Vol 13 No 10, October 2004
1009-1963/2004/13(10)/1770-07

Chinese Physics

© 2004 Chin. Phys. Soc.
and IOP Publishing Ltd

Numerical solution of Helmholtz equation of barotropic atmosphere using wavelets*

Wang Ping(汪 萍)† and Dai Xin-Gang(戴新刚)

Key Laboratory of Regional Climate-Environment for Temperate East Asia, Institute of Atmospheric Physics, Chinese Academy of Sciences, START Regional Center for Temperate East Asia, Beijing 100029, *China*

(Received 18 February 2004; revised manuscript received 20 May 2004)

The numerical solution of the Helmholtz equation for barotropic atmosphere is estimated by use of the wavelet-Galerkin method. The solution involves the decomposition of a circulant matrix consisting up of 2-term connection coefficients of wavelet scaling functions. Three matrix decompositions, i.e. fast Fourier transformation (FFT), Jacobian and QR decomposition methods, are tested numerically. The Jacobian method has the smallest matrix-reconstruction error with the best orthogonality while the FFT method causes the biggest errors. Numerical result reveals that the numerical solution of the equation is very sensitive to the decomposition methods, and the QR and Jacobian decomposition methods, whose errors are of the order of 10^{-3}, much smaller than that with the FFT method, are more suitable to the numerical solution of the equation. With the two methods the solutions are also proved to have much higher accuracy than the iteration solution with the finite difference approximation. In addition, the wavelet numerical method is very useful for the solution of a climate model in low resolution. The solution accuracy of the equation may significantly increase with the order of Daubechies wavelet.

Keywords: wavelet-Galerkin method, Helmholtz equation, matrix decomposition, barotropic atmosphere

PACC: 9260X

1. Introduction

The numerical solution of Helmholtz equation is one of the most important problems encountered in atmospheric modelling and weather forecasting.[1−5] Several effective methods are often used in solving the equation, such as iteration, Fourier transform, variational and finite elementary methods.[6] The choice of method depends on the problem and requirement considered. The wavelet-Galerkin method, based on wavelet theory, is a new and rapidly developing one.[7] Wavelet bases, different from others, are compactly supported in both physical and time spaces. A signal or a function can be accurately represented by use of them at a high compression rate.[8−14] That is why their application has expanded to the numerical solution of differential equations soon after the successful construction of the compactly supported orthogonal wavelets. There are mainly three methods, i.e. wavelet-Galerkin[15,16], wavelet collocation,[17−19] and wavelet-capacitance matrix methods,[20,21] used for numerical solution of differential equations in connection with wavelets. For the first one the main difficulty is the boundary problem, especially under complex boundary conditions. The others can obtain high accuracy solutions at the boundary nearby, but are not suitable for extensive calculation. Qian and Weiss acquired a numerical solution of a Poisson equation using the wavelet-Galerkin method, but there remain some things needed to be studied in detail. For instance, how does the matrix decomposition affect the orthogonal and reconstruction errors of the circulant matrix and the solution accuracy of the equation considered as well? This paper focuses on the numerical solution of the Helmholtz equation for barotropic atmosphere using the wavelet-Galerkin method in association with matrix decomposition and orthogonal wavelet bases.

Section 2 includes a brief introduction to periodic wavelet bases. The wavelet spectrum Helmholtz equation is derived in section 3. The reconstruction and orthogonal errors for different matrix decomposi-

*Project supported by the National Key Planning Development for Basic Research (Grant No 1999043403) and the National Natural Science Foundation of China (Grant No 49875024).

†E-mail: wp668@sohu.com

tions are investigated in section 4. Section 5 focuses on the numerical solutions in terms of different matrix decompositions. The solution accuracy and its relationship with the order of the Daubechies wavelet are presented in section 6. The final section is for the conclusion.

