Vol 13 No 11, November 2004
1009-1963/2004/13(11)/1790-06

Chinese Physics

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Non-Noether symmetrical conserved quantity for nonholonomic Vacco dynamical systems with variable mass*

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(Received 16 October 2003; revised manuscript received 7 April 2004)

Using a form invariance under special infinitesimal transformations in which time is not variable, the non-Noether conserved quantity of the nonholonomic Vacco dynamical system with variable mass is studied. The differential equations of motion of the systems are established. The definition and criterion of the form invariance of the system under special infinitesimal transformations are studied. The necessary and sufficient condition under which the form invariance is a Lie symmetry is given. The Hojman theorem is established. Finally an example is given to illustrate the application of the result.

Keywords: variable mass, nonholonomic constraint, Vacco dynamical system, non-Noether conserved quantity

PACC: 0320, 0200

1. Introduction

The conservation laws for a dynamical system play an important role in the study of mechanics. In particular, the existence of a certain conserved quantity will help us to understand the partial physical state and the quality of the system studied when it is hard to solve the differential equations of motion of the system. This has become one of the most important directions in modern analytical mechanics. There are four main methods used to seek the conserved quantities of a mechanical system: namely, the Noether theory,[1] the Lie method,[2] the theory of integrating factors[3] and the Mei form invariance[4] (or Mei symmetry). Significant achievements have been obtained in the study of this method in recent years.[5−17] In 1992, a new kind of conserved quantity was derived by Hojman from the differential equation of motion without using either the Lagrangian or Hamiltonian.[18] Pillay and Leach demonstrated that the invariance derived from the Hojman conservation law was trivial, for all of the Noether symmetry:[19] i.e. the conserved quantity for the Noether symmetry was identically equal to zero. Therefore, the requisite symmetry does not belong to Neother, while the significant nontrivial conserved quantity is derived using Hojman's method. This conserved quantity is called the non-Noether conserved quantity. In Refs.[20–22], the non-Noether conserved quantity of the variable mass system, the generalized Hamiltonian system and the relativistic Hamiltonian system were studied successively.

It is well known that the Vacco equation is the dynamical equation of the non-ideal nonholonomic system[23] and its integral is currently under research by few people.[24] In this paper, Hojman's method is used to seek the conserved quantity of a variable mass nonholonomic Vacco dynamical system.

2. Differential equations of motion of system

We consider a mechanical system composed of N particles, and its configuration is determined by n generalized coordinates $q_1, q_2, \ldots, q_n$. The motion of the system is subjected to the following g non-ideal bilat-

*Project supported by the Heilongjiang Natural Science Foundation, China (Grant No 9507).

eral nonholonomic constraints:

$$f_\beta(t,q,\dot{q})=0,\quad(\beta=1,2,\ldots,g). \tag{1}$$

The mass of a representative point i of the system is

$$m_i=m_i(t,q,\dot{q}),\quad(i=1,2,\ldots,N). \tag{2}$$

Then, the Vacco dynamical equations of the system can be written as follows:[18]

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}-\frac{\partial L}{\partial q_s}=&Q_s+P_s+\lambda_\beta\left(\frac{\partial f_\beta}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f_\beta}{\partial\dot{q}_s}\right)\\&-\dot{\lambda}_\beta\frac{\partial f_\beta}{\partial\dot{q}_s},\end{aligned}$$

$$(s=1,2,\ldots,n;\beta=1,2,\ldots,g), \tag{3}$$

where L is the Lagrangian of the system, Q_s are non-potential generalized forces, P_s are the generalized counter thrusts, and we have

$$\begin{aligned}P_s=&\left(\bar{R}_i+\dot{m}_i\dot{\bar{r}}_i\right)\cdot\frac{\partial\bar{r}_i}{\partial q_s}-\frac{1}{2}\dot{\bar{r}}_i\cdot\dot{\bar{r}}_i\frac{\partial m_i}{\partial g_s}\\&+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{\bar{r}}_i\cdot\dot{\bar{r}}_i\frac{\partial m_i}{\partial\dot{q}_s}\right)\end{aligned}$$

$$(s=1,2,\ldots,n;i=1,2,\ldots,N). \tag{4}$$

Here $\bar{r}_i$ and $\dot{\bar{r}}_i$ are the vector radius and the velocity of the particle i, respectively. Then

$$R_i=\frac{\mathrm{d}m_i}{\mathrm{d}t}\bar{u}_i, \tag{5}$$

where $\bar{u}_i$ is the relative velocity of the small particle corresponding to the particle i.

