_{2x}/\omega_{1x}$, $G_{11}=1$, $G_{12}=g_{12}/g_{11}$, $G_{21}=g_{21}/g_{11}$, and $G_{22}=$

g_{22}/g_{11}. The condensate density, time, axial spatial coordinate and transverse spatial coordinates are measured respectively in the units of $n_0 = N_1/(l_0 a_{1\perp}^2)$, $t_0 = \omega_{1\perp}^{-1}$, $l_0 = \hbar/(m_1 n_0 g_{11})^{1/2} = (4\pi n_0 a_{11})^{-1/2}$ (healing length), and $a_{1\perp} = [\hbar/(m_1\omega_{1\perp})]^{1/2}$ (harmonic oscillator length in the transverse directions). $\nabla_\perp = (\partial/\partial y, \partial/\partial z)$ is the transverse gradient operator. The normalization conditions for Ψ_1 and Ψ_2 now read $\int \mathrm{d}\boldsymbol{r}\, n_1 = 1$ and $\int \mathrm{d}\boldsymbol{r}\, n_2 = N_2/N_1$, respectively. We see that, in addition to γ_m, γ_ω, γ_x, and G_{jl} $(j,l = 1,2)$, the system is characterized by two other dimensionless parameters $\varepsilon = n_0 g_{11}/(\hbar\omega_{1\perp})$ and $\delta = \omega_{1x}/\omega_{1\perp}$. The parameter ε describes the ratio between the atomic interaction and the strength of the transverse confinement of the system, while the parameter δ reflects the extent of the anisotropy of the trapping potentials. Equations (4)–(7) are equivalent to the equations of motion in hydrodynamics for a two-fluid mixture. Note that when transferring Eqs.(1) and (2) into Eqs.(4)–(7), no approximation has been made.

2.2. Ground state

We first study the ground state of the system. We consider a long cigar-shaped trap, for which $\omega_{jx} \ll \omega_{j\perp}$ $(j = 1,2)$, and assume that the transverse confinement is strong enough so that the conditions $a_{1\perp} \ll l_0$ and $\hbar\omega_{1x} \ll n_0 g_{11} \ll \hbar\omega_{1\perp}$ can be satisfied. In this situation we have $\delta \ll \varepsilon \ll 1$. We assume also that $\delta = \Omega_{11}\varepsilon^2$ with Ω_{11} of order unity. This means that the trapping potentials along the axial direction are slowly-varying functions of x because $U_1 \equiv \varepsilon^{-2}\delta^2 x^2/2$ appearing in Eqs.(6) and (7) reads as $U_1 = \Omega_{11}^2 X^2/2$ with $X = \varepsilon x$ (a slow variable). Thus if the dimensionless parameters γ_m, γ_ω, γ_x and G_{jl} $(j,l = 1,2)$ are fixed and are assumed to be of the order of unity, then in Eqs.(4)–(7) we have only *one* small parameter ε, which can be taken as an expansion parameter in the perturbation expansion given below.[1)] On the other hand, because of the strong confinement in the transverse directions, the system can be taken as quasi-one-dimensional. This implies that at sufficiently low temperature the transverse motion of the atoms is essentially 'frozen' and is governed by the ground-state wavefunctions of corresponding transverse harmonic oscillators.[26,27] Thus one can assume $\sqrt{n_1} = A_1(x,t)\psi_0(y)\psi_0(z)$, $\sqrt{n_2} = A_2(x,t)\psi_0(y_0 y)\psi_0(y_0 z)$, $\phi_1 = \phi_1(x,t)$ and $\phi_2 = \phi_2(x,t)$, where $\psi_0(y)\psi_0(z)$ is the ground state wavefunction of the two-dimensional harmonic oscillator satisfying the equation $[-\nabla_\perp^2/2 + (y^2 + z^2)/2]\psi_0(y)\psi_0(z) = \psi_0(y)\psi_0(z)$. Then, by setting $y_0 = \sqrt{\gamma_m\gamma_\omega}$ and using the normalized ground harmonic oscillator wavefunction $\psi_0(y) = \pi^{-1/4}\exp(-y^2/2)$, Eqs.(4)–(7) are reduced to

$$\frac{\partial A_1}{\partial t} + \varepsilon\left[\frac{\partial A_1}{\partial x}\frac{\partial \phi_1}{\partial x} + \frac{1}{2}A_1\frac{\partial^2\phi_1}{\partial x^2}\right] = 0, \tag{8}$$

$$\frac{\partial A_2}{\partial t} + \varepsilon\frac{1}{\gamma_m}\left[\frac{\partial A_2}{\partial x}\frac{\partial \phi_2}{\partial x} + \frac{1}{2}A_2\frac{\partial^2\phi_2}{\partial x^2}\right] = 0, \tag{9}$$

$$\left(\frac{\partial\phi_1}{\partial t} + 1\right)A_1 + \varepsilon\left[-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\left(\frac{\partial\phi_1}{\partial x}\right)^2 + U_1(X) + G_{11}I_0A_1^2 + G_{12}J_0A_2^2\right]A_1 = 0, \tag{10}$$

$$\left(\frac{\partial\phi_2}{\partial t} + \gamma_\omega\right)A_2 + \varepsilon\left[-\frac{1}{2\gamma_m}\frac{\partial^2}{\partial x^2} + \frac{1}{2\gamma_m}\left(\frac{\partial\phi_2}{\partial x}\right)^2 + U_2(X) + G_{21}J_0A_1^2 + G_{22}I_0A_2^2\right]A_2 = 0, \tag{11}$$

where

$$I_0 = \int \mathrm{d}y\mathrm{d}z\psi_0^4(y)\psi_0^4(z) = 1/(2\pi),$$

$$J_0 = \int \mathrm{d}y\mathrm{d}z\psi_0^2(y)\psi_0^2(z)\psi_0^2(\sqrt{\gamma_m\gamma_\omega}y)\psi_0^2(\sqrt{\gamma_m\gamma_\omega}z) = 1/[\pi(1+\gamma_m\gamma_\omega)],$$

$U_1(X) = \Omega_{11}^2X^2/2$, $U_2(X) = \gamma_m\gamma_x^2U_1(X)$ with $X = \varepsilon x$.