2. Periodic wavelet bases

The boundary condition treatment is, as is well known, a difficult problem for the Galerkin method as well as for the wavelet-Galerkin method used for numerical solution of differential equations. The classical wavelet is defined on a real line, while the interval wavelet is still under development.[22] For simplicity, we take the periodic wavelets supported on [0,1] as the bases for a function expansion. The periodic wavelets can be directly obtained through Poisson summation of classical ones.[23] The periodic scaling function and wavelet are expressed as

$$\tilde{\varphi}_{j,k}(x) = \sum_{l \in ZZ} \varphi_{j,k}(x-l), \tag{1}$$

$$\tilde{\psi}_{j,k}(x) = \sum_{l \in ZZ} \psi_{j,k}(x-l), \tag{2}$$

where $\varphi_{j,k}(x)$ and $\psi_{j,k}(x)$ respectively denote the orthogonal scaling function and wavelet. For instance, the Daubechies scaling function and wavelet, which was constructed by Daubechies in 1988,[9] respectively are $\tilde{\varphi}_{j,k}(x) \in \tilde{V}_j$ and $\tilde{\psi}_{j,k}(x) \in \tilde{W}_j$, where $\tilde{W}_j$ is the orthogonal complement of $\tilde{V}_j$, and

$$\tilde{V}_{j+1} = \tilde{V}_j \oplus \tilde{W}_j, \tag{3}$$

$$L^2(IR) = \mathrm{close}_{L^2(IR)}(\tilde{V}_0 \oplus_{j=0}^{\infty} \tilde{W}_j). \tag{4}$$

In this paper we pursue the solution of Helmholtz equation or Poisson equation by using the periodic scaling function $\tilde{\varphi}_{j,k}(x)$ under low resolution.

A two-dimensional stream function ψ can be expanded by the periodic scaling function as

$$\psi \approx P_{Vr}(\psi) = \sum_{j=0}^{2^r-1} \sum_{k=0}^{2^r-1} \psi_{r,j,k} \tilde{\varphi}_{r,j}(x) \tilde{\varphi}_{r,k}(y), \tag{5}$$

where $P_{Vr}(f)$ is the orthogonal projection of f on $\tilde{V}_r$ with resolution r, and the definition of the expansion coefficient is as follows:

$$\psi_{r,j,k} = \int\int \psi(x,y) \tilde{\varphi}_{r,j}(x) \tilde{\varphi}_{r,k}(y) \mathrm{d}x\mathrm{d}y. \tag{6}$$

Similarly, the derivative of the stream function can be mapped onto $\tilde{V}_r$

$$\frac{\partial^2 \psi}{\partial x^2} \approx P_{Vr}\left(\frac{\partial^2 \psi}{\partial x^2}\right) = \sum_{j=0}^{2^r-1} \sum_{k=0}^{2^r-1} CX_{r,j,k} \tilde{\varphi}_{r,j}(x) \tilde{\varphi}_{r,k}(y), \tag{7}$$

$$\frac{\partial^2 \psi}{\partial y^2} \approx P_{Vr}\left(\frac{\partial^2 \psi}{\partial y^2}\right) = \sum_{j=0}^{2^r-1} \sum_{k=0}^{2^r-1} CY_{r,j,k} \tilde{\varphi}_{r,j}(x) \tilde{\varphi}_{r,k}(y), \tag{8}$$

where their expansion coefficients are defined as

$$CX_{r,j,k} = 2^{2r} \sum_{l=2-ND}^{ND-2} \psi_{r,l+j,k} \Omega_l^2, \tag{9}$$

$$CY_{r,j,k} = 2^{2r} \sum_{m=2-ND}^{ND-2} \psi_{r,j,m+k} \Omega_m^2, \tag{10}$$

where the index ND represents the order of Daubechies wavelet, and the 2-term connection coefficient is

$$\Omega_i^2 = \int \tilde{\varphi}(x) \tilde{\varphi}_i^{(2)}(x) \mathrm{d}x, \tag{11}$$

where $\tilde{\varphi}_i^{(2)}(x)$ represents the second order derivative of the periodic scaling function $\tilde{\varphi}_i(x)$. The connection coefficients, Eq.(11), can be accurately calculated by an algebraic technique.[24]

