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Non-Noether symmetries and conserved quantities of the Lagrange mechano-electrical systems*

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This paper focuses on studying non-Noether symmetries and conserved quantities of Lagrange mechano-electrical dynamical systems. Based on the relationships between the motion and Lagrangian, we present conservation laws on non-Noether symmetries for Lagrange mechano-electrical dynamical systems. A criterion is obtained on which non-Noether symmetry leads to Noether symmetry of the systems. The work also gives connections between the non-Noether symmetries and Lie point symmetries, and further obtains Lie invariants to form a complete set of non-Noether conserved quantity. Finally, an example is discussed to illustrate these results.

Keywords: mechano-electrical system, non-Noether symmetry, non-Noether conserved quantity, infinitesimal transformation

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1. Introduction

The symmetric principle is one of the key topics in mathematics, mechanics and physics. Research on the conserved quantities of the mechanical systems has important theoretic and practical significance, and reveals profoundly the physical essence. There are many methods to find the conserved quantities of the mechanical system. A modern one is the Noether theory,[1] which is based on the invariance of the Hamilton action under the infinitesimal transformation. Djukić and Vujanović studied the Noether symmetries and the conserved quantities of the non-conservative holonomic systems by expanding Noether theory to the case including generalized velocity.[2] Li deduced the general Noether theory of linear nonholonomic dynamical systems.[3] Afterwards the Noether theory of the non-linear nonholonomic dynamical systems was disseminated.[4–6] Lately, the studies of Noether symmetry have given many results.[7–9] Another modern one is the Lie symmetries that leaves invariant the equation of motion under the infinitesimal transformation.[10] Cicogna and Gaeta obtained some conclusions on the existence and the properties of Lie point symmetries in mechanics, but they did not obtain the conserved quantity corresponding to those systems[11]. Mei made major progress with Lie symmetries for constrained mechanical systems.[12] Recently, the study of Lie symmetry has been very active.[13–15] Nevertheless, in the foregoing studies only the conserved quantities of Noether symmetrical type are considered. Lutzky has given non-Noether conserved quantities of both Lie point symmetries and Lie velocity-dependent symmetries for Lagrangian systems, and pointed out that a given symmetry, if conserving the action integral, leads to Noether symmetry.[16–18] Without using either a Lagrangian or Hamiltonian, Hojman has presented a theorem to obtain directly conserved quantities with Lie symmetry.[19] Later, Mei derived non-Noether conserved quantities for Hamiltonian systems in phase space.[20,21] Recently, the work on the non-Noether symmetry of the nonconservative dynamical

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system has been accomplished by the present authors, which leads straight to the conserved quantity of non-Noether type.[22] In the present paper, the authors first give a connection between the motion and the Lagrangian for a Lagrange mechano-electrical system. Then non-Noether symmetries of Lagrangian systems are extended to Lagrangian mechano-electrical systems, and a formula for the non-Nether conserved quantities of the Lagrangian mechano-electrical systems is derived. The authors further establish the condition under which non-Noether symmetries deduce Noether symmetries. Finally, an example is presented to demonstrate the applications of results.

2. Lagrange equations of mechano-electrical systems

These systems are called the mechano-electrical dynamical systems when the mechanical process and electromagnetic process in the systems are related to each other. The mechanical part of N particle components is described with the model of mechanics. If the system is subject to d ideal, bilateral, holonomic constraints, the position in space of the system is determined by $n = 3N - d$ generalized coordinates q_s $(s = 1, \ldots, n)$. The electrodynamical part of electric circuit and magnetic circuit is described with the model of electricity by m generalized electric quantities. Suppose the mechano-electrical system has m closed circuits, every closed circuit consists of the line conductor and capacitance, there is not any relation between electric circuits, and the electromagnetic process of closed circuits are not independent. Suppose i_k is the current, V_k is the electric potential, e_k is the electric quantity in the capacitance ($\dot{e}_k = i_k$), R_k is the resistance, C_k is the capacitance, then the Lagrangian of the mechano-electrical systems is

$$L = T(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{e}, \dot{\boldsymbol{e}}) - V(\boldsymbol{q}) + W_{\mathrm{m}}(\boldsymbol{q}, \dot{\boldsymbol{e}}) - W_{\mathrm{e}}(\boldsymbol{q}, \boldsymbol{e}), \quad (1)$$