The ground state of such a quasi-one-dimensional system corresponds to taking $\partial\phi_j/\partial t = -\tilde{\mu_j}$ ($\tilde{\mu_j}$ is the dimensionless chemical potential for the species j) and $\partial\phi_j/\partial x = 0$ (i.e. no flow in the system). This, from Eqs.(8) and (9), results in $A_j = A_{j\mathrm{GS}}$ being time-independent in the ground state. Letting $\tilde{\mu_j} = \mu_j^{(0)} + \varepsilon\mu_j^{(1)}$, Eqs.(10) and (11) become

$$(-\mu_1^{(0)} + 1)A_{1\mathrm{GS}} + \varepsilon\left[-\frac{1}{2}\frac{\partial^2}{\partial x^2} + U_1(X) - \mu_1^{(1)} + G_{11}I_0A_{1\mathrm{GS}}^2 + G_{12}J_0A_{2\mathrm{GS}}^2\right]A_{1\mathrm{GS}} = 0, \tag{12}$$

$$(-\mu_2^{(0)} + \gamma_\omega)A_{2\mathrm{GS}} + \varepsilon\left[-\frac{1}{2\gamma_m}\frac{\partial^2}{\partial x^2} + U_2(X) - \mu_2^{(1)} + G_{21}J_0A_{1\mathrm{GS}}^2 + G_{22}I_0A_{2\mathrm{GS}}^2\right]A_{2\mathrm{GS}} = 0. \tag{13}$$

[1)] We stress that to obtain a controllable and consistent perturbation expansion for a nonlinear system with many variables and parameters, it is helpful and sometimes necessary to make a non-dimensionalization of the equations of motion and compare with each other the relative orders of magnitude of the dimensionless parameters appearing in the system, see Ref.[25].

To solve Eqs.(12) and (13) we make the Taylor expansion $A_{j\mathrm{GS}} = A_{j\mathrm{GS}}^{(0)} + \varepsilon A_{j\mathrm{GS}}^{(1)} + \varepsilon^2 A_{j\mathrm{GS}}^{(2)} + \cdots$ with $A_{j\mathrm{GS}}^{(l)}$ $(j = 1,2; l = 0,1,2,...)$ being functions of the slow variable X. At ε^0-order we obtain $\mu_1^{(0)} = 1$ and $\mu_2^{(0)} = \gamma_\omega$. At ε-order one has the solution for $A_{j\mathrm{GS}}^{(0)}$:

$$(A_{1\mathrm{GS}}^{(0)})^2 = \frac{I_0 G_{22}\mu_1^{(1)} - J_0 G_{12}\mu_2^{(1)} - (I_0 G_{22} - J_0 G_{12}\gamma_m\gamma_x^2)U_1(X)}{I_0^2 G_{11}G_{22} - J_0^2 G_{12}G_{21}}, \tag{14}$$

$$(A_{2\mathrm{GS}}^{(0)})^2 = \frac{I_0 G_{11}\mu_2^{(1)} - J_0 G_{21}\mu_1^{(1)} - (I_0 G_{11}\gamma_m\gamma_x^2 - J_0 G_{21}))U_1(X)}{I_0^2 G_{11}G_{22} - J_0^2 G_{12}G_{21}}. \tag{15}$$

We find that high-order $A_{j\mathrm{GS}}^{(l)}$ can be taken as zero up to $l = 3$. Other higher-order terms are too small to play any role in the SHG process discussed in Section 4 below. Accordingly, we can safely take Eqs.(14) and (15) as the ground-state solution in the axial direction of the condensate.

Depending on the parameters of the system, the ground state of a multi-species BEC can have very rich structures. One of them is the phase separation if the interaction between different species (represented by $G_{12} = G_{21}$ in our model) is big enough.[28] Here, however, we assume that G_{12} is not very big so that $I_0^2 G_{11}G_{22} - J_0^2 G_{12}G_{21} > 0$ (i. e. $(1+\gamma_m\gamma_\omega)^2 G_{11}G_{22} - 4G_{12}G_{21} > 0$). Under this condition, the ground state solution given above is modulationally stable and hence the phase separation does not occur (see the discussion in Section 3 below). From Eqs.(14) and (15), if both $I_0 G_{22} - J_0 G_{12}\gamma_m\gamma_x^2$ and $I_0 G_{11}\gamma_m\gamma_x^2 - J_0 G_{21}$ are positive one can easily obtain the radius R_j $(j = 1,2)$ of the ground-state condensate in the axial direction:

$$R_1 = \frac{\sqrt{2}}{\Omega_{11}}\left(\frac{I_0 G_{22}\mu_1^{(1)} - J_0 G_{12}\mu_2^{(1)}}{I_0 G_{22} - J_0 G_{12}\gamma_m\gamma_x^2}\right)^{1/2}, \tag{16}$$

$$R_2 = \frac{\sqrt{2}}{\Omega_{11}}\left(\frac{I_0 G_{11}\mu_2^{(1)} - J_0 G_{21}\mu_1^{(1)}}{I_0 G_{11}\gamma_m\gamma_x^2 - J_0 G_{21}}\right)^{1/2}, \tag{17}$$

for the species 1 and the species 2, respectively. The modified chemical potentials $\mu_j^{(1)}$ can be obtained from the normalized conditions $\int \mathrm{d}x A_{1\mathrm{GS}}^2 = 1$ and $\int \mathrm{d}x A_{2\mathrm{GS}}^2 = \gamma_m\gamma_\omega N_2/N_1$. For the harmonic potentials given above we have

$$\mu_1^{(1)} = \left(\frac{3\Omega_{11}}{4\sqrt{2}}\varepsilon\right)^{2/3} \frac{I_0 G_{11}(I_0 G_{22} - J_0 G_{12}\gamma_m\gamma_x^2)^{1/3} + J_0 G_{12}(\gamma_m\gamma_\omega\frac{N_2}{N_1})^{2/3}(I_0 G_{11}\gamma_m\gamma_x^2 - J_0 G_{21})^{1/3}}{(I_0^2 G_{11}G_{22} - J_0^2 G_{12}G_{21})^{1/3}}, \tag{18}$$

$$\mu_2^{(1)} = \left(\frac{3\Omega_{11}}{4\sqrt{2}}\varepsilon\right)^{2/3} \frac{I_0 G_{22}(\gamma_m\gamma_\omega\frac{N_2}{N_1})^{2/3}(I_0 G_{11}\gamma_m\gamma_x^2 - J_0 G_{21})^{1/3} + J_0 G_{21}(I_0 G_{22} - J_0 G_{12}\gamma_m\gamma_x^2)^{1/3}}{(I_0^2 G_{11}G_{22} - J_0^2 G_{12}G_{21})^{1/3}}. \tag{19}$$

If there is no confinement in the axial direction, i.e. $U_1(X) = 0$, the modified chemical potentials take the simple form

$$\mu_1^{(1)} = I_0 G_{11}/L_1 + J_0 G_{12}\gamma_m\gamma_\omega N_2/(N_1 L_2), \tag{20}$$

$$\mu_2^{(1)} = J_0 G_{21}/L_1 + I_0 G_{22}\gamma_m\gamma_\omega N_2/(N_1 L_2), \tag{21}$$

where L_j is the condensate length in the x-direction of the species j $(j = 1,2)$. In practice any condensate has a finite length. But if the length is long enough one can for simplicity neglect its boundary effect in the axial direction and the quantity L_j can be approximated by R_j given in Eqs.(16) and (17). From Eqs.(18), (19) and Eqs.(20), (21) we see that the correction terms of the chemical potentials $\mu_j^{(1)}$ are the functions of the parameters γ_m, γ_ω, γ_x, G_{jl} and N_2/N_1. Note that the chemical potential for the species j in physical units is given by $\mu_j = \hbar\omega_{1\perp}\tilde{\mu}_j$. Thus we have $\mu_1 = \hbar\omega_{1\perp}(1 + \varepsilon\mu_1^{(1)})$ and $\mu_2 = \hbar\omega_{1\perp}(\gamma_\omega + \varepsilon\mu_2^{(1)})$.

