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# Variable separation solutions and new solitary wave structures to the (1+1)-dimensional equations of long-wave–short-wave resonant interaction*

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A variable separation approach is proposed and extended to the (1+1)-dimensional physical system. The variable separation solutions of (1+1)-dimensional equations of long-wave–short-wave resonant interaction are obtained. Some special type of solutions such as soliton solution, non-propagating solitary wave solution, propagating solitary wave solution, oscillating solitary wave solution are found by selecting the arbitrary function appropriately.

**Keywords:** (1+1)-dimensional equations of long-wave–short-wave resonant interaction equation, variable separation approach, new solitary wave structures

**PACC:** 0230, 0340

## 1. Introduction

The Fourier transformation and the variable separation approach (VSA) are two important methods in linear mathematical physics. The well known inverse scattering transformation for the nonlinear integrable systems can be considered as an extension of the Fourier transformation in nonlinear mathematical physics.[1] However, it is relatively difficult to extend the VSA to the nonlinear mathematical physics. Recently, the study of variable separation in nonlinear science has achieved great success and has established various kinds of separation methods, such as the differential geometry method,[2] the ansatz-based methods,[3] the symmetry constraints[4] or nonlinearization of the Lax pairs[5] or the formal variable separation approach,[6] the functional variable separation approach,[7] the multi-linear variable separations approach,[8] and so on.

Since Lou and Lu[9] successfully applied a kind of variable separation method, now called the multi-linear variable separations approach (MLVSA), firstly in 1995 for the Davey–Stewartson (DS) equation, the MLVSA has been revised and developed for various (2+1)-dimensional nonlinear evolution systems. A nearly systematic process of the MLVSA to solve (2+1)-dimensional nonlinear evolution systems has been established,[10−23] but the application of the MLVSA to the (1+1)-dimensional nonlinear evolution systems is restricted to only a few examples. In this paper, we further investigate the extension of the MLVSA to (1+1)-dimensional nonlinear models by considering a class of (1+1)-dimensional equations of long-wave–short-wave resonant interaction (LSW)

$$\mathrm{i}S_t + \beta S_{xx} = \gamma LS, \tag{1}$$

$$L_t + \alpha(SS^*)_x = 0, \tag{2}$$

where $L$ is the profile of the long wave and $S$ is the envelope of the short wave. $\alpha, \beta$ and $\gamma$ are positive real model constants. Equations (1) and (2) were proposed for the first time by Zakharov to describe the interaction of Langmuir oscillations with ionic sound in a plasma. The universal equations were obtained for waves on the water surface by Djordjevic and Redekopp[24] and for the model of a molecular chain in the form of an $\alpha$-helix by Davydov.[25] These equations were also derived in bubbly liquids by Akhatov and Khismatulin.[26] Refs.[27–29] discussed the

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Cauchy problem and global solution for Eqs.(1) and (2).

## 2. Variable separation procedure for Eqs.(1) and (2)

To use the variable separation approach, we take the following transformation:

$$L = -2\gamma^{-1}\beta(\ln f)_{xx} + l_0, \quad S = \frac{g}{f} + s_0, \tag{3}$$

which can be obtained from the standard Painlevé truncated expansion. Here $f$ is real, $g$ is complex, and $\{l_0, s_0\}$ is an arbitrary known seed solution of the LSW equation. Substituting Eq.(3) into Eqs.(1) and (2) and integrating Eq.(2) once to the argument $x$, we obtain the bilinear forms of Eqs.(1) and (2):

$$(\mathrm{i}D_t + \beta D_x^2)g \cdot f + \beta s_0 D_x^2 f \cdot f - \gamma l_0 g f = 0, \tag{4}$$

$$-\gamma^{-1}\beta D_x D_t f \cdot f = -\alpha(gg^* + fgs_0^* + fg^* s_0), \tag{5}$$

where $D_x, D_t$ are defined as

$$D_x^m D_t^n g \cdot f \equiv (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n g\ f|_{(x=x', t=t')}, \tag{6}$$

which are the usual bilinear operators introduced first by Hirota.[30]