where

$$W_{\mathrm{e}} = \frac{1}{2}\frac{e_k^2}{C_k},$$
$$W_{\mathrm{m}} = \frac{1}{2}L_{kr}i_k i_r \quad (k, r = 1, \ldots, m) \qquad (2)$$

are electric field energy and magnetic field energy of the mth circuit, respectively, in which $C_k = C_k(q_s)$ is the capacitance of the kth circuit, $L_{kr}(k \neq r)$ is the mutual inductance between kth and rth circuit, $L_{kr} = L_{rk}(q_s)$, and L_{kk} is self-inductance of the kth circuit.

Let $q_s(s = 1, \ldots, n, n+1, \ldots, m)$ denote the generalized coordinates. $q_s(s = 1, \ldots, n)$ are called coordinate components and $q_s(s = n+1, \ldots, n+m)$ are electric components. The equations of motion of the system is written in the form[23,24]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} + \frac{\partial F}{\partial \dot{q}_s} = Q_s$$
$$(s = 1, \ldots, n, \ldots, n+m), \qquad (3)$$

where $L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is the Lagrangian, F is the dissipative function, Q'_s is the non-potential generalized forces, $u_s(s = n+1, \ldots, n+m)$ is the electric potential energy and $Q_s = Q'_s + u_s$. Equation (3) is called the Lagrange–Maxwell equation of mechano-electrical systems. In Eq.(3),

$$F = F_{\mathrm{e}}(\dot{\boldsymbol{e}}) + F_{\mathrm{m}}(\boldsymbol{x}, \dot{\boldsymbol{x}}), \qquad (4)$$

where

$$F_{\mathrm{e}} = \frac{1}{2}R_k i_k^2 = \frac{1}{2}R_k \dot{e}_k^2 \qquad (5)$$

is the lead-through electric dissipative function, F_{m} is the dissipative function of the viscous frictional damping forces, and $-\dfrac{\partial F}{\partial \dot{q}_s}$ are called the dissipative forces.

When the mechano-electrical system satisfies conditions $Q_s - \dfrac{\partial F}{\partial \dot{q}_s} = 0$, Eq.(3) is reduced to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = 0 \quad (s = 1, \ldots, n, \ \ldots, n+m). \quad (6)$$

Eq.(6) is called the dynamical equation of the Lagrangian mechano-electrical system.

3. Non-Noether symmetries and conserved quantities of the Lagrange mechano-electrical system

Suppose that the Lagrange equation of the Lagrangian mechano-electrical system may be put in the form

$$\ddot{q}_s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \ \ (s = 1, \ldots, n, n+1, \ldots, n+m). \quad (7)$$

Let us introduce the infinitesimal transformations with respect to the generalized coordinates and time:

$$t^{\bullet} = t + \varepsilon\xi_0(\boldsymbol{q}, t), \quad q_s^{\bullet} = q_s + \varepsilon\xi_s(\boldsymbol{q}, t), \qquad (8)$$

where ε is a small parameter, and $\xi_0(\boldsymbol{q}, t)$ and $\xi_s(\boldsymbol{q}, t)$ are infinitesimal generators.

The invariance of motion equations (7) under the infinitesimal transformation (8) leads to the satisfaction of the determining equations

$$\ddot{\xi}_s - \dot{q}_s\ddot{\xi}_0 - 2\alpha_s\dot{\xi}_0 = X^{(1)}(\alpha_s)$$
$$(s = 1, \ldots, n, n+1, \ldots, n+m). \tag{9}$$

The infinitesimal generators $\xi_0(\boldsymbol{q}, t)$ and $\xi_s(\boldsymbol{q}, t)$ form a complete set of the Lie point symmetries for the Lagrange mechano-electrical dynamical system. The operator $X^{(1)}$ is the generator of the first extended group given by

$$X^{(1)} = \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} + (\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\frac{\partial}{\partial \dot{q}_s}, \tag{10}$$

and the vector field

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s\frac{\partial}{\partial q_s} + \alpha_s\frac{\partial}{\partial \dot{q}_s} \tag{11}$$

represents the total time derivatives along the trajectories of Eq.(7). Then for any function ϕ,

$$\dot{\phi} = \frac{\partial \phi}{\partial t} + \dot{q}_s\frac{\partial \phi}{\partial q_s} + \alpha_s\frac{\partial \phi}{\partial \dot{q}_s}. \tag{12}$$

