Vol 13 No 11, November 2004
1009-1963/2004/13(11)/1862-04

Chinese Physics

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Preparation of nonclassical states and measurement of the Wigner function for the collective motion of N trapped ions*

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(Received 9 March 2004; revised manuscript received 16 April 2004)

We propose a scheme for generating nonclassical states for the centre-of-mass vibrational mode of N trapped ions, including superpositions of several coherent states on a circle and Fock states. In the scheme N trapped ions are driven by a laser beam tuned to the carrier. The scheme also provides a new prospect for laser cooling. The scheme can be used to measure the Wigner function of the collective vibrational mode.

Keywords: nonclassical state, Wigner function, trapped ion
PACC: 4250

Recently, there has been much interest in the preparation and reconstruction of quantum states. In cavity quantum electrodynamics, schemes have been proposed for the generation of nonclassical states.[1−3] The ion trap is another qualified system for quantum state preparation and reconstruction. An ion trapped in a harmonic potential can be regarded as a particle with a quantized centre-of-mass motion. One can manipulate the ion motion by driving it by classical fields which couple its internal and external states. Recently, motional Fock, squeezed, coherent,[4] and Schrödinger cat states[5] have been observed. Experimental reconstruction of motional quantum state of a trapped ion has also been reported.[6] However, the scheme involves a complex data analysis. Lutterbach and Davidovich have presented a scheme for direct measurement of the Wigner function for both a cavity field and a trapped ion.[7]

On the other hand, much attention has been paid to the manipulation of vibronic states of two or more trapped ions. The preparation in the ground state of collective modes of motion of two trapped ions has been reported.[8] Schemes have been presented for realizing quantum computation using trapped ions.[9,10] Entanglement of two or more trapped ions has been reported.[11] Recently, a scheme has been proposed for generating superpositions of several coherent states along a straight line for the collective vibrational mode of N trapped ions through the bichromatic excitations of the ions and conditional measurement of the electronic states.[12,13] In this paper we propose a scheme for the generation of superpositions of $N+1$ coherent states on a circle for the collective motion of N trapped ions using a laser beam tuned to the carrier. The scheme can also be used to generate vibrational Fock states for the collective motion of N trapped ions. As far as we know, this is the first scheme for generating superpositions of multiple coherent states with the same amplitude but different phases and Fock states for the collective motion of N trapped ions. The scheme also provides a possibility of measuring the Wigner function of the collective vibrational mode for the first time.

We consider N two-level ions confined to move along the x axis in a Paul trap. We first simultaneously drive the N ions with a laser beam tuned to the ion transition. We assume that the inter-ion distance is much smaller than the laser beam width, thus the Rabi frequency of the laser field at the position of each ion is the same.[11] In the interaction picture the

*Project supported by Fok Ying Tung Education Foundation (Grant No 81008), the National Natural Science Foundation of China (Grant Nos 60008003 and 100225421), and Funds from Fuzhou University.
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interaction Hamiltonian is given by[14,15]

$$\hat{H} = \sum_{l=1}^{N} \hat{H}_l, \tag{1}$$

where $\hat{H}_l$ is the Hamiltonian describing the interaction of the lth ion with the laser,

$$\hat{H}_l = \left[\sum_{j=0}^{\infty} \mathrm{e}^{-\eta^2/2} \Omega \mathrm{e}^{-\mathrm{i}\phi_l} \frac{(\mathrm{i}\eta)^{2j}}{(j!)^2} \hat{a}^{+j} \hat{a}^j \right] \times |e_l\rangle \langle g_l| + \mathrm{H.c.}, \tag{2}$$

where $\hat{a}^+$ and $\hat{a}$ are the creation and annihilation operators for the collective centre-of-mass vibrational mode in the x axis, $|e_l\rangle$ and $|g_l\rangle$ are the excited and ground electronic states of the lth two-level ion, Ω is the Rabi frequency of the laser, and ϕ_l is the phase of the laser at the position of the lth ion. The Lamb–Dicke parameter η is defined by $\eta = k/\sqrt{2\nu M}$ with k being the projection of the wavevector on the x axis, ν being the frequency of the centre-of-mass mode, and M being the total mass of the ion collection. We here have assumed that besides the centre-of-mass vibrational mode the other modes are in the vacuum state and thus take no effect on the evolution of the system. In the Lamb–Dicke regime, i.e. $\eta \ll 1$, we can expand Eq.(2) up to the second order in η. Then we have

$$\hat{H}_l = \left[\Omega \mathrm{e}^{-\mathrm{i}\phi_l} \left(1 - \eta^2 \hat{a}^{+j} \hat{a}^j\right) \right] |e_l\rangle \langle g_l| + \mathrm{H.c.} \tag{3}$$