3. Linear excitations

Now we begin to consider the elementary excitations, i.e. collective modes, created from the ground state obtained in the preceding section. Because the system is considered to be strongly confined in the transverse directions and the trapping potentials in the axial direction are slow-varying functions of x, the collective modes with wavelength smaller than the axial size of the condensate can be created and propagate in the axial direction. To investigate such excitations we take $A_j = A_j(x,\tau)$, $\phi_j = -\tilde{\mu}_j t + \tilde{\phi}_j(x,\tau)$ with $\tau = \varepsilon t$, then Eqs.(8)–(11) take the form

$$\frac{\partial A_1}{\partial\tau} + \frac{\partial A_1}{\partial x}\frac{\partial\tilde{\phi}_1}{\partial x} + \frac{1}{2}A_1\frac{\partial^2\tilde{\phi}_1}{\partial x^2} = 0, \tag{22}$$

$$\frac{\partial A_2}{\partial \tau}+\frac{1}{\gamma_m}\left(\frac{\partial A_2}{\partial x}\frac{\partial \tilde{\phi}_2}{\partial x}+\frac{1}{2}A_2\frac{\partial^2\tilde{\phi}_2}{\partial x^2}\right)=0, \tag{23}$$

$$\left[\frac{\partial\tilde{\phi}_1}{\partial\tau}-\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{1}{2}\left(\frac{\partial\tilde{\phi}_1}{\partial x}\right)^2+U_1(X)-\mu_1^{(1)} + G_{11}I_0A_1^2+G_{12}J_0A_2^2\right]A_1=0, \tag{24}$$

$$\left[\frac{\partial\tilde{\phi}_2}{\partial\tau}-\frac{1}{2\gamma_m}\frac{\partial^2}{\partial x^2}+\frac{1}{2\gamma_m}\left(\frac{\partial\tilde{\phi}_2}{\partial x}\right)^2+U_2(X)-\mu_2^{(1)} + G_{21}J_0A_1^2+G_{22}I_0A_2^2\right]A_2=0. \tag{25}$$

Note that these equations are valid for both linear and nonlinear excitations. For linear excitations we take $A_j - A_{j\mathrm{GS}}(X) = \varepsilon f_j(x,X,\tau)$ and $\tilde{\phi}_j = \varepsilon\varphi_j(x,X,\tau)$. Substituting them into Eqs.(22)–(25) and keeping the terms only to the power of ε, one obtains the linear equations for f_j and φ_j $(j=1,2)$. Then by assuming the plane-wave solution for the fluctuation $(f_1,f_2,\varphi_1,\varphi_2) = (f_{10},f_{20},\varphi_{10},\varphi_{20})\exp(\mathrm{i}\theta)+\mathrm{c.c.}$, where $\theta = qx-\omega\tau$ is the phase of the plane wave, c.c. represents the complex conjugate, f_{j0} and φ_{j0} are independent of x and τ but may be functions of the slow variable X, we get the linear dispersion relation of the collective modes:

$$\begin{aligned}\omega^2(q)/q^2 =&\omega_\pm^2(q)/q^2\\ =&\frac{1}{2}(\tilde{G}_{11}+\gamma_m^{-1}\tilde{G}_{22})+\frac{1}{8}(1+\gamma_m^{-2})q^2\\ &\pm\frac{1}{4}\left\{\left[2(\tilde{G}_{11}-\gamma_m^{-1}\tilde{G}_{22})+\frac{1}{2}(1-\gamma_m^{-2})q^2\right]^2+16\gamma_m^{-1}\tilde{G}_{12}\tilde{G}_{21}\right\}^{1/2},\end{aligned} \tag{26}$$

where $\tilde{G}_{11} = I_0G_{11}(A_{1\mathrm{GS}}^{(0)})^2$, $\tilde{G}_{12} = J_0G_{12}(A_{2\mathrm{GS}}^{(0)})^2$, $\tilde{G}_{21} = J_0G_{21}(A_{1\mathrm{GS}}^{(0)})^2$, and $\tilde{G}_{22} = I_0G_{22}(A_{2\mathrm{GS}}^{(0)})^2$. $(A_{j\mathrm{GS}}^{(0)})^2$ are given by Eqs.(14) and (15). From Eq.(26) we see that the dispersion curve of the collective modes has two branches, i. e. the upper branch $\omega_+(q)$ and the lower branch $\omega_-(q)$.

On the basis of Eqs.(26) we can discuss the modulational stability of the ground state. Note that (26) can be rewritten as

$$\beta 2\omega_\pm^2/Q^2 = b\pm\sqrt{b^2-4c}, \tag{27}$$

where $b = 2\tilde{G}_{11}+2\gamma_m^{-1}\tilde{G}_{22}+(1+\gamma_m^{-2})Q^2$ and $c = (2\tilde{G}_{11}+Q^2)(2\gamma_m^{-1}\tilde{G}_{22}+\gamma_m^{-2}Q^2)-4\gamma_m^{-1}\tilde{G}_{12}\tilde{G}_{21}$ with $Q = q/\sqrt{2}$. Because both b and b^2-4c are positive, $\omega_+(q)$ is always real and thus there is no instability of the ground state under the perturbation with frequency $\omega_+(q)$. However, $\omega_-(q)$ will be imaginary if $c<0$. This will happen if $\tilde{G}_{12}$, i.e. the repulsive interaction between two different condensate species, becomes larger. In this case, the fluctuation will grow exponentially and thus the ground state of the system is modulationally unstable. This type of instability is in fact a type of cross-phase modulational instability, well known in nonlinear optics.[29] Through this instability the system will undergo a symmetry breaking and hence a phase transition into a new ground state. This may result in the appearance of a phase separation in space for two condensates with different species. However, the modulational instability and hence the phase separation does not occur if $c>0$. It is obvious that the quantity c is always positive when $\tilde{G}_{11}\tilde{G}_{22} > \tilde{G}_{12}\tilde{G}_{21}$. This is just the stability condition $(1+\gamma_m\gamma_\omega)^2G_{11}G_{22}-4G_{12}G_{21}>0$ mentioned in Section 2.2. We see that the parameters of both the trapping potentials and the interactions between like and unlike species can be used to control the modulational stability and hence the phase separation. In this work, in order to have an efficient SHG discussed in the next section, we assume that there is no phase separation in the ground state of the system. For small q the collective modes are actually the low-energy phonons of the system. From Eq.(26) one obtains the sound speed c of the system as

$$\begin{aligned}c^2 =&c_\pm^2=\left(\frac{\mathrm{d}\omega_\pm}{\mathrm{d}q}\right)_{q\to0}^2=\frac{1}{2}(\tilde{G}_{11}+\gamma_m^{-1}\tilde{G}_{22})\\ &\pm\frac{1}{2}\left[(\tilde{G}_{11}-\gamma_m^{-1}\tilde{G}_{22})^2+4\gamma_m^{-1}\tilde{G}_{12}\tilde{G}_{21}\right]^{1/2}.\end{aligned} \tag{28}$$

We see that the system has two sound speeds, c_+ and c_-, originating from the two components in the condensate.