If Eq.(7) remains invariant under the transformation of continuous Lie group (8), the invariance is called the non-Noether symmetry that is represented by a complete set of generators $\xi_0(\boldsymbol{q}, t)$ and $\xi_s(\boldsymbol{q}, t)$ satisfying determining equation (9) of the Lagrange mechano-electrical system.

Equation (9) may be regarded as a criterion of non-Noether symmetries, as follows.

Criterion 1 The symmetry is called non-Noether symmetry corresponding the Lagrange mechano-electrical systems (7) if the infinitesimal generators $\xi_0(\boldsymbol{q}, t)$ and $\xi_s(\boldsymbol{q}, t)$ satisfy the determining equation (8).

To derive non-Noether conserved quantities of the systems, two relations for the Lagrange mechano-electrical systems are needed.

First, the $n+m$ dimensional Lagrange mechano-electrical system (6) is written in the form

$$\frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_l}\ddot{q}_l = \frac{\partial L}{\partial q_s} - \frac{\partial^2 L}{\partial \dot{q}_s \partial t} - \frac{\partial^2 L}{\partial \dot{q}_s \partial q_l}\dot{q}_l$$
$$(s, l = 1, \ldots, n, n+1, \ldots, n+m).$$

One can easily demonstrate the relations between α_s and L as

$$\frac{\partial \alpha_s}{\partial \dot{q}_s} + \frac{\mathrm{d}}{\mathrm{d}t}(\ln D) = 0, \tag{13}$$

where

$$D = \det[\partial^2 L/\partial \dot{q}_s \partial \dot{q}_l]$$
$$(s, l = 1, \ldots, n, n+1, \ldots, n+m). \tag{14}$$

Second, we remark that if ξ_0 and ξ_s satisfy Eq.(9), it can be shown that[9]

$$\dot{X}^{(1)}(\phi) = X^{(1)}(\dot{\phi}) + \dot{\xi}_0\dot{\phi} \tag{15}$$

is satisfied for any function $\phi(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$. With these results in hand, one can prove the following theorem.

Theorem 1 The Lagrange mechano-electrical system (7) possesses a non-Noether conserved quantity in the form

$$\Phi = 2\left(\frac{\partial \xi_s}{\partial q_s} - \dot{q}_s\frac{\partial \xi_0}{\partial q_s}\right) - (n+m)\dot{\xi}_0 + X^1(\ln(D)) \tag{16}$$

when the infinitesimal transformation generators ξ_0 and ξ_s satisfy determining equation (9).

Proof. Move the right-hand side of equation (9) to the left, and represent it by Π_s and Π_k, which have the partial derivative

$$\frac{\partial \Pi_s}{\partial \dot{q}_s} = \frac{\mathrm{d}}{\mathrm{d}t}\left[2\left(\frac{\partial \xi_s}{\partial q_s} - \dot{q}_s\frac{\partial \xi_0}{\partial q_s}\right) - n\dot{\xi}_0\right] - X^{(1)}\left(\frac{\partial \alpha_s}{\partial \dot{q}_s}\right) - \frac{\partial \alpha_s}{\partial \dot{q}_s}\dot{\xi}_0$$
$$(s = 1, \ldots, n, n+1, \ldots n+m). \tag{17}$$

If ξ_0 and ξ_s satisfy Eq.(9), we may apply Eqs.(13) and (15) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}X^{(1)}(\ln D) = -X^{(1)}\left(\frac{\partial \alpha_s}{\partial \dot{q}_s}\right) - \dot{\xi}_0\frac{\partial \alpha_s}{\partial \dot{q}_s}, \tag{18}$$

which allows Eq.(17) to be written in the form

$$\frac{\partial \Pi_s}{\partial \dot{q}_s} = \frac{\mathrm{d}}{\mathrm{d}t}\left[2\left(\frac{\partial \xi_s}{\partial q_s} - \dot{q}_s\frac{\partial \xi_0}{\partial q_s}\right) - (n+m)\dot{\xi}_0 + X^{(1)}(\ln D)\right]. \tag{19}$$