Assume the electronic subsystem is initially in the state $|e_1, e_2, \ldots e_N\rangle$ and the centre-of-mass motional mode initially in the coherent state $|\alpha\rangle$.[4] After an interaction time interval τ the system evolves to

$$|\psi(\tau)\rangle = \prod_{l=1}^{N} [\cos(\hat{O}\tau) |e_l\rangle - \mathrm{i}\mathrm{e}^{\mathrm{i}\phi_l} \sin(\hat{O}\tau) |g_l\rangle] |\alpha\rangle, \tag{4}$$

where

$$\hat{O} = \Omega(1 - \eta^2 \hat{a}^+ \hat{a}). \tag{5}$$

We now detect the internal state of the ion. If we find the ions in the state $|e_1, e_2 \ldots e_N\rangle$ the centre-of-mass vibrational mode collapses to

$$|\psi_\mathrm{c}\rangle = N_\mathrm{c} \cos^N(\hat{O}\tau) |\alpha\rangle, \tag{6}$$

where N_C is a normalization factor. Substituting Eq.(5) into Eq.(6) we obtain

$$|\psi_\mathrm{c}\rangle = N_\mathrm{c} \sum_{n=0}^{N} C_N^n \mathrm{e}^{-\mathrm{i}(2n-N)\Omega\tau} \left| \alpha \mathrm{e}^{\mathrm{i}(2n-N)\eta^2\Omega\tau} \right\rangle. \tag{7}$$

Choosing the interaction time τ appropriately so that $2\eta^2\Omega\tau = \dfrac{2\pi}{N}$ we obtain a superposition of $N+1$ coherent states on a circle in the phase space

$$|\psi_\mathrm{c}\rangle = N_\mathrm{c} \sum_{n=0}^{N} C_N^n \mathrm{e}^{-\mathrm{i}(2n-N)\pi/(N\eta^2)} \left| -\alpha \mathrm{e}^{\mathrm{i}\frac{2n\pi}{N}} \right\rangle. \tag{8}$$

Due to interference between the coherent components, these states can exhibit various nonclassical properties, such as squeezing and sub-Poissonian photon statistics.[16]

On the other hand, with the choice $2\eta^2\Omega\tau = \pi$ we obtain a superposition of two coherent states with the same amplitudes but opposite phases. For the case $N = 4k$ we obtain

$$|\psi_\mathrm{c}\rangle = N_\mathrm{c} \left[\sum_{m=0}^{2k} C_{4k}^{2m} \mathrm{e}^{-\mathrm{i}(2m-2k)\pi/\eta^2} |\alpha\rangle + \sum_{m=0}^{2k-1} C_{4k}^{2m+1} \mathrm{e}^{-\mathrm{i}(2m-2k+1)\pi/\eta^2} |-\alpha\rangle \right]. \tag{9}$$

For the case $N = 4k+1$ we obtain

$$|\psi_\mathrm{c}\rangle = N_\mathrm{c} \left[\sum_{m=0}^{2k} C_{4k+1}^{2m} \mathrm{e}^{-\mathrm{i}(4m-4k-1)\pi/(2\eta^2)} |-\mathrm{i}\alpha\rangle + \sum_{m=0}^{2k} C_{4k+1}^{2m+1} \mathrm{e}^{-\mathrm{i}(4m-4k+1)\pi/(2\eta^2)} |\mathrm{i}\alpha\rangle \right]. \tag{10}$$

For the case $N = 4k+2$ we obtain

$$|\psi_\mathrm{c}\rangle = N_\mathrm{c} \left[\sum_{m=0}^{2k} C_{2(2k+1)}^{2m+1} \mathrm{e}^{-\mathrm{i}(2m-2k)\pi/\eta^2} |\alpha\rangle + \sum_{m=0}^{2k+1} C_{2(2k+1)}^{2m} \mathrm{e}^{-\mathrm{i}(2m-2k-1)\pi/\eta^2} |-\alpha\rangle \right]. \tag{11}$$

For the case $N = 4k+3$ we obtain

$$|\psi_\mathrm{c}\rangle = N_\mathrm{c} \left[\sum_{m=0}^{2k+1} C_{4k+3}^{2m} \mathrm{e}^{-\mathrm{i}(4m-4k-3)\pi/(2\eta^2)} |\mathrm{i}\alpha\rangle + \sum_{m=0}^{2k+1} C_{4k+3}^{2m+1} \mathrm{e}^{-\mathrm{i}(4m-4k-1)\pi/(2\eta^2)} |-\mathrm{i}\alpha\rangle \right]. \tag{12}$$