Now, the seed solutions are selected as follows:

$$l_0 = l_0(x), \quad s_0 = 0, \tag{7}$$

then Eqs.(4) and (5) can be simplified to

$$(\mathrm{i}D_t + \beta D_x^2)g \cdot f - \gamma l_0 g f = 0, \tag{8}$$

$$-\gamma^{-1}\beta D_x D_t f \cdot f = -\alpha g g^*. \tag{9}$$

To solve Eqs.(8) and (9) by using a variable separation approach, we construct the ansatz

$$f = a_0 + a_1 p(x) + a_2 q(t) + a_3 p(x) q(t), \tag{10}$$

$$g = g_1(x) g_2(t) \exp[\mathrm{i}r_1(x) + \mathrm{i}r_2(t)], \tag{11}$$

where $a_0, a_1, a_2$ and $a_3$ are arbitrary real constants. The variable separation functions $p \equiv p(x)$, $g_1 \equiv g_1(x)$, $r_1 \equiv r_1(x)$ and $q \equiv q(t)$, $g_2 \equiv g_2(t)$, $r_2 \equiv r_2(t)$ are only real functions of $\{x\}$ and $\{t\}$, respectively. It is clear that the variables $x$ and $t$ now have been separated totally. Substituting Eqs.(10) and (11) into Eqs.(8) and (9), separating the real part and the imaginary part of the resulting equation, and further setting the coefficients of $f^i (i = 0, 1)$ to zero, then we have

$$2p_x g_{1x} - g_1 p_{xz} = 0, \tag{12}$$

$$-r_{2t} g_1 + \beta g_{1xx} - \beta g_1 r_{1x}^2 - \gamma l_0 g_1 = 0, \tag{13}$$

$$-(a_2 + a_3 p)q_t - 2\beta(a_1 + a_3 q)p_x r_{1x} = 0, \tag{14}$$

$$g_{1t} g_2 + 2\beta g_{1x} g_2 r_{1x} + \beta g_1 g_2 r_{1xx} = 0, \tag{15}$$

$$\alpha g_1^2 g_2^2 = 2\beta(a_0 a_3 - a_1 a_2)p_x q_t. \tag{16}$$

Solving partial differential Eqs.(12)–(16), we obtain the following results:

$$g_1 = \delta_1 \sqrt{2(\alpha\gamma)^{-1}\lambda_0 \beta(a_0 a_3 - a_1 a_2)p_x}, \tag{17}$$

$$g_2 = \delta_2 \sqrt{\lambda_0^{-1} q_t}, \quad (\delta_1^2 = 1, \delta_2^2 = 1), \tag{18}$$

$$r_1 = -k \int [(2\beta p_x)^{-1}(a_2 + a_3 p)]\mathrm{d}x, \tag{19}$$

$$l_0 = -\gamma^{-1} c_1 + (4\gamma\beta p_x^2)^{-1}[-k^2(a_2 + a_3 p)^2 + \beta^2(2p_{xxx}p_x - p_{xx}^2)], \tag{20}$$

$$r_2(t) = c_0 + c_1 t, \tag{21}$$

$$q_t = k(a_1 + a_3 q). \tag{22}$$

In Eqs.(17)–(21) $\lambda_0$, $k$, $c_0$, $c_1$ are arbitrary real constants. From Eqs.(17) and (18), we know the real conditions of $g_1$ and $g_2$ require

$$(\alpha\gamma)^{-1}\lambda_0\beta(a_0 a_3 - a_1 a_2)p_x \geq 0, \tag{23}$$

$$\lambda_0^{-1} q_t \geq 0. \tag{24}$$

Finally, we obtain a general excitation of system (1) and (2):

$$S(x,t) = \pm \frac{\sqrt{2(\alpha\gamma)^{-1}\beta(a_0 a_3 - a_1 a_2)p_x q_t}}{a_0 + a_1 p + a_2 q + a_3 pq} \exp\{\mathrm{i}[c_0 + c_1 t - k \int (2\beta p_x)^{-1}(a_2 + a_3 p)\mathrm{d}x]\}, \tag{25}$$

$$L(x,t) = -2\gamma^{-1}\beta \frac{(a_1 + a_3 q)(a_0 + a_2 q)p_{xx} + (a_1 + a_3 q)^2(pp_{xx} - p_x^2)}{(a_0 + a_1 p + a_2 q + a_3 pq)^2} + \frac{-k^2(a_2 + a_3 p)^2 + \beta^2(2p_{xxx}p_x - p_{xx}^2)}{4\gamma\beta p_x^2} - \gamma^{-1} c_1, \tag{26}$$

where $p(x)$ is an arbitrary function of $\{x\}$, and $q$ can be expressed by Eq.(22) as the arbitrary function of $\{t\}$. Especially, the module square of $S$ can also be expressed as

$$G = |S|^2 = \frac{2(\alpha\gamma)^{-1}\beta(a_0a_3 - a_1a_2)p_xq_t}{(a_0 + a_1p + a_2q + a_3pq)^2}. \qquad (27)$$