Furthermore, if ξ_0 and ξ_s satisfy both $\Pi_s = 0$ and $\dfrac{\partial \Pi_s}{\partial \dot{q}_s} = 0$, consequently conserved quantity Eq.(16) leads to the fact that Φ should be a conserved quantity. This is the main result of the present paper. We should note that in order to get this conserved quantity, it is necessary to postulate that the equation of motion is derived from a Lagrangian, but the action integral invariant is not essential. We now show that if the symmetry group does preserve the action, then the quantity Φ vanishes.

Equation (16) gives non-Noether conserved quantities. At the same time, Eq.(16) can give the conserved quantity directly. However, it may reduce to

Noether conserved quantities on the following condition.

Theorem 2 For the Lagrange mechano-electrical systems, if the non-Noether conserved quantity Φ vanishes, then the symmetry group preserves the action, the non-Noether symmetry will lead to a Noether symmetry and possess the classical Noether conserved quantity

$$I = (\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\frac{\partial L}{\partial \dot{q}_s} + (\dot{\eta}_k - \dot{e}_k\dot{\xi}_0)\frac{\partial L}{\partial \dot{e}_k} + \xi_0 L + G \quad (20)$$

when there exists a function $G(\boldsymbol{q}, t)$ such that

$$X^{(1)}(L) + \dot{\xi}_0 L + \dot{G} = 0 \quad (s = 1, \ldots, n; k = 1, \ldots, m). \quad (21)$$

Thus we see that a conserved quantity can be associated with any symmetry of a Lagrange mechano-electrical system.

Proof: For any functions ξ_0 and ξ_s one can easily verify the following equation by straightforward calculation:

$$X^{(1)}\left(\frac{\partial^2 L}{\partial \dot{q}_l \partial \dot{q}_s}\right) = \frac{\partial^2 X^{(1)}(L)}{\partial \dot{q}_l \partial \dot{q}_s} - \frac{\partial L}{\partial \dot{q}_r}\frac{\partial (A_{rs})}{\partial \dot{q}_l} - A_{rs}\frac{\partial^2 L}{\partial \dot{q}_r \partial \dot{q}_l} - A_{rl}\frac{\partial^2 L}{\partial \dot{q}_r \partial \dot{q}_s},$$

$$(s, l, r = 1, \ldots, n;\ n+1, \ldots, n+m), \quad (22)$$

where

$$A_{rl} = \frac{\partial \xi_r}{\partial q_l} - \dot{q}_r\frac{\partial \xi_0}{\partial q_l} - \dot{\xi}_0\delta_{rl}. \quad (23)$$

Noether theorem points out that if ξ_0 and ξ_s generate a Noether symmetry corresponding to the Lagrange mechano-electrical systems, there exists a function $G_N(t, \boldsymbol{q})$ such that

$$X^{(1)}(L) + \dot{\xi}_0 L = -\dot{G}_N. \quad (24)$$

Since the right-hand side of Eq.(24) is linear in terms of the generalized velocities and electric current, it leads to

$$\frac{\partial^2 X^{(1)}(L)}{\partial \dot{q}_l \partial \dot{q}_s} = -\frac{\partial^2 (\dot{\xi}_0 L)}{\partial \dot{q}_l \partial \dot{q}_s}. \quad (25)$$

Substitution of Eq.(25) into Eq.(22) gives

$$X^{(1)}\left(\frac{\partial^2 L}{\partial \dot{q}_l \partial \dot{q}_s}\right) = \dot{\xi}_0\frac{\partial^2 L}{\partial \dot{q}_l \partial \dot{q}_s} - B_{ls}\frac{\partial^2 L}{\partial \dot{q}_r \partial \dot{q}_l} - B_{rl}\frac{\partial^2 L}{\partial \dot{q}_r \partial \dot{q}_s}$$

$$(l, s, r = 1, \ldots, n, n+1, \ldots, n+m), \quad (26)$$

where

$$B_{ls} = \frac{\partial \xi_l}{\partial q_s} - \dot{q}_s\frac{\partial \xi_0}{\partial q_s}. \quad (27)$$