If the system is initially in the vibronic state $|e_1, e_2, \ldots e_N\rangle|n\rangle$, with $|n\rangle$ being a vibrational Fock state of the centre-of-mass mode, after an interaction time τ the system evolves to

$$|\psi(\tau)\rangle = \prod_{l=1}^{N} [\cos(g\tau)|e_l\rangle - \mathrm{i}\mathrm{e}^{\mathrm{i}\phi_l} \sin(g\tau)|g_l\rangle]|n\rangle, \tag{13}$$

where

$$g = \Omega(1 - \eta^2 n). \tag{14}$$

We now detect the internal state of the ion. The probability of finding the ions in the state $|e_1,e_2...e_N\rangle$ is $\cos^{2N}(g\tau)$. Suppose that the initial phonon-number distribution for the centre-of-motion is P_n, i.e. the probability of the collective motion in the Fock state $|n\rangle$ is P_n. The initial electronic state is $|e_1,e_2,...e_N\rangle$ again and one desires to generate the Fock state $|n_0\rangle$. We perform K interaction-measurement cycles. During the jth cycle we choose the interaction time appropriately so that $\Omega(1-\eta^2n_0)\tau_j = k_j\pi$, where k_j is an appropriate positive integer. In general when $n \neq n_0$ $\Omega(1-\eta^2n)\tau_j \neq p_{n,j}\pi$, $p_{n,j}$ is a positive integer. Suppose that we find the ions in the state $|e_1,e_2...e_N\rangle$ in each cycle. After an appropriate number of interaction-measurement cycles, the centre-of-mass mode collapses to the Fock state $|n_0\rangle$. The probability of getting the Fock state $|n_0\rangle$ is given by the probability P_n for the initial occupation of the desired state. The fidelity is

$$F = \frac{P_{n_0}}{P_{n_0} + \sum_{n\neq n_0} P_n \prod_{j=1}^{K} \cos^{2N}[\Omega(1-\eta^2n)\tau_j]}. \quad (15)$$

We here take an example. Suppose that there are four ions in the trap and the centre-of-mass mode is initially in a thermal state with the mean phonon-number $\bar{n}$ equal to 1 and we desire to generate the Fock state $|1\rangle$. We perform two interaction-detection cycles. Set $\eta = 0.2$. With the choice $\Omega(1-\eta^2)\tau_1 = 10\pi$ and $\Omega(1-\eta^2)\tau_2 = 2\pi$ the fidelity is about 0.98.

The scheme also provides a possibility of laser cooling down to the collective vibrational ground state for N trapped ions. As an example, again we suppose that there are four ions in the trap initially in the electronic state $|e_1,e_2,...e_N\rangle$ and the centre-of-mass mode is initially in a thermal state with the mean phonon-number $\bar{n}$ equal to 1. We also perform two interaction-detection cycles and suppose we detect the ions in the state $|e_1,e_2,...e_N\rangle$ in each cycle. With the choice $\eta = 0.2$, $\Omega\tau_1 = 10\pi$ and $\Omega\tau_2 = 2\pi$ the possibility of the ions being collapsed to the vibrational ground state is about 0.99.

We note the scheme can be used to measure the Wigner function for the centre-of-mass vibrational mode. Assume that the initial density operator of the centre-of-mass motional mode is $\hat{\rho}$. The electronic subsystem is initially in the state $|e_1,e_2,...e_N\rangle$. We first displace this mode by α,[4] leading to

$$\hat{\rho}_d(\alpha) = \hat{D}(\alpha)\hat{\rho}\hat{D}^+(\alpha), \quad (16)$$

where $\hat{D}(\alpha)$ is the displacement operator

$$\hat{D}(\alpha) = \mathrm{e}^{\alpha\hat{a}^+ - \alpha^*\hat{a}}. \quad (17)$$

Then the ions are excited by the laser tuned to the carrier. After an interaction time τ the density operator for the whole system combined by the electronic subsystem and the centre-of-mass vibrational mode is

$$\begin{aligned}\hat{\rho}_{\mathrm{w}}(\tau) =& \prod_{l=1}^{N}[\cos(\hat{O}\tau)|e_l\rangle\langle e_l| \\ & - \mathrm{i}\mathrm{e}^{\mathrm{i}\phi_l}\sin(\hat{O}\tau)|g_l\rangle\langle g_l|] \\ & \times |e_1,e_2,...e_N\rangle\hat{\rho}_d(\alpha) \\ & \times \langle e_1,e_2,...e_N|\prod_{l=1}^{N}[\cos(\hat{O}\tau)|e_l\rangle\langle e_l| \\ & - \mathrm{i}\mathrm{e}^{\mathrm{i}\phi_l}\sin(\hat{O}\tau)|g_l\rangle\langle g_l|]^+, \end{aligned} \quad (18)$$