4. Second harmonic generation

4.1. Phase-matching condition

We are interested in a possible SHG based on the interaction of the propagating collective modes in the system. As mentioned in the Introduction, for a SHG to occur, two necessary conditions must be satisfied. One of them is the phase-matching condition, i.e.

$$q_2 = 2q_1, \qquad \omega_2 = 2\omega_1, \tag{29}$$

where q_1 (q_2) and ω_1 (ω_2) are the wave vector and frequency of the fundamental (second harmonic) waves,

respectively. Note that, for a single-component condensate, one obtains a single branch of Bogoliubov excitation spectrum, i.e. $\omega(q) = q(c^2 + q^2/4)^{1/2}$, (see Ref.[13]), which cannot satisfy the SHG phase-matching condition (29) for any finite q and hence a SHG is impossible in the single-component condensate. We now show that the two-component BEC, which has two branches of excitation spectrum, can provide this possibility.

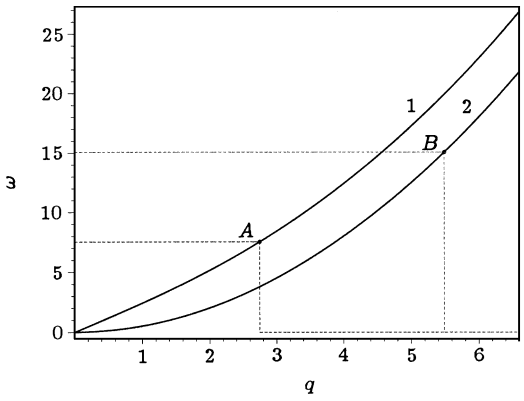


Fig.1. Dispersion curve of a two-component BEC consisting of two different hyperfine spin states $|1,-1\rangle$ and $|2,2\rangle$ of ^{87}Rb in a trap with $\gamma_m = \gamma_\omega = \gamma_x = 1.0$ and the particle numbers $N_1 = N_2 = 2 \times 10^6$. Curves 1 and 2 represent the upper branch, $\omega_+(q)$, and the lower branch, $\omega_-(q)$, respectively. The interaction parameters of the system are $G_{11} = 1.0$, $G_{12} = G_{21} = 0.9926$ and $G_{22} = 1.0027$. For the fundamental wave and the second harmonic wave, the phase-matched wave vectors and frequencies for the SHG are respectively $(q_1 = 2.743, \omega_1 = 7.536)$ and $(q_2 = 5.486, \omega_2 = 15.072)$, which have been illustrated by the points $A = (q_1, \omega_1)$ and $B = (q_2, \omega_2)$ in the figure.

According to the linear dispersion relation given by (26), we choose $\omega_1 = \omega_+(q_1)$ and $\omega_2 = \omega_-(q_2) = \omega_-(2q_1)$. Then the condition (29) is equivalent to $\omega_-(2q_1) = 2\omega_+(q_1)$. It results in the equation for determining q_1:

$$\begin{aligned}&\frac{3}{2}(1+\gamma_m^{-2})q_1^2\\ =&2\{[\tilde{G}_{11}-\gamma_m^{-1}\tilde{G}_{22}+(1-\gamma_m^{-2})q_1^2]^2\\ &+4\gamma_m^{-1}\tilde{G}_{12}\tilde{G}_{21}\}^{1/2}\\ &+\left\{\left[2(\tilde{G}_{11}-\gamma_m^{-1}\tilde{G}_{22})+\frac{1}{2}(1-\gamma_m^{-2})q_1^2\right]^2\right.\\ &\left.+16\gamma_m^{-1}\tilde{G}_{12}\tilde{G}_{21}\right\}^{1/2}. \qquad (30)\end{aligned}$$

This allows a solution of a finite q_1. In particular, when $\gamma_m = 1$, the solution takes the simple form

$$q_1 = \frac{2}{\sqrt{3}}[(\tilde{G}_{11}-\tilde{G}_{22})^2 + 4\tilde{G}_{12}\tilde{G}_{21}]^{1/4}. \qquad (31)$$

Thus the SHG phase-matching condition can be satisfied in the two-component BEC due to the multi-value property of the linear dispersion relation. We see that one can control experimentally the system parameters γ_m, γ_ω, γ_x, N_2/N_1 and G_{jl} $(j,l = 1,2)$. Shown in Fig.1 is the dispersion curve of the collective modes of a two-component BEC consisting of different hyperfine spin states $|1,-1\rangle$ and $|2,2\rangle$ of ^{87}Rb,[30] in a trap with $\gamma_m = \gamma_\omega = \gamma_x = 1$,[1)] and the particle numbers $N_1 = N_2 = 2 \times 10^6$. For this system one has $G_{11} = 1.0$, $G_{12} = 0.9926$ and $G_{22} = 1.0027$. Note that both curves are acoustic. The modes satisfying the phase-matching condition (29) for the SHG have been clearly shown as the point $A = (q_1, \omega_1)$ (the fundamental wave) and the point $B = (q_2, \omega_2)$ (the second harmonic wave). From Eq.(31) we obtain $q_1 = 2.743$ and hence $\omega_1 = 7.536$, $q_2 = 5.486$ and $\omega_2 = 15.072$. Because the locality (i.e. the dependence on the slow variable X) in the phase-matching condition (29) in the SHG process is not significant if the condensate is long enough, for simplicity here and after we assume that the system is uniform in the axial direction.

4.2. Nonlinear amplitude equations

We know that an optical SHG occurs in active media with no inversion symmetry. For trapped atoms this symmetry is not broken and hence the SHG in the BEC is possible only when the ground state (condensate) is not depleted by the excitations. This imposes a constraint that the amplitude of the excitations can not be too large. Here we develop a weak nonlinear theory for the SHG in the BEC by making the asymptotic expansion,[32] $A_j - A_{j\mathrm{GS}} = A_{j\mathrm{GS}}^{(0)}(\varepsilon F_j^{(1)} + \varepsilon^2 F_j^{(2)} + \cdots)$ and $\tilde{\phi}_j = \varepsilon\phi_j^{(1)} + \varepsilon^2\phi_j^{(2)} + \cdots$ with $F_j^{(l)}$ and $\phi_j^{(l)}$ being functions of x, τ (fast variables) and $X = \varepsilon x$, $T = \varepsilon\tau$ (slow variables), Eqs.(22)–(25) are transferred into a set of linear equations for $F_j^{(l)}$ and $\phi_j^{(l)}$ $(j = 1,2;\ l = 1,2,3,\ldots)$ (see Appendix A). Note that due to the asymptotic expansion from the ground state, these equations become linear but inhomogeneous. The nonlinear terms in Eqs.(22)–(25) now become a driving source and some quadratic nonlinear terms appear on the right hand side of the equations on $F_j^{(l)}$ and $\phi_j^{(l)}$ (see Appendix A). This is the

[1)] In this way, two species have maximum overlap in the ground state. By adjusting the parameters of rotating magnetic field of TOP trap, one can obtain the same radial and axial oscillation frequencies for the trapping potentials V_1 and V_2 for the species 1 and the species 2, see Ref.[31].

reason why a condition for the SHG mentioned in the Introduction, i. e. the interaction between the excitations must be characterized by a quadratic nonlinearity, can be satisfied under the mechanism based on the collective-mode interaction.