Due the arbitrariness of the function of $p(x)$, the field $G(x,t)$ shown by Eq.(27) possesses abundant soliton structures. Here we list some examples from solution (27).

# 3. Some special types of solitary wave structures

## 3.1. Soliton solutions

If we select the function

$$p(x) = \sum_{i=1}^{n} \exp(k_ix + k_{0i}),$$
$$q(t) = -\frac{a_1}{a_3} + \exp(a_3kt + w_0), \qquad (28)$$

where $w_0$, $k_i$, $k_{0i}$ are arbitrary constants and $n$ an arbitrary positive integer, then we have the soliton solution. This soliton not only has not any dispersive relation between $(a_3k)$ and $k_i$, but also can be either left-travelling or right-travelling.

Figure 1 shows evolution plots of left-travelling soliton for solution $G(x,t)$ (27) with (28) and $a_0 = 1$, $a_1 = a_2 = 0$, $a_3 = \alpha = \gamma = 1$, $\beta = \frac{1}{2}$, $n = 1$, $k_1 = 3$, $k_{01} = 0$, $k = 1$, $w_0 = 0$, at time $t = -5$, $t = 0$ and $t = 5$.

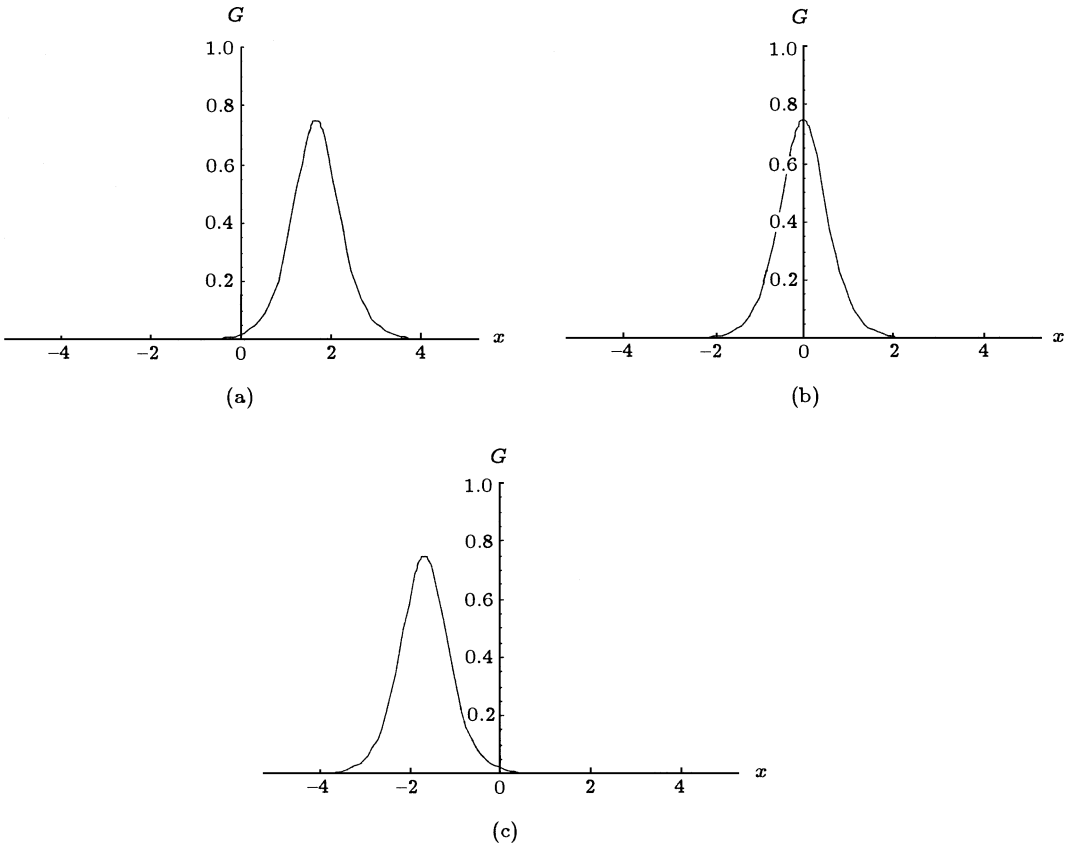



**Fig.1.** Evolution plots of solutions $G(x,t)$ of single soliton expressed by Eq.(27) with the condition (28) and $\{a_0, a_1, a_2, a_3, \alpha, \gamma, \beta, n, k_1, k_{01}, k, w_0\} = \{1, 0, 0, 1, 1, 1, \frac{1}{2}, 1, 3, 0, 1, 0\}$ at time (a) $t = -5$ (b) $t = 0$ and (c) $t = 5$.

## 3.2. Non-propagating solitary wave solution

When $p$ and $q$ are selected as

$$p(x) = \exp(\tanh(x^3 - 0.2)), \quad q = \exp\left(t + \frac{1}{2}\right), \qquad (29)$$

we can obtain a kind of non-propagating solitary wave for Eqs.(1) and (2). Figure 2 is the evolution plots of $G(x,t)$ expressed by Eq.(27) with condition (29) and $a_0$=1, $a_1 = a_2 = 0$, $a_3 = \alpha = \gamma = 1$, $\beta = \frac{1}{2}$, at time $t$=–4, $t$=–3, $t$=–1, $t$=2 and $t$=3. From Fig.2(a)–(e), we find that the non-propagating solitary wave is fixed in the $x$-axis but the amplitude changes with time $t$.