λ_β are constraint multipliers. Before we integrate the differential equations of motion, we obtain λ_β from Eqs.(1) and (3), and express it as a function of t, q and $\dot{q}$, i.e.

$$\lambda_\beta=\lambda_\beta(t,q,\dot{q}),\quad(\beta=1,2,\ldots,g). \tag{6}$$

Thus, Eqs.(3) can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\dot{q}_s}-\frac{\partial L}{\partial q_s}=Q_s+P_s+\Lambda_s,\quad(s=1,2,\ldots,n), \tag{7}$$

where

$$\begin{aligned}\Lambda_s=&\Lambda_s\left(t,q,\dot{q}\right)\\=&\lambda_\beta\left(\frac{\partial f_\beta}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f_\beta}{\partial\dot{q}_s}\right)-\dot{\lambda}_\beta\frac{\partial f_\beta}{\partial\dot{q}_s}\end{aligned} \tag{8}$$

are the generalized nonholonomic reactive forces of the constraint.

Equations (7) are called differential equations of motion of holonomic system corresponding to the variable mass nonholonomic Vacco dynamical system (1) and (3). The integrals of the nonholonomic system (1) and (3) can be found according to the corresponding holonomic system (7), only if we apply the restriction of the nonholonomic constraints (1) to the initial condition.[6]

Suppose that the system is non-singular, i.e.

$$\det\left(\frac{\partial^2 L}{\partial\dot{q}_s\partial\dot{q}_k}\right)\neq 0, \tag{9}$$

then all of the generalized accelerations can be obtained from Eqs.(7) and can be written as

$$\ddot{q}_s=\alpha_s(t,q,\dot{q}). \tag{10}$$

3. Definition and criterion of the form invariance of system

Introduce the special infinitesimal transformations in which time is not variable as

$$\begin{aligned}&t^*=t,\\&q_s^*(t^*)=q_s(t)+\varepsilon\xi_s(t,q,\dot{q})\end{aligned} \tag{11}$$

where ε is an infinitesimal parameter and ξ_s are infinitesimal generators. Under transformations (11), L, Q_s, P_s and Λ_s become

$$L^*=L\left(t^*,q^*,\frac{\mathrm{d}q^*}{\mathrm{d}t^*}\right),\quad Q_s^*=Q_s\left(t^*,q^*,\frac{\mathrm{d}q^*}{\mathrm{d}t^*}\right),$$

$$P_s^*=P_s\left(t^*,q^*,\frac{\mathrm{d}q^*}{\mathrm{d}t^*}\right),\quad \Lambda_s^*=\Lambda_s\left(t^*,q^*,\frac{\mathrm{d}q^*}{\mathrm{d}t^*}\right).$$

Expanding L^*, Q_s^*, P_s^* and Λ_s^*, we have

$$L^*=L\left(t,q,\dot{q}\right)+\varepsilon X^{(1)}(L)+o(\varepsilon^2), \tag{12}$$

$$\begin{aligned}Q_s^*=&Q_s(t,q,\dot{q})+\varepsilon X^{(1)}(Q_s)+o(\varepsilon^2),\\P_s^*=&P_s(t,q,\dot{q})+\varepsilon X^{(1)}(P_s)+o(\varepsilon^2),\end{aligned}$$

$$\Lambda_s^*=\Lambda_s(t,q,\dot{q})+\varepsilon X^{(1)}(\Lambda_s)+o(\varepsilon^2), \tag{13}$$

where

$$\begin{aligned}X^{(1)}=&\xi_s\frac{\partial}{\partial q_s}+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\cdot\frac{\partial}{\partial\dot{q}_s},\\\frac{\bar{\mathrm{d}}}{\mathrm{d}t}=&\frac{\partial}{\partial t}+\dot{q}_s\frac{\partial}{\partial q_s}+\alpha_s\frac{\partial}{\partial\dot{q}_s}.\end{aligned} \tag{14}$$

Introducing an Euler operator

$$E_s=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_s}-\frac{\partial}{\partial q_s},\quad(s=1,2,\ldots,n), \tag{15}$$

then Eqs.(7) can be expressed as

$$E_s(L)=Q_s+P_s+\Lambda_s,\quad(s=1,2,\ldots,n). \tag{16}$$

3.1. Definition

Under the infinitesimal transformations (11), if the form of the equation (16) remains invariant, i.e.

$$E_s(L^*) = Q_s^* + P_s^* + R_s^*, \quad (s = 1, 2, \ldots, n), \quad (17)$$

then the invariance is called the form invariance of the equation (16) of the corresponding holonomic system.