At the first-order ($l = 1$), we have the plane-wave solution:

$$F_1^{(1)} = U\exp(\mathrm{i}\theta) + \text{c.c.}, \tag{32}$$

$$F_2^{(1)} = \left(\tilde{G}_{12}q^2\right)^{-1} L_1(\omega, q)U\exp(\mathrm{i}\theta) + \text{c.c.}, \tag{33}$$

$$\phi_1^{(1)} = -\,\mathrm{i}(2\omega/q^2)U\exp(\mathrm{i}\theta) + \text{c.c.}, \tag{34}$$

$$\phi_2^{(1)} = -\,\mathrm{i}2\gamma_m\omega\left(\tilde{G}_{12}q^4\right)^{-1} \times L_1(\omega, q)U\exp(i\theta) + \text{c.c.}, \tag{35}$$

where $\theta = qx - \omega\tau$ is the phase of the plane-wave, $L_1(\omega, q) = \omega^2 - (\tilde{G}_{11} + q^2/4)q^2$ and U is a function of X and T, called the amplitude (or envelope) function. ω is given by (26) (the linear dispersion relation). Because the linear superposition of plane waves with different wave vector q is also a solution, we can take

$$F_1^{(1)} = U\exp(\mathrm{i}\theta_1) + U_2\exp(\mathrm{i}\theta_2) + \text{c.c.}, \tag{36}$$

$$F_2^{(1)} = \left(\tilde{G}_{12}q_1^2\right)^{-1} L_1(\omega_1, q_1)U_1\exp(\mathrm{i}\theta_1) + \left(\tilde{G}_{12}q_2^2\right)^{-1} L_1(\omega_2, q_2)U_2\exp(\mathrm{i}\theta_2) + \text{c.c.}, \tag{37}$$

$$\phi_1^{(1)} = -\,\mathrm{i}(2\omega_1/q_1^2)U_1\exp(\mathrm{i}\theta_1) - \mathrm{i}(2\omega_2/q_2^2)U_2\exp(\mathrm{i}\theta_2) + \text{c.c.}, \tag{38}$$

$$\phi_2^{(1)} = -\,\mathrm{i}2\gamma_m\omega_1\left(\tilde{G}_{12}q_1^4\right)^{-1} L_1(\omega_1, q_1)U_1\exp(\mathrm{i}\theta_1) - \mathrm{i}2\gamma_m\omega_2\left(\tilde{G}_{12}q_2^4\right)^{-1} \times L_1(\omega_2, q_2)U_2\exp(\mathrm{i}\theta_2) + \text{c.c.}, \tag{39}$$

where q_1, q_2, ω_1 and ω_2 are chosen according to the SHG phase-matching condition (29), i.e. $\omega_1 = \omega_+(q_1)$ and $\omega_2 = \omega_-(q_2)$ with $q_2 = 2q_1$. U_1 and U_2 are the amplitude functions of the fundamental wave (with the phase $\theta_1 = q_1x - \omega_1\tau$) and the second harmonic wave (with the phase $\theta_2 = q_2x - \omega_2\tau$), respectively.

At the second-order ($l = 2$), by solvability conditions we obtain the closed equations for U_1 and U_2:

$$\frac{\partial U_1}{\partial T} + v_{\mathrm{g}1}\frac{\partial U_1}{\partial X} + \mathrm{i}\Gamma_1 U_1^* U_2\exp(-\mathrm{i}\delta qX) = 0, \tag{40}$$

$$\frac{\partial U_2}{\partial T} + v_{\mathrm{g}2}\frac{\partial U_2}{\partial X} + \mathrm{i}\Gamma_2 U_1^2\exp(\mathrm{i}\delta qX) = 0, \tag{41}$$

where

$$v_{\mathrm{g}1} = (\mathrm{d}\omega_+/\mathrm{d}q)_{q=q_1} \quad \text{and} \quad v_{\mathrm{g}2} = (\mathrm{d}\omega_-/\mathrm{d}q)_{q=q_2}$$

are respectively the group velocity of the fundamental and second harmonic waves. The explicit expressions of the group velocities $v_{\mathrm{g}j}$ $(j = 1, 2)$ and the nonlinear coefficients Γ_1 and Γ_2 have been given in Appendix B. In Eqs.(40) and (41) we have included a small wave-vector mismatch $q_2 - 2q_1 = \varepsilon\delta q$ (δq is of order unity). Equations (40) and (41) are the nonlinear amplitude equations for the SHG between two collective modes in the system.

4.3. SHG solutions

We now consider the solutions of Eqs.(40) and (41) corresponding to the SHG of the system. By the transformation $U_j = \varepsilon u_j$ and note that $X = \varepsilon x$ and $T = \varepsilon\tau$, Eqs.(40) and (41) can be written as

$$\frac{\partial u_1}{\partial \tau} + v_{\mathrm{g}1}\frac{\partial u_1}{\partial x} + \mathrm{i}\Gamma_1 u_1^* u_2\exp(-\mathrm{i}\Delta qx) = 0, \tag{42}$$

$$\frac{\partial u_2}{\partial \tau} + v_{\mathrm{g}2}\frac{\partial u_2}{\partial x} + \mathrm{i}\Gamma_2 u_1^2\exp(\mathrm{i}\Delta qx) = 0, \tag{43}$$

where $\Delta q = \varepsilon\delta q$. For a stationary case, i.e. $\partial/\partial\tau = 0$, and for $\delta q = 0$, Eqs.(42) and (43) admit the solutions[18]

$$u_1 = \left(\frac{-\Gamma_1 W}{v_{\mathrm{g}1}}\right)^{1/2} \mathrm{sech}\left[\frac{\Gamma_1}{v_{\mathrm{g}1}}\left(\frac{-\Gamma_2 W}{v_{\mathrm{g}2}}\right)^{1/2} x\right] \mathrm{e}^{\mathrm{i}\varphi_0}, \tag{44}$$

$$u_2 = \left(\frac{-\Gamma_2 W}{v_{\mathrm{g}2}}\right)^{1/2} \tanh\left[\frac{\Gamma_1}{v_{\mathrm{g}1}}\left(\frac{-\Gamma_2 W}{v_{\mathrm{g}2}}\right)^{1/2} x\right] \mathrm{e}^{\mathrm{i}(\varphi_0 + \pi/2)}, \tag{45}$$