Let M_{ts} be the cofactor of $\dfrac{\partial^2 L}{\partial \dot{q}_t \partial \dot{q}_s}$ in the matrix formed by these second derivatives, then the properties of determinants yield

$$M_{ts}\frac{\partial^2 L}{\partial \dot{q}_t \partial \dot{q}_r} = D\delta_{sr}, \quad (28)$$

and

$$M_{ts}\frac{\partial}{\partial \rho}\frac{\partial^2 L}{\partial \dot{q}_t \partial \dot{q}_s} = \frac{\partial D}{\partial \rho}, \quad (29)$$

where D is given by Eq.(14), and ρ may be one of $q_s, \dot{q}_s$ or t. Multiplying two equation in formula (26) by M_{ts} respectively, summing over repeated indices, and using Eqs.(28) and (29), one obtains

$$X^{(1)}(\ln D) = n\dot{\xi}_0 - 2B_{rr}. \quad (30)$$

From Eqs.(30) and (16), one knows that the conservation of Φ can be proved if the symmetry group leaves the action invariant. In this case, however, we have the classical Noether invariant on the Lagrange mechano-electrical system.

It should be pointed out that the conserved quantity Φ (see Eq.(16)) need not be a 'new' conserved quantity; that is, it may be expressible in terms of the Noether invariants of the system. In fact, for a system in which the Noether invariants form a complete set of conserved quantities, any additional constant of the motion must necessarily be a function of these Noether invariants.

4. Example

Capacitor C is shown in Fig.1. One of its electrode planks (of unit mass m=1) is hanged on a spring with stiffness k. The plank is vibrating along vertical direction under the action of gravity, elastic and electric field force. C is the capacitance, A is a constant, S is the distance, L is a unit inductance (L=1), R is the resistance and electromotive force $E = \dot{e}$. Let us study non-Noether symmetry of the system.

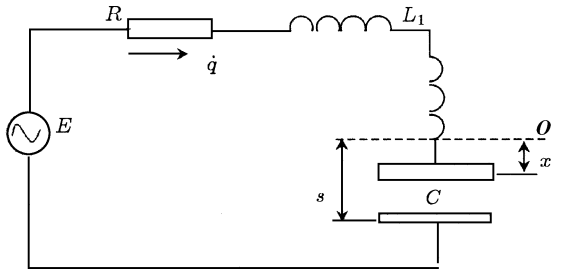


Fig.1. Example.

The system has two degrees of freedom. We choose the position of equilibrium (O point) of active plank of capacitor as the origin, and choose the coordinate x and electric quantity e as generalized coordinates. $T = \frac{1}{2}\dot{q}^2$ is kinetic energy of the system, $W_{\rm m} = \frac{1}{2}\dot{e}^2$ is magnetic energy, $V = \frac{1}{2}kq^2$ is potential of spring, $W_{\rm e} = \frac{1}{2}Ae^2$ is electric energy of capacitor. Then the Lagrange function L of the system is written as

$$L = \frac{1}{2}\dot{q}^2 + \frac{1}{2}\dot{e}^2 - \frac{1}{2}kq^2 - \frac{1}{2}Ae^2, \tag{31}$$

The dissipative function is

$$F = F_{\rm e} = \frac{1}{2}R\dot{e}^2, \tag{32}$$

and the electric energy is

$$u_r = R\dot{e}. \tag{33}$$

Its equations of motion are

$$\ddot{q} = -kq = \alpha_1, \tag{34}$$

$$\ddot{e} = -Ae = \alpha_2. \tag{35}$$

The determining equations of the non-Noether symmetry of Eqs.(34) and (35) under the infinitesimal transformation group ξ_0, ξ and η are

$$\ddot{\xi} - \dot{q}\ddot{\xi}_0 + 2kq\dot{\xi}_0 = -k\xi, \tag{36}$$

$$\ddot{\eta} - \dot{e}\ddot{\xi}_0 + 2Ae\dot{\xi}_0 = -A\eta. \tag{37}$$

The determining Eqs.(36) and (37) have the following solutions:

$$\begin{aligned}
\xi_0 =& q\sin\omega_1 t \quad (\omega_1^2 = k),\\
\xi =& \omega_1 q^2\cos\omega_1 t + \sin\omega_1 t\ ,\\
& (\text{or } \xi = \omega_1 q^2\cos\omega_1 t + \cos\omega_1 t)\\
\eta =& \omega_1 qe\cos\omega_1 t + c_1 q + c_2 e\\
& + c_3\cos\omega_2 t + c_4\sin\omega_2 t \quad (\omega_2^2 = A),
\end{aligned} \tag{38}$$

and

$$\begin{aligned}
\xi_0 =& e\sin\omega_2 t \quad (\omega_2^2 = A),\\
\eta =& \omega_2 e^2\cos\omega_2 t + \sin\omega_2 t\\
& (\text{or } \eta = \omega_2 e^2\cos\omega_2 t + \cos\omega_2 t),\\
\xi =& \omega_2 qe\cos\omega_2 t + c_5 q + c_6 e\\
& + c_7\cos\omega_1 t + c_8\sin\omega_1 t \quad (\omega_1^2 = k).
\end{aligned} \tag{39}$$

where $k = A$ in Eqs.(38) and (39). Substitution of Eq.(38) into Eq.(16) leads to the conserved quantity of non-Noether symmetry of the mechano-electrical system

$$\Phi_1 = 4\omega_1 q\cos\omega_1 t - 4\dot{q}\sin\omega_1 t = \text{const}, \tag{40}$$

where $c_3 = c_4 = 0$. Substituting generator (39) into Eq.(16) yields the conserved quantity of non-Noether symmetry of the system

$$\Phi_2 = 4\omega_2 e\cos\omega_2 t - 4\dot{e}\sin\omega_2 t = \text{const}, \tag{41}$$

where $c_5 = c_6 = 0$.

Actually, Eqs.(36) and (37) in the system have solutions as follows:

$$\xi_0 = a,\ \xi = q,\ \eta = e, \quad \Phi = 4; \tag{42}$$

$$\xi_0 = a,\ \xi = \sin\omega_1 t,\ \eta = \sin\omega_2 t, \quad \Phi = 0; \tag{43}$$

$$\xi_0 = a,\ \xi = \sin\omega_1 t,\ \eta = \cos\omega_2 t, \quad \Phi = 0; \tag{44}$$

$$\xi_0 = a,\ \xi = \cos\omega_1 t,\ \eta = \sin\omega_2 t, \quad \Phi = 0; \tag{45}$$

$$\xi_0 = a,\ \xi = \cos\omega_1 t,\ \eta = \cos\omega_2 t, \quad \Phi = 0; \tag{46}$$

$$\begin{aligned}
\xi_0 =& q\sin\omega_1 t,\ \xi = \omega q^2\cos\omega_1 t,\\
\eta =& \omega_1 qe\cos\omega_1 t + c_1 q + c_2 e,\\
\Phi =& 4\omega_1 q\cos\omega_1 t - 4\dot{q}\sin\omega_1 t + c_1 + c_2
\end{aligned} \tag{47}$$

$$\begin{aligned}
\xi_0 =& q\sin\omega_1 t,\ \xi = \omega q^2\cos\omega_1 t,\\
\eta =& \omega_1 qe\cos\omega_1 t + c_3\cos\omega_2 t,\\
\Phi =& 4\omega_1 q\cos\omega_1 t - 4\dot{q}\sin\omega_1 t = \text{const};
\end{aligned} \tag{48}$$

$$\begin{aligned}
\xi_0 =& q\sin\omega_1 t,\ \xi = \omega q^2\cos\omega_1 t,\\
\eta =& \omega_1 qe\cos\omega_1 t + c_4\sin\omega_2 t,\\
\Phi =& 4\omega_1 q\cos\omega_1 t - 4\dot{q}\sin\omega_1 t = \text{const};
\end{aligned} \tag{49}$$

$$\begin{aligned}
\xi_0 =& q\sin\omega_1 t,\ \xi = \omega q^2\cos\omega_1 t + \sin\omega_1 t,\\
\eta =& \omega_1 qe\cos\omega_1 t + c_1 d + c_2 e,\\
\Phi =& 4\omega_1 q\cos\omega_1 t - 4\dot{q}\sin\omega_1 t + c_1 + c_2 = \text{const};
\end{aligned} \tag{50}$$