The form invariance of the corresponding holonomic system can lead to a conserved quantity of the system under certain conditions.

3.2. Criterion

For the holonomic system (16) corresponding to the variable mass nonholonomic Vacco dynamical system (1) and (3), if the infinitesimal generators ξ_s satisfy the following relation:

$$E_s\{X^{(1)}(L)\} = X^{(1)}(Q_s) + X^{(1)}(P_s) + X^{(1)}(\Lambda_s), \quad (18)$$

then it is form invariant under the infinitesimal transformations (11).

Proof Applying Euler operator (15) to the equation (12), and ignoring ε^2 and the higher infinitesimal terms, we have

$$E_s(L^*) = E_s(L) + \varepsilon E_s\{X^{(1)}(L)\}. \quad (19)$$

Substituting Eqs.(16) into Eq.(19), and using Eqs.(13), we have

$$\begin{aligned} E_s(L^*) =& Q_s + P_s + \Lambda_s + \varepsilon E_s\{X^{(1)}(L)\} \\ =& Q_s^* - \varepsilon X^{(1)}(Q_s) + P_s^* - \varepsilon X^{(1)}(P_s) \\ & + \Lambda_s^* - \varepsilon X^{(1)}(\Lambda_s) + \varepsilon E_s\{X^{(1)}(L)\} \\ =& Q_s^* + P_s^* + \Lambda_s^* + \varepsilon\{E_s[X^{(1)}(L)] \\ & - X^{(1)}(Q_s) - X^{(1)}(P_s) - X^{(1)}(\Lambda_s)\}. \end{aligned} \quad (20)$$

Now substituting Eq.(18) into Eq.(20), the Eqs.(17) will be obtained.

4. Relation between form invariance and Lie symmetry

In order that the non-Noether conserved quantity is derived by the form invariance of the mechanical system, the relation between the form invariance and the Lie symmetry of the corresponding holonomic system under the infinitesimal transformations (11) is studied.

Under the infinitesimal transformations (11), the determining equation of Lie symmetry of Eqs.(16) is

$$X^{(2)}\{E_s(L)\} = X^{(1)}(Q_s) + X^{(1)}(P_s) + X^{(1)}(\Lambda_s), \quad (21)$$

where

$$X^{(2)} = X^{(1)} + \frac{\bar{\mathrm{d}}^2}{\mathrm{d}t^2}\xi_k \cdot \frac{\partial}{\partial \ddot{q}_k}. \quad (22)$$

Equation (21) can be expressed equivalently as

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right] = \frac{\partial \alpha_s}{\partial q_k}\xi_k + \frac{\partial \alpha_s}{\partial \dot{q}_k}\frac{\mathrm{d}}{\mathrm{d}t}\xi_k. \quad (23)$$

Consider the following relations:

$$\begin{aligned} E_s(X^{(1)}) =& E_s\left(\xi_k\frac{\partial}{\partial q_k} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k \cdot \frac{\partial}{\partial \dot{q}_k}\right) \\ =& E_s\left(\xi_k\frac{\partial}{\partial q_k}\right) + E_s\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k \cdot \frac{\partial}{\partial \dot{q}_k}\right) \\ =& \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{q}_s}\left(\xi_k\frac{\partial}{\partial q_k}\right)\right] - \frac{\partial}{\partial q_s}\left(\xi_k\frac{\partial}{\partial q_k}\right) \\ & + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{q}_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k \cdot \frac{\partial}{\partial \dot{q}_k}\right)\right] \\ & - \frac{\partial}{\partial q_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k \cdot \frac{\partial}{\partial \dot{q}_k}\right) \\ =& \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\partial \xi_k}{\partial \dot{q}_s}\frac{\partial}{\partial q_k}\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left[\xi_k\frac{\partial}{\partial \dot{q}_s}\left(\frac{\partial}{\partial q_k}\right)\right] \\ & - \frac{\partial \xi_k}{\partial q_s}\frac{\partial}{\partial q_k} - \xi_k\frac{\partial}{\partial q_s}\left(\frac{\partial}{\partial q_k}\right) \\ & + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{q}_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k\right)\frac{\partial}{\partial \dot{q}_k}\right] \\ & + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k \cdot \frac{\partial}{\partial \dot{q}_s}\left(\frac{\partial}{\partial \dot{q}_k}\right)\right] \\ & - \frac{\partial}{\partial q_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k\right)\frac{\partial}{\partial \dot{q}_k} \\ & - \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k \cdot \frac{\partial}{\partial q_s}\left(\frac{\partial}{\partial \dot{q}_k}\right) \\ =& \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left[\frac{\partial \xi_k}{\partial \dot{q}_s}\frac{\partial}{\partial q_k}\right] - \frac{\partial \xi_k}{\partial q_s}\frac{\partial}{\partial q_k} \\ & + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{q}_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k\right)\frac{\partial}{\partial \dot{q}_k}\right] \\ & - \frac{\partial}{\partial q_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k\right)\frac{\partial}{\partial \dot{q}_k} + \xi_k\left[\frac{\partial^3}{\partial t \partial \dot{q}_s \partial q_k}\right. \\ & \left. + \dot{q}_r\frac{\partial^3}{\partial q_r \partial \dot{q}_s \partial q_k} + \ddot{q}_r\frac{\partial^3}{\partial \dot{q}_r \partial \dot{q}_s \partial q_k}\right] \\ & - \xi_k\frac{\partial}{\partial q_s}\left(\frac{\partial}{\partial q_k}\right) + \frac{\bar{\mathrm{d}}^2}{\mathrm{d}t^2}\xi_k \cdot \frac{\partial^2}{\partial \dot{q}_s \partial \dot{q}_k} \\ & + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k \cdot \left[\frac{\partial^3}{\partial t \partial \dot{q}_s \partial \dot{q}_k} + \dot{q}_r\frac{\partial^3}{\partial q_r \partial \dot{q}_s \partial \dot{q}_k}\right. \\ & \left. + \ddot{q}_r\frac{\partial^3}{\partial \dot{q}_r \partial \dot{q}_s \partial \dot{q}_k}\right], \end{aligned} \quad (24)$$

$$
\begin{aligned}
X^{(2)}(E_s) =& \xi_r \frac{\partial E_s}{\partial q_r} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_r \cdot \frac{\partial E_s}{\partial \dot{q}_r} + \frac{\bar{\mathrm{d}}^2}{\mathrm{d}t^2}\xi_r \cdot \frac{\partial E_s}{\partial \ddot{q}_r} \\
=& \xi_r \frac{\partial}{\partial q_r}\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s} - \frac{\partial}{\partial q_s}\right] \\
&+ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_r \cdot \frac{\partial}{\partial \dot{q}_r}\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s} - \frac{\partial}{\partial q_s}\right] \\
&+ \frac{\bar{\mathrm{d}}^2}{\mathrm{d}t^2}\xi_r \cdot \frac{\partial}{\partial \ddot{q}_r}\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s} - \frac{\partial}{\partial q_s}\right] \\
=& \xi_r \frac{\partial}{\partial q_r}\left[\frac{\partial^2}{\partial t \partial \dot{q}_s} + \dot{q}_k \frac{\partial^2}{\partial q_k \partial \dot{q}_s}\right. \\
&\left. + \ddot{q}_k \frac{\partial^2}{\partial \dot{q}_k \partial \dot{q}_s} - \frac{\partial}{\partial q_s}\right] \\
&+ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_r \cdot \frac{\partial}{\partial \dot{q}_r}\left[\frac{\partial^2}{\partial t \partial \dot{q}_s} + \dot{q}_k \frac{\partial^2}{\partial q_k \partial \dot{q}_s}\right. \\
&\left. + \ddot{q}_k \frac{\partial^2}{\partial \dot{q}_k \partial \dot{q}_s} - \frac{\partial}{\partial q_s}\right] + \frac{\bar{\mathrm{d}}^2}{\mathrm{d}t^2}\xi_r \\
&\times \frac{\partial}{\partial \ddot{q}_r}\left[\frac{\partial^2}{\partial t \partial \dot{q}_s} + \dot{q}_k \frac{\partial^2}{\partial q_k \partial \dot{q}_s}\right. \\
&\left. + \ddot{q}_k \frac{\partial^2}{\partial \dot{q}_k \partial \dot{q}_s} - \frac{\partial}{\partial q_s}\right] \\
=& \xi_r \left[\frac{\partial^3}{\partial t \partial q_r \partial \dot{q}_s} + \dot{q}_k \frac{\partial^3}{\partial q_r \partial q_k \partial \dot{q}_s}\right. \\
&\left. + \ddot{q}_k \frac{\partial^3}{\partial \dot{q}_r \partial \dot{q}_k \partial \dot{q}_s}\right] - \xi_r \frac{\partial}{\partial q_r}\left(\frac{\partial}{\partial q_s}\right) \\
&+ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_r \cdot \left[\frac{\partial^3}{\partial t \partial \dot{q}_r \partial \dot{q}_s} + \frac{\partial^2}{\partial q_k \partial \dot{q}_s}\right. \\
&\left. + \dot{q}_k \frac{\partial^3}{\partial q_k \partial \dot{q}_r \partial \dot{q}_s} + \ddot{q}_k \frac{\partial^3}{\partial \dot{q}_r \partial \dot{q}_k \partial \dot{q}_s}\right] \\
&- \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_r \cdot \frac{\partial}{\partial \dot{q}_r}\left(\frac{\partial}{\partial q_s}\right) \\
&+ \frac{\bar{\mathrm{d}}^2}{\mathrm{d}t^2}\xi_r \cdot \frac{\partial^2}{\partial \dot{q}_k \partial \dot{q}_s}. \qquad (25)
\end{aligned}
$$