where $\hat{O}$ is given by Eq.(5). We now detect the internal state of the ion. The probability of measuring the ions in the state $|e_1,e_2...e_N\rangle$ is

$$P(\alpha,\tau) = \mathrm{Tr}\{\cos^N(\hat{O}\tau)\hat{\rho}_d(\alpha)[\cos^N(\hat{O}\tau)]^+\}. \quad (19)$$

Choosing the interaction time appropriately so that $\eta^2\Omega\tau = \frac{\pi}{2}$, we have

$$\begin{aligned}P(\alpha,\tau) =& \frac{1}{2^{2N}}[|A|^2 + |B|^2 \\ & + 2\mathrm{Re}\{A^*B\}\mathrm{Tr}\{\mathrm{e}^{\mathrm{i}\pi\hat{a}^+\hat{a}}\hat{\rho}_d(\alpha)\}. \end{aligned} \quad (20)$$

For the case $N = 4k$, A and B are given by

$$\begin{aligned}A &= \sum_{m=0}^{2k} C_{4k}^{2m}\mathrm{e}^{-\mathrm{i}(2m-2k)\pi/\eta^2}, \\ B &= \sum_{m=0}^{2k} C_{4k}^{2m+1}\mathrm{e}^{-\mathrm{i}(2m-2k+1)\pi/\eta^2}. \end{aligned} \quad (21)$$

For the case $N = 4k+1$, A and B are given by

$$\begin{aligned}A &= \sum_{m=0}^{2k} C_{4k+1}^{2m}\mathrm{e}^{-\mathrm{i}(4m-4k-1)\pi/(2\eta^2)}, \\ B &= \sum_{m=0}^{2k} C_{4k+1}^{2m+1}\mathrm{e}^{-\mathrm{i}(4m-4k+1)\pi/(2\eta^2)}. \end{aligned} \quad (22)$$

For the case $N = 4k+2$, A and B are given by

$$\begin{aligned}A &= \sum_{m=0}^{2k} C_{2(2k+1)}^{2m+1}\mathrm{e}^{-\mathrm{i}(2m-2k)\pi/\eta^2}, \\ B &= \sum_{m=0}^{2k} C_{2(2k+1)}^{2m}\mathrm{e}^{-\mathrm{i}(2m-2k-1)\pi/\eta^2}. \end{aligned} \quad (23)$$

For the case $N = 4k + 3$, A and B are given by

$$A = \sum_{m=0}^{2k+1} C_{4k+3}^{2m} \mathrm{e}^{-\mathrm{i}(4m-4k-3)\pi/(2\eta^2)},$$
$$B = \sum_{m=0}^{2k+1} C_{4k+3}^{2m+1} \mathrm{e}^{-\mathrm{i}(4m-4k-1)\pi/(2\eta^2)}. \quad (24)$$

The Wigner function for the centre-of-mass vibrational mode is expressed as[6]

$$W(\alpha) = \frac{2}{\pi}\mathrm{Tr}[\mathrm{e}^{\mathrm{i}\pi\hat{a}^+\hat{a}}\hat{D}(-\alpha)\hat{\rho}\hat{D}(\alpha)]$$
$$= \frac{2}{\pi}\mathrm{Tr}[\mathrm{e}^{\mathrm{i}\pi\hat{a}^+\hat{a}}\hat{\rho}_a(-\alpha)]. \quad (25)$$

Thus we obtain

$$W(-\alpha) = \frac{2^{2N}[P(\alpha,\tau) - |A|^2 - |B|^2]}{\pi \mathrm{Re}\{A^* B\}}. \quad (26)$$

Therefore the measurement of the probability at the point α directly yields the Wigner function at the point $-\alpha$ for the Wigner function of the centre-of-mass vibrational mode.

In conclusion, we have proposed a scheme for generating nonclassical states for the centre-of-mass vibrational mode of N trapped ions. In the scheme N trapped ions are driven by a laser beam dispersively tuned to the carrier. Under certain conditions, Fock states, superpositions of $N + 1$ coherent states on a circle, and superpositions of two coherent states with the same amplitudes but opposite phases can be generated. The scheme also provides a prospect for laser cooling. Furthermore, the scheme can be used to measure the Wigner function of the collective vibrational mode. The scheme does not require the addressing of individual ionic laser.

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