$$\begin{aligned}
\xi_0 =& q\sin\omega_1 t,\ \xi = \omega q^2\cos\omega_1 t + \sin\omega_1 t,\\
\eta =& \omega_1 qe\cos\omega_1 t + c_3\cos\omega_2 t,\\
\Phi =& 4\omega_1 q\cos\omega_1 t - 4\dot{q}\sin\omega_1 t = \text{const};
\end{aligned} \tag{51}$$

$$\begin{aligned}
\xi_0 =& q\sin\omega_1 t,\ \xi = \omega q^2\cos\omega_1 t + \sin\omega_1 t,\\
\eta =& \omega_1 qe\cos\omega_1 t + c_4\sin\omega_2 t,\\
\Phi =& 4\omega_1 q\cos\omega_1 t - 4\dot{q}\sin\omega_1 t = \text{const};
\end{aligned} \tag{52}$$

$$\begin{aligned}
\xi_0 =& q\sin\omega_1 t,\ \xi = \omega q^2\cos\omega_1 t + \cos\omega_1 t,\\
\eta =& \omega_1 qe\cos\omega_1 t + c_1 d + c_2 e,\\
\Phi =& 4\omega_1 q\cos\omega_1 t - 4\dot{q}\sin\omega_1 t + c_1 + c_2 = \text{const};
\end{aligned} \tag{53}$$

$$\begin{aligned}
\xi_0 =& q\sin\omega_1 t,\ \xi = \omega_1 q^2\cos\omega_1 t + \cos\omega_1 t,\\
\eta =& \omega_1 qe\cos\omega_1 t + c_3\cos\omega_2 t,\\
\Phi =& 4\omega_1 q\cos\omega_1 t - 4\dot{q}\sin\omega_1 t = \text{const};
\end{aligned} \tag{54}$$

$$\begin{aligned}\xi_0 =& q\sin\omega_1 t,\ \xi = \omega q^2\cos\omega_1 t + \cos\omega_1 t,\\ \eta =& \omega_1 qe\cos\omega_1 t + c_4\sin\omega_2 t,\\ \Phi =& 4\omega_1 q\cos\omega_1 t - 4\dot{q}\sin\omega_1 t = \text{const}.\end{aligned} \tag{55}$$

Similarly, from of Eqs.(42)–(55), we can get the following symmetries and conserved quantities which are written as compendia forms:

$$\begin{aligned}\xi_0 =& a,\ e\sin\omega_2 t,\\ \eta =& \sin\omega_2 t,\ \cos\omega_2 t,\ \omega_2 e^2\sin\omega_2 t,\\ \xi =& \omega_2 qe\cos\omega_2 t,\ \omega_2 qe\cos\omega_2 t + c_5 d + c_6 e,\\ & \omega_2 qe\cos\omega_2 t + c_7\cos\omega_1 t,\\ & \omega_2 qe\cos\omega_2 t + c_8\sin\omega_1 t,\\ \Phi =& 4,\ 0,\ 4\omega_2 e\cos\omega_2 t - 4\dot{e}\sin\omega_2 t,\\ & 4\omega_2 e\cos\omega_2 t - 4\dot{e}\sin\omega_2 t + c_6.\end{aligned} \tag{56}$$

where Lie point symmetries (42)–(46) preserve action of the system, and possess Noether-type conserved quantities; while Lie point symmetries (47)–(56) possess non-Noether conserved quantities. That indicates that the Lutzky-type non-Noether symmetries (38) and (39) are a complete set of Lie point symmetries of the determining equations (36) and (37) for the mechano-electrical system, and the Lutzky-type non-Noether conserved quantities (40) and (41) need not be a "new' conserved quantity. That is, it may be expressible in terms of the Noether invariants of the system. In fact, for a mechano-electrical system in which the Lie point symmetries form a complete set of symmetry, any additional symmetries of the motion must necessarily be a function of these Lie point symmetries. Similarly, for the system where the Noether invariants form a complete set of conserved quantities, any additional constant of the motion must necessarily be a function of these Noether invariants.

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