Using Eqs.(24) and (25), the differential operator is obtained as follows:

$$
\begin{aligned}
& E_s\left(X^{(1)}\right) - X^{(2)}(E_s) \\
=& \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left[\frac{\partial \xi_k}{\partial \dot{q}_s}\frac{\partial}{\partial q_k}\right] - \frac{\partial \xi_k}{\partial \dot{q}_s}\frac{\partial}{\partial q_k} \\
&+ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{q}_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k\right)\frac{\partial}{\partial \dot{q}_k}\right] \\
&- \frac{\partial}{\partial q_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k\right)\frac{\partial}{\partial \dot{q}_k}. \qquad (26)
\end{aligned}
$$

Then

$$
\begin{aligned}
& E_s\left\{X^{(1)}(L)\right\} - X^{(1)}(Q_s) \\
&- X^{(1)}(P_s) - X^{(1)}(\Lambda_s) \\
&- \left\{X^{(2)}[E_s(L)]\right\} - X^{(1)}(Q_s) \\
&- X^{(1)}(P_s) - X^{(1)}(\Lambda_s) \\
=& \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left[\frac{\partial \xi_k}{\partial \dot{q}_s}\frac{\partial L}{\partial q_k}\right] - \frac{\partial \xi_k}{\partial q_s}\frac{\partial L}{\partial q_k} \\
&+ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{q}_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k\right)\frac{\partial L}{\partial \dot{q}_k}\right] \\
&- \frac{\partial}{\partial q_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k\right)\frac{\partial L}{\partial \dot{q}_k}. \qquad (27)
\end{aligned}
$$

Hence we have the following theorem.

Theorem 1 For the holonomic system (16) corresponding to the variable mass nonholonomic Vacco dynamical system (1) and (3), the necessary and sufficient condition under which the form invariance is a Lie symmetry is that the infinitesimal generators ξ_s satisfy the following relation:

$$
\begin{aligned}
& \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left[\frac{\partial \xi_k}{\partial \dot{q}_s}\frac{\partial L}{\partial q_k}\right] - \frac{\partial \xi_k}{\partial q_s}\frac{\partial L}{\partial q_k} \\
&+ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{q}_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k\right)\frac{\partial L}{\partial \dot{q}_k}\right] - \frac{\partial}{\partial q_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k\right)\frac{\partial L}{\partial \dot{q}_k} = 0. \qquad (28)
\end{aligned}
$$