where $W = -(v_{\mathrm{g}1}/\Gamma_1)|u_1|^2 - (v_{\mathrm{g}2}/\Gamma_2)|u_2|^2$ is a constant denoting the total power of the excitation and φ_0 is an arbitrary constant. At $x = 0$, the fundamental wave takes the total power W of the system and thus the power of the second harmonic wave is zero. As x increases the power of the fundamental wave in converted gradually to the second harmonic wave. At the distance x, the conversion efficiency of the power from the fundamental wave to the second harmonic wave is given by

$$\eta = \frac{W_2(x)}{W_1(0)} = \tanh^2\left[\frac{\Gamma_1}{v_{\mathrm{g}1}}\left(\frac{-\Gamma_2 W}{v_{\mathrm{g}2}}\right)^{1/2} x\right], \tag{46}$$

where $W_j(x) = -(v_{\mathrm{g}j}/\Gamma_j)|u_j|^2$ is the power of jth wave. Thus the conversion efficiency for the SHG is determined by $v_{\mathrm{g}j}$, Γ_j $(j = 1, 2)$, and the propagating distance x.

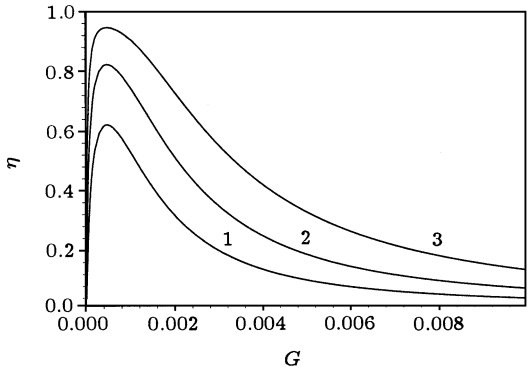


Fig.2. Conversion efficiency as a function of interspecies interaction parameter G ($= G_{12} = G_{21}$) and input power W for a stationary SHG in the case of $\gamma_m = \gamma_\omega = \gamma_x = 1.0$, $G_{11} = 1.0$, $G_{22} = 1.0027$. Curves 1, 2, and 3 correspond to the propagating distance $x = 10.0$ with the input power W taking the values 5.0, 10.0 and 20.0, respectively.

Note that the solutions (44) and (45) are valid only for $(\Gamma_1/v_{g1})(\Gamma_2/v_{g2}) > 0$. Because both v_{g1} and v_{g2} are positive, it is required that Γ_1 and Γ_2 have the same sign. It can be shown that, for the case of $\gamma_m = \gamma_\omega = \gamma_x = 1.0$, both Γ_1 and Γ_2 are negative for the interaction parameters $G_{11} = 1.0$, $G_{22} = 1.0027$ when G_{12} lies in the interval between zero and 1. Thus the solutions (44) and (45) are physically realizable.

Shown in Fig.2 is the conversion efficiency η as a function of G_{12} ($= G_{21}$) and the input power W when taking $\gamma_m = \gamma_\omega = \gamma_x = 1.0$, $G_{11} = 1.0$, $G_{22} = 1.0027$. The propagating distance x is taken as 10.0 (i.e. ten times of the healing length). The curves 1, 2, and 3 correspond to the input power W taking the values 5.0, 10.0 and 20.0, respectively. For $G_{12} = 0.9926$,[28] η will be 1.465×10^{-5} if $x = 10.0$ and $W = 20.0$. We see that, when increasing the interspecies interaction parameter G_{12}, the conversion efficiency of the power from the fundamental wave to the second harmonic wave increases rapidly from zero to a maximum first and then decreases to a smaller value. This property of the conversion efficiency can be explained as follows. If G_{12} is zero or near zero, the SHG is impossible or the interaction between the excitations excited from two species is not significant and hence the conversion efficiency η is also small. Increasing G_{12} results in larger interaction and thus η increases. The reason for the decrease of η after passing over a maximum is that the interspecies interaction is repulsive. For larger G_{12} the overlapping between two species decreases and hence the decrease of the conversion efficiency.

Figure 3 shows the conversion efficiency as a function of G_{12} and the propagating distance (or sample length) x, in which the parameters γ_m, γ_ω, γ_x, G_{11} and G_{22} are chosen the same as in Fig.2 but the input power is fixed ($W = 10.0$). The curves 1, 2, and 3 correspond to the propagating distance x taking the values 5.0, 10.0 and 20.0, respectively. From Figs.2 and 3 we see that to obtain a significant conversion efficiency of the SHG, in addition to a large propagating distance (equivalent to large sample length) and a large input power (equivalent to large-amplitude fundamental wave excitation), one must choose an appropriate interspecies interaction strength G_{12}. This provides us also a possibility for determining G_{12} by measuring the SHG conversion efficiency η.

For the situation when u_j ($j = 1, 2$) depend only on τ, similar results can be obtained as those given by (44), (45), and (46) by replacing x with τ and formally taking $v_{gj} = 1$ ($j = 1, 2$). If $\delta q \neq 0$, i.e. the fundamental wave and the second harmonic wave are not exactly phase-matched, the energy will be exchanged periodically between two wave modes[18] which is not discussed here.

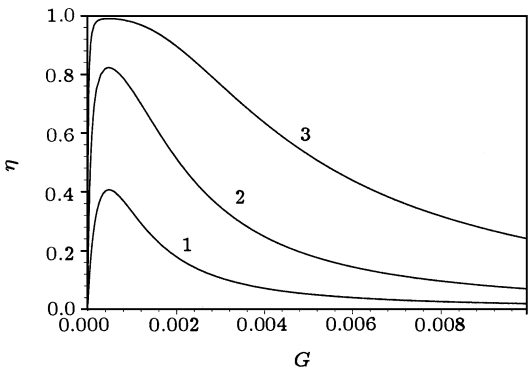


Fig.3. Conversion efficiency as a function of interspecies interaction parameter G ($= G_{12} = G_{21}$) and propagating distance x in a stationary SHG for the case of $\gamma_m = \gamma_\omega = \gamma_x = 1.0$, $G_{11} = 1.0$, $G_{22} = 1.0027$. Curves 1, 2, and 3 correspond to the input power $W = 10.0$ with the propagating distance x taking the values 5.0, 10.0 and 20.0, respectively.