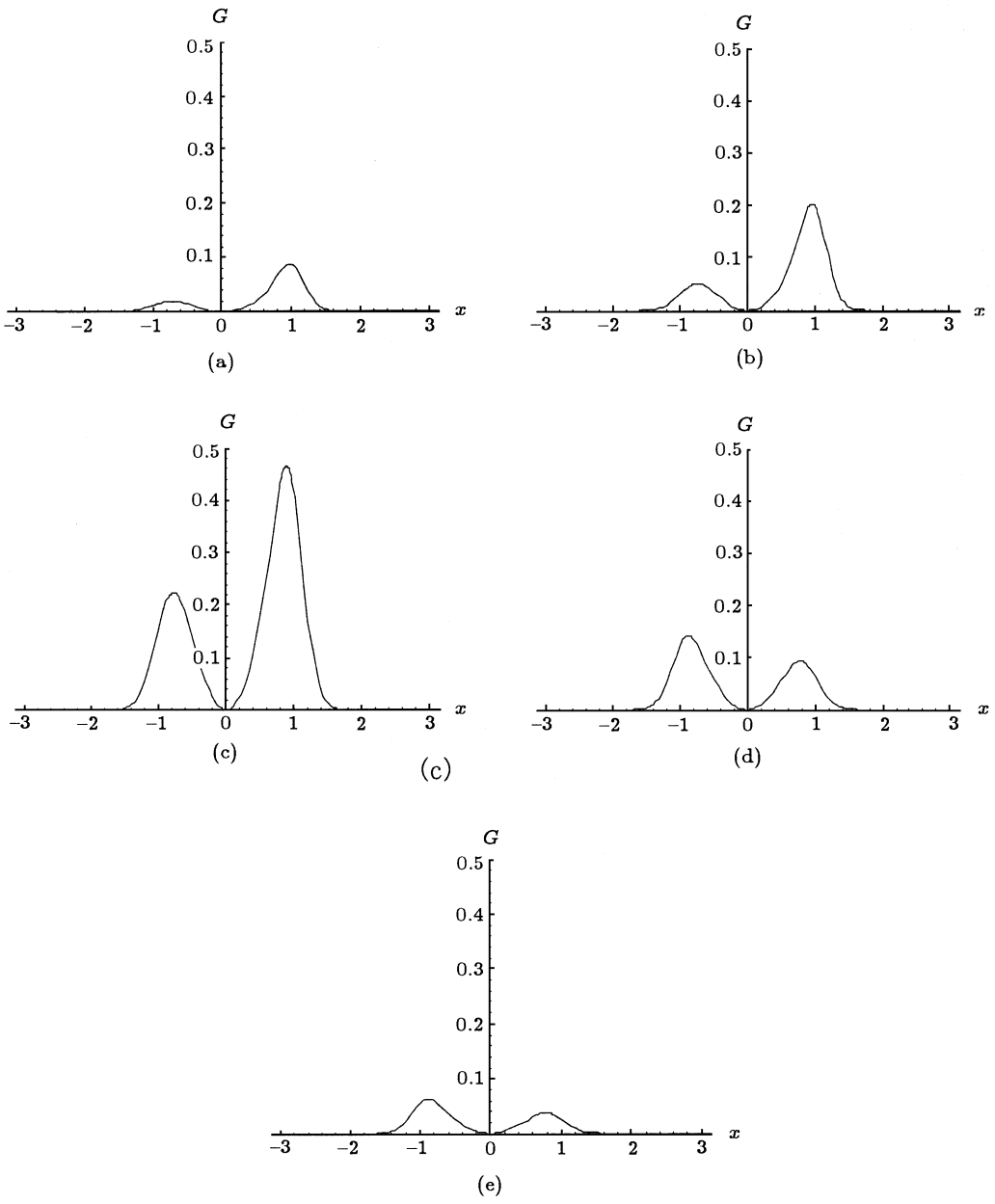



**Fig.2.** Evolution plots of non-propagating solitary wave $G(x,t)$ expressed by Eq.(27) with condition (29) and $\{a_0, a_1, a_2, a_3, \alpha, \gamma, \beta\} = \{1, 0, 0, 1, 1, 1, \frac{1}{2}\}$ at (a) $t = -4$ (b) $t = -3$, (c) $t = -1$, (d) $t = 2$ and (e) $t = 3$.

### 3.3. Propagating solitary wave solution

When $p$ and $q$ are chosen as

$$p(x) = \exp\left(\frac{x^5}{5} + \frac{x^3}{3} + x\right), \quad q = \exp(2t + 2), \quad (30)$$

we can obtain a kind of propagating solitary wave for Eqs.(1) and (2). Figure 3 shows the evolution plots of $G(x,t)$ expressed by Eq.(27) with condition (30) and $a_0 = 13$, $a_1 = a_2 = 0$, $a_3 = 2, \alpha = \gamma = 1$, $\beta = \frac{1}{4}$, at time $t = -2$, $t = -0.7$, $t$=0, $t$=0.7 and $t$=2. From Figs.3(a)–3(e), we can see that the amplitude of propagating solitary wave changes with time $t$. This solitary wave can be either left-travelling or right-travelling.