5. Hojman theorem of the corresponding holonomic system

Theorem 2 For the holonomic system (16) corresponding to the variable mass nonholonomic Vacco dynamical system (1) and (3), if the infinitesimal generator ξ_s satisfies relation (28), and there is a function $\mu = \mu(t, q, \dot{q})$ reduced to

$$
\frac{\partial \alpha_s}{\partial \dot{q}_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln \mu = 0, \qquad (29)
$$

then the form invariance of the system can be led to a non-Noether conserved quantity as follows:

$$
I_{\mathrm{H}} = \frac{1}{\mu}\frac{\partial}{\partial q_s}(\mu \xi_s) + \frac{1}{\mu}\frac{\partial}{\partial \dot{q}_s}\left(\mu \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) = \mathrm{const}. \qquad (30)
$$

Proof According to the theorem of Ref.[15], if the ξ_s are infinitesimal generators of Lie symmetry then there exists a conserved quantity (30). The ξ_s satisfies the relation (28), from the supposed condition of the theorem, thus the system is not only form invariant but also Lie symmetrical.

It should be noticed that the conserved quantity derived from Theorem 2 does not satisfy

$$I_{\mathrm{H}} \equiv 0. \tag{31}$$

6. Example

We consider a particle of variable mass in the space, its mass is

$$m = m_0 \mathrm{e}^{-\beta t}, (m_0 = \mathrm{const}, \quad \beta = \mathrm{const}, \beta > 0). \tag{32}$$

The Lagrangian is

$$L = \frac{1}{2} m \left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_2^2 \right). \tag{33}$$

The absolute velocity of separate corpuscle is zero, i.e.

$$u = -\dot{\bar{r}}_i = -\dot{q}_1 \bar{i} - \dot{q}_2 \bar{j} - \dot{q}_3 \bar{k}. \tag{34}$$

The motion of the particle is subjected to the following non-ideal bilateral nonholonomic constraint:

$$f = \frac{1}{2} \left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_2^3 \right) - C = 0. \tag{35}$$

Let us study the non-Noether conserved quantity of the system.

The first step is to establish the equations of motion of the corresponding holonomic system.

According to Eqs.(32), (34) and (4), we obtain

$$P_1 = P_2 = P_3 = 0. \tag{36}$$

Equations (3) give

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{q}_1) &= -\lambda \ddot{q}_1 - \dot{\lambda} \dot{q}_1, \\ \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{q}_2) &= -\lambda \ddot{q}_2 - \dot{\lambda} \dot{q}, \\ \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{q}_3) &= -\lambda \ddot{q}_3 - \dot{\lambda} \dot{q}_3. \end{aligned} \tag{37}$$

Differentiating the constraint equation (35) with respect to time t, we obtain

$$\dot{q}_1 \ddot{q}_1 + \dot{q}_2 \ddot{q}_2 + \dot{q}_3 \ddot{q}_3 = 0. \tag{38}$$

From Eqs.(35), (37) and (38), we obtain

$$\lambda = -m. \tag{39}$$

Substituting Eq.(39) into Eqs.(37), we have

$$\ddot{q}_1 = \beta \dot{q}_1, \quad \ddot{q}_2 = \beta \dot{q}_2, \quad \ddot{q}_3 = \beta \dot{q}_3. \tag{40}$$

According to Eqs.(10), we have

$$\alpha_1 = \beta \dot{q}_1, \quad \alpha_2 = \beta \dot{q}_2, \quad \alpha_3 = \beta \dot{q}_3. \tag{41}$$

The determinate equation (23) gives respectively

$$\begin{aligned} \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_1 \right] &= \beta \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_1, \\ \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_2 \right] &= \beta \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_2, \\ \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_3 \right] &= \beta \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_3. \end{aligned} \tag{42}$$

The second step is to seek the conserved quantity of the system.

Taking

$$\begin{aligned} \xi_1 &= \frac{1}{2} \left(\dot{q}_1 - \beta q_1 \right)^2, \\ \xi_2 &= \frac{1}{2} \left(\dot{q}_2 - \beta q_2 \right)^2, \\ \xi_3 &= \frac{1}{2} \left(\dot{q}_3 - \beta q_3 \right)^2, \end{aligned} \tag{43}$$

it is easy to prove that Eqs.(43) satisfy the determining equations (42) and the relation (28), thus the system (40) is not only form invariant but also Lie symmetrical.

The equation (29) gives

$$\mu = \mathrm{e}^{-3\beta t}. \tag{44}$$

Substituting Eqs.(41), (43) and (44) into Eq.(30), we obtain the conserved quantity as

$$I_{\mathrm{H}} = \beta(q_1 + q_2 + q_3) - (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) = \mathrm{const}. \tag{45}$$

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