For short-pulse excitations, one cannot assume $\partial/\partial\tau = 0$ (i.e. a non-stationary case) and hence a walk-off effect due to different group velocity for the fundamental and the second harmonic wave must be taken into account.[18] By the transformation $u_1 = [v_{g1}v_{g2}/(\Gamma_1\Gamma_2)]^{1/2} w_1 \exp(\mathrm{i}\varphi)$ and $u_2 = (v_{g1}/\Gamma_1) w_2 \exp[\mathrm{i}(2\varphi + \pi/2)]$ (w_1, w_2 and φ are real functions of x and τ) and assumption $\Delta q = 0$, Eqs.(42) and (43) become

$$\frac{\partial w_1}{\partial x} + \frac{1}{v_{g1}} \frac{\partial u_1}{\partial \tau} = w_1 w_2, \tag{47}$$

$$\frac{\partial w_2}{\partial x} + \frac{1}{v_{g2}} \frac{\partial w_2}{\partial \tau} = -w_1^2. \tag{48}$$

Consider a travelling-wave solution, i.e. take w_j $(j = 1, 2)$ as functions of x and $\zeta = \tau - x/v_{g1}$, Eqs.(47) and (48) are transferred as

$$\frac{\partial w_1}{\partial x} = w_1 w_2, \tag{49}$$

$$\frac{\partial w_2}{\partial x} + \nu \frac{\partial w_2}{\partial \zeta} = -w_1^2, \tag{50}$$

where $\nu = 1/v_{g2} - 1/v_{g1}$ is a parameter denoting the walk-off (or group-velocity dispersion) effect. If at $x = 0$ the fundamental wave and the second harmonic wave take the form:

$$w_1(0, \tau) = \frac{A_0}{1 + \tau^2/\tau_0^2},$$
$$w_2(0, \tau) = 0, \tag{51}$$

where τ_0 is the initial pulse-width and A_0 is a constant representing the initial amplitude of the fundamental wave, then Eqs.(49) and (50) allow the following solution:

$$w_1(\zeta, x) = \frac{A_0}{(1 + \tilde{\zeta}^2)^{1/2}[1 + (\tilde{\zeta} - \tilde{x})^2]^{1/2}} \times \frac{1}{\cosh \xi + \tilde{\zeta}/f_0 \sinh \xi}, \tag{52}$$

$$w_2(\zeta, x) = \frac{\tau_{\rm cr}}{\tau} \frac{A_0}{1 + (\tilde{\zeta} - \tilde{x})^2} \times \frac{\tilde{x} \cosh \xi + [f_0 - \tilde{\zeta}(\tilde{\zeta} - \tilde{x})/f_0] \sinh \xi}{\cosh \xi + \tilde{\zeta}/f_0}, \tag{53}$$

where $\tilde{\zeta} = \zeta/\tau_0$, $\tilde{x} = x/L_\nu$, $f_0 = (\tau_0^2/\tau_{\rm cr}^2 - 1)^{1/2}$, $\tau_{\rm cr} = \nu L_{\rm NL}$, $\xi = f_0[\tan^{-1} \tilde{\zeta} - \tan^{-1}(\tilde{\zeta} - \tilde{x})]$ with $L_\nu = \tau_0/\nu$ (walk-off or dispersion length) and $L_{\rm NL} = A_0^{-1}$ (nonlinear length).

When the walk-off length is much larger than the nonlinear length, i.e. $L_\nu \gg L_{\rm NL}$, the walk-off effect can be neglected. In this case the solutions (52) and (53) is simplified to

$$w_1(\zeta, x) = \frac{A_0}{1 + \tilde{\zeta}^2} \operatorname{sech}\left[\frac{A_0}{(1 + \tilde{\zeta})} x\right], \tag{54}$$

$$w_2(\zeta, x) = \frac{A_0}{1 + \tilde{\zeta}^2} \tanh\left[\frac{A_0}{(1 + \tilde{\zeta})} x\right]. \tag{55}$$

Obviously, this situation corresponds to a quasi-stationary SHG process and only in this case the conversion efficiency of the power from the fundamental wave to the second harmonic wave is significant.

5. Discussion and summary

We have made a theoretical prediction of a SHG for propagating collective excitations in a long cigar-shaped two-component Bose–Einstein condensate with a repulsive atom–atom interaction. We have investigated the ground state of the system and shown that the linear dispersion relation of the collective excitations in such system consists of two branches and thus provides a possibility to fulfil the phase-matching condition of the SHG if the wave vectors and frequencies of the fundamental and the second harmonic waves are chosen adequately from different dispersion branches. Because the collective excitations under consideration are created from the ground state of the system, the interaction between these excitations is characterized by a quadratic nonlinearity and hence another necessary condition for the SHG can also be satisfied. We have derived the nonlinear amplitude equations for the SHG using the method of multiple-scales and presented the SHG solutions similar to those obtained for a photon SHG in nonlinear optical media. The conversion efficiency of power from the fundamental wave to the second harmonic wave has also been discussed.

Note that in the experiment of Myatt *et al* an elongated magnetic trap was used to create condensates of 2×10^6 atoms in either of the $|2, 2\rangle$ or $|1, -1\rangle$ spin states of ^{87}Rb.[30] In such a two-component BEC system one has $\gamma_m = 1.0$ and $G_{11} = 1.0$, $G_{12} = G_{21} = 0.9926$ and $G_{22} = 1.0027$. In order to make the two species have maximum overlap in the ground state hence large power conversion efficiency for the SHG based on the collective modes in the condensate, one can adjust the parameters of rotating magnetic field of TOP trap to obtain almost the same radial and axial oscillation frequencies for the trapping potentials and hence to obtain $\gamma_x = \gamma_\omega = 1.0$.[31] In this way the SHG resonance condition shown in Fig.1 can be easily satisfied.

In fact, the quasi-one-dimensional approximation used above is not necessary and one can easily extend the analysis developed here to the case including higher-order eigen-modes in the transverse directions as done for the soliton dynamics in BECs,[33] and to a two-dimensional or a three-dimensional two-component condensate. To experimentally test the prediction predicted in this work, one can use the method developed in Ref.[34] to generate high frequency and weak nonlinear excitations of small size by suddenly modifying the trapping potential using optical dipole force of a focused laser beam. To get a larger conversion efficiency one can adjust the parameters of the trapping potential. It is better to use a conden-

sate long enough (as in Ref.[34]) so that the two wave modes can have a significant energy transfer. Another way is to change the interspecies interaction parameter G_{12} using Feshbach resonance technique.[35] Inversely, the measurement of the conversion efficiency η in the SHG may provide a possibility to determine the interspecies interaction parameter G_{12} for a two-component BEC.