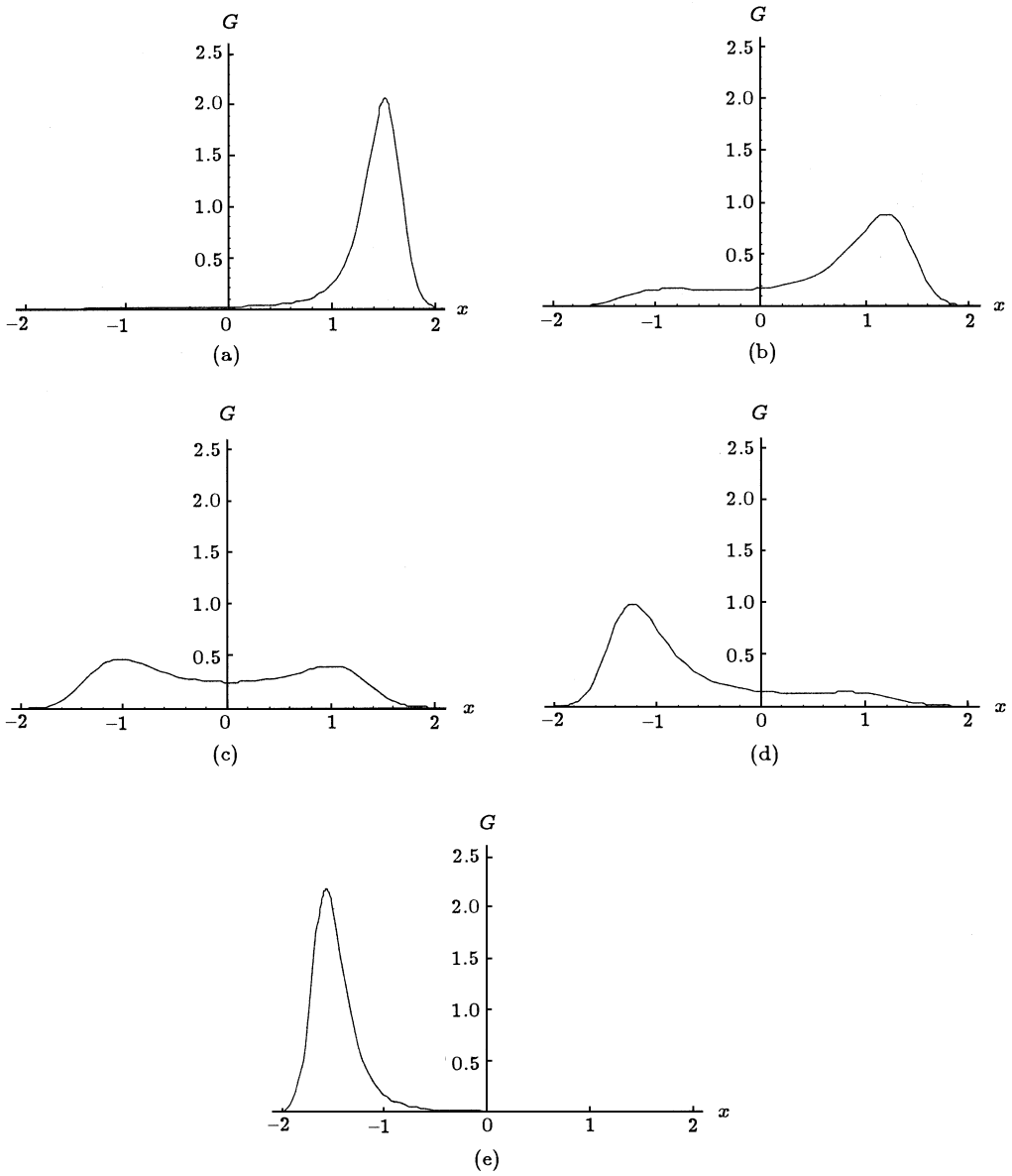



**Fig.3.** Evolution plots of propagating solitary wave $G(x,t)$ expressed by Eq.(27) with condition (30) and $\{a_0,a_1,a_2,a_3,\alpha,\gamma,\beta\}=\{13,0,0,2,1,1,\frac{1}{4}\}$, at (a) $t=-2$, (b) $t=-0.7$, (c) $t=0$, (d) $t=0.7$ and (e) $t=2$.

### 3.4. Oscillating solitary wave solution

Selecting the function

$$p(x)=\exp(x+\sin x),\quad q(t)=\exp(wt), \tag{31}$$

we can obtain a kind of oscillating solitary wave solution. Figure 4 shows the evolution plots of solutions of $G(x,t)$ expressed by Eq.(27) with condition (31) and $a_0=2$, $a_1=a_2=0$, $a_3=1$, $\alpha=\gamma=1$, $\beta=\frac{1}{2}$, $w=1$, at time $t=-3$, $t=-1$, $t=1$, $t=3$, $t=5$ and $t=7$.

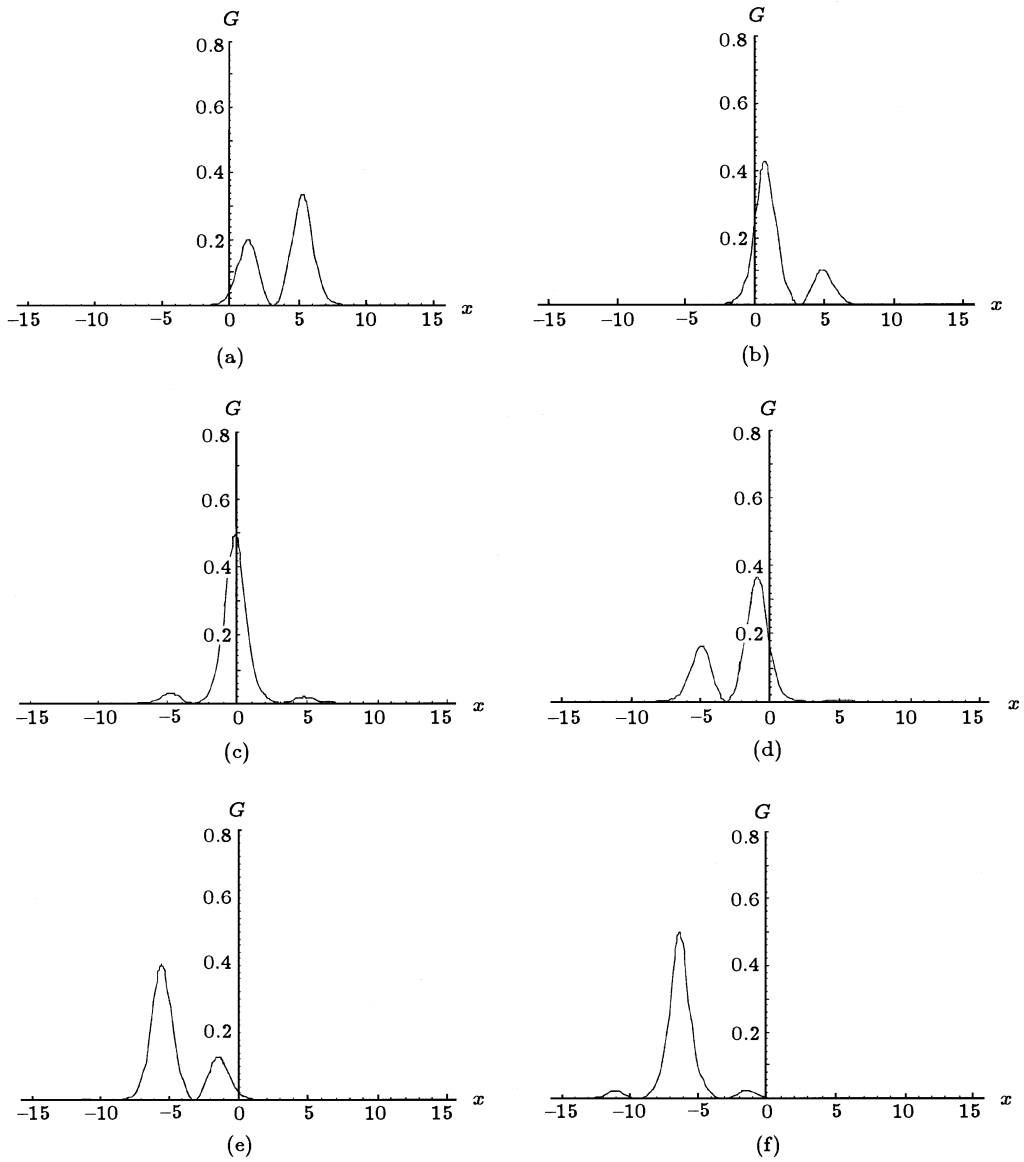



**Fig.4.** Evolution plots of oscillating solitary solution $G(x,t)$ expressed by Eq.(27) with condition (31) and $\{a_0, a_1, a_2, a_3, \alpha, \gamma, \beta, w\} = \{2, 0, 0, 1, 1, 1, \frac{1}{2}, 1\}$, at (a) $t = -3$, (b) $t = -1$, (c) $t = 1$, (d) $t = 3$, (e) $t = 5$ and (f) $t = 7$.

## 4. Summary and discussion

In summary, using the variable separation approach, the new variable separation solution of (1+1)-dimensional equation of long-wave–short-wave resonant interaction is obtained. By selecting the arbitrary function appropriately, the soliton solution, the non-propagating solitary wave solution, the propagating solitary wave solution, and the oscillating solitary solution can be constructed. To our knowledge, the non-propagating soliton solution of (1+1)-dimensional equations of long-wave–short-wave resonant interaction found here have not been reported in the literature. The non-propagating soliton solution and further applications of the MLVSA to other (1+1)-dimensional physical models is worthy of further study.

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