Acknowledgements

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Appendix A

The equations controlling the motion of $F_j^{(l)}$ and $\phi_j^{(l)}$ are given by

$$\frac{\partial F_1^{(l)}}{\partial \tau} + \frac{1}{2}\frac{\partial^2 \phi_1^{(l)}}{\partial x^2} = P^{(l)}, \tag{56}$$

$$\frac{\partial F_2^{(l)}}{\partial \tau} + \frac{1}{2}\gamma_m^{-1}\frac{\partial^2 \phi_2^{(l)}}{\partial x^2} = Q^{(l)}, \tag{57}$$

$$\frac{1}{2}\frac{\partial^2 F_1^{(l)}}{\partial x^2} - \frac{\partial \phi_1^{(l)}}{\partial \tau} - 2\tilde{G}_{11}F_1^{(l)} - 2\tilde{G}_{12}F_2^{(l)} = R^{(l)}, \tag{58}$$

$$\frac{1}{2}\gamma_m^{-1}\frac{\partial^2 F_2^{(l)}}{\partial x^2} - \frac{\partial \phi_2^{(l)}}{\partial \tau} - 2\tilde{G}_{21}F_1^{(l)} - 2\tilde{G}_{22}F_2^{(l)} = S^{(l)}, \tag{59}$$

$l = 1, 2, ...$ with

$$P^{(1)} = Q^{(1)} = R^{(1)} = S^{(1)} = 0, \tag{60}$$

$$P^{(2)} = -\frac{\partial F_1^{(1)}}{\partial T} - \frac{\partial F_1^{(1)}}{\partial x}\frac{\partial \phi_1^{(1)}}{\partial x} - (A_{1\mathrm{GS}}^{(0)})^{-1}\frac{\partial A_{1\mathrm{GS}}^{(0)}}{\partial X}\frac{\partial \phi_1^{(1)}}{\partial x} - \frac{\partial^2 \phi_1^{(1)}}{\partial x \partial X} - \frac{1}{2}F_1^{(1)}\frac{\partial^2 \phi_1^{(1)}}{\partial x^2}, \tag{61}$$

$$Q^{(2)} = -\frac{\partial F_2^{(1)}}{\partial T} - \gamma_m^{-1}\left[\frac{\partial F_1^{(1)}}{\partial x}\frac{\partial \phi_1^{(1)}}{\partial x} + (A_{1\mathrm{GS}}^{(0)})^{-1}\frac{\partial A_{1\mathrm{GS}}^{(0)}}{\partial X}\frac{\partial \phi_1^{(1)}}{\partial x} + \frac{\partial^2 \phi_1^{(1)}}{\partial x \partial X} + \frac{1}{2}F_1^{(1)}\frac{\partial^2 \phi_1^{(1)}}{\partial x^2}\right], \tag{62}$$

$$\begin{aligned} R^{(2)} =& \frac{\partial \phi_1^{(1)}}{\partial T} + F_1^{(1)}\frac{\partial \phi_1^{(1)}}{\partial \tau} - \frac{\partial^2 F_1^{(1)}}{\partial x \partial X} - \frac{1}{2}(A_{1\mathrm{GS}}^{(0)})^{-1}\frac{\partial^2 A_{1\mathrm{GS}}^{(0)}}{\partial X^2} + \frac{1}{2}\left(\frac{\partial \phi_1^{(1)}}{\partial x}\right)^2 \\ & + 3\tilde{G}_{11}(F_1^{(1)})^2 + 2\tilde{G}_{12}F_1^{(1)}F_2^{(1)} + \tilde{G}_{12}(F_2^{(1)})^2, \end{aligned} \tag{63}$$

$$\begin{aligned} S^{(2)} =& \frac{\partial \phi_2^{(1)}}{\partial T} + F_2^{(1)}\frac{\partial \phi_2^{(1)}}{\partial \tau} - \gamma_m^{-1}\left[\frac{\partial^2 F_2^{(1)}}{\partial x \partial X} + \frac{1}{2}(A_{1\mathrm{GS}}^{(0)})^{-1}\frac{\partial^2 A_{1\mathrm{GS}}^{(0)}}{\partial X^2} - \frac{1}{2}\left(\frac{\partial \phi_2^{(1)}}{\partial x}\right)^2\right] \\ & + 3\tilde{G}_{22}(F_2^{(1)})^2 + 2\tilde{G}_{21}F_1^{(1)}F_2^{(1)} + \tilde{G}_{21}(F_1^{(1)})^2, \end{aligned} \tag{64}$$

where $A_{1\mathrm{GS}}^{(0)}$ $(j = 1, 2)$ are given by (14) and (15). Higher-order $P^{(l)}$, $Q^{(l)}$, $R^{(l)}$ and $S^{(l)}$ $(l = 3, 4, ...)$ are not needed for the SHG process. Note that some *quadratic* nonlinear terms appear in the expressions of $P^{(2)}$, $Q^{(2)}$, $R^{(2)}$ and $S^{(2)}$.

Appendix B

The group velocity $v_{\mathrm{g}j}$ is given by

$$v_{\mathrm{g}j} = \frac{L_1(\omega_j, q_j)(\omega_j^2 + \gamma_m^{-2}q_j^4/4) + L_2(\omega_j, q_j)(\omega_j^2 + q_j^4/4)}{q_j\omega_j[L_1(\omega_j, q_j) + L_2(\omega_j, q_j)]}, \tag{65}$$

$j = 1, 2$ with $L_1(\omega, q) = \omega^2 - (\tilde{G}_{11} + q^2/4)q^2$ and $L_2(\omega, q) = \omega^2 - (\gamma_m^{-1}\tilde{G}_{22} + \gamma_m^{-2}q^2/4)q^2$.

The expressions of the nonlinear coefficients appearing in Eqs.(40) and (41) read

$$\Gamma_1 = \frac{\Delta_1}{2\omega_1[L_1(\omega_1, q_1) + L_2(\omega_1, q_1)]}, \tag{66}$$

$$\Gamma_2 = \frac{\Delta_2}{2\omega_2[L_1(\omega_2, q_2) + L_2(\omega_2, q_2)]}, \tag{67}$$

where

$$\Delta_1 = L_2(\omega_1,q_1)\left[3(\omega_1^2+\tilde{G}_{11}q_1^2)+L_1(\omega_1,q_1)+\frac{1}{4}L_1(\omega_2,q_2)+\frac{L_1(\omega_1,q_1)L_1(\omega_2,q_2)}{4\tilde{G}_{12}q_1^2}\right]$$
$$+\gamma_m^{-1}\tilde{G}_{12}\tilde{G}_{21}q_1^4+\frac{3(\omega_1^2+\gamma_m^{-1}\tilde{G}_{22}q_1^2)}{4\tilde{G}_{12}q_1^2}L_1(\omega_1,q_1)L_1(\omega_2,q_2)$$
$$+\gamma_m^{-1}\tilde{G}_{21}q_1^2\left[L_1(\omega_1,q_1)+\frac{1}{4}L_1(\omega_2,q_2)\right], \tag{68}$$

$$\Delta_2 = \frac{24}{\tilde{G}_{12}}L_1^2(\omega_1,q_1)\left(\frac{\omega_1^2}{q_1^2}+\gamma_m^{-1}\tilde{G}_{22}\right)+8\gamma_m^{-1}\tilde{G}_{12}\tilde{G}_{21}q_1^4+16\gamma_m^{-1}\tilde{G}_{21}q_1^2L_1(\omega_1,q_1)$$
$$+L_2(\omega_2,q_2)\left[6\omega_1^2+6\tilde{G}_{11}q_1^2+4L_1(\omega_1,q_1)+\frac{2L_1^2(\omega_1,q_1)}{\tilde{G}_{12}q_1^2}\right]. \tag{69}$$

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