3. Wavelet-spectrum Helmholtz equation

The non-dimensional equation of the barotropic atmosphere is a Helmholtz equation,[25] i.e.

$$(\nabla^2 - \lambda_0^2)\psi_t = -F, \tag{12}$$

$$(x,y) \in G, \quad G = \{(x,y) | (x,y) \in [0,1] \otimes [0,1]\},$$

where F is the force including the vorticity advection, topographic term, friction, and diabatic heating. $\lambda_0^2 = f_0^2 L^2/(gH)$, $\pi L = 5 \times 10^6$m, $H = 10^4$m, $f_0 = 2\Omega \sin\phi_0$, f is Coriolis parameter, $\phi_0 = 45°$N; 'g' is the gravity acceleration. The solution of Eq.(12) is the trend of the stream function. For simplicity we remove the subscript 't' from ψ_t. Thus ψ replaces ψ_t hereafter and is regarded as a stream function.

Substituting (5), (7) and (8) into (12), multiplying both sides of the resulting equation by $\varphi_{r,p}\varphi_{r,q}$, and integrating it on domain $\boldsymbol{G}$, we obtain

$$\sum_{l=2-ND}^{ND-2} \psi_{r,l+p,q} \Omega_l^2 + \sum_{m=2-ND}^{ND-2} \psi_{r,p,m+q} \Omega_m^2 - 2^{-2r} \lambda_0^2 \psi_{r,p,q} = -2^{-2r} F_r(p,q), \tag{13}$$

where

$$F_r(p,q)=\int\int F(x,y)\tilde{\varphi}_{r,p}(x)\tilde{\varphi}_{r,q}(y)\mathrm{d}x\mathrm{d}y. \quad (14)$$

Under the periodic boundary condition, Eq.(13) can be transformed into a matrix form

$$\Omega\Psi+\Psi\Omega=C, \quad (15)$$

where $C=\{-2^{-2r}F_r(p,q)\}$, Ω is a circulant matrix defined as

$$\Omega=\mathrm{circ}(\Omega_0^2-2^{-2r-1}\lambda_0^2,\Omega_1^2,\Omega_2^2,\cdots,\Omega_{ND-2}^2,0,\cdots,0,\Omega_{2-ND}^2,\cdots,\Omega_{-2}^2,\Omega_{-1}^2). \quad (16)$$

Equation (15) is a matrix equation of Helmholtz equation in terms of wavelet spectrum under the periodic boundary condition. According to the algebraic theory of matrix a circulant matrix Ω can be diagonalized by[26]

$$\Omega=\Phi D_\Omega\Phi^t, \quad (17)$$

where

$$D_\Omega=\begin{pmatrix}\lambda_1&0&\cdots&0\\0&\lambda_2&\cdot&\cdot\\\cdot&&\cdot&0\\\cdot&&&\\0&\cdots&0&\lambda_N\end{pmatrix}, \quad (18)$$

where λ_i $(i=1,\ldots,N)$ are the eigenvalues of the circulant matrix Ω. By means of FFT, the eigenvalues are[15,26]

$$\lambda_i=\Omega_0^2-2^{-2r-1}+\sum_{k=-p}^{p}(\Omega_0^2-2^{-2r-1})\cos\left(\frac{2\pi ik}{N}\right) \quad (19)$$

and the sets of eigenvectors[15] are

$$\Phi=\frac{\sqrt{2}}{\sqrt{N}}\begin{pmatrix}1/\sqrt{2}&1&\cdots&1&1/\sqrt{2}&0&\cdots&0\\1/\sqrt{2}&c_1&\cdots&c_{m-1}&-1/\sqrt{2}&s_{m-1}&\cdots&s_1\\\cdot&\cdot&\cdots&\cdot&\cdot&\cdot&\cdots&\cdot\\\cdot&\cdot&\cdots&\cdot&\cdot&\cdot&\cdots&\cdot\\\cdot&\cdot&\cdots&\cdot&\cdot&\cdot&\cdots&\cdot\\1/\sqrt{2}&c_{N-1}&\cdots&c_{(m-1)(N-1)}&-1/\sqrt{2}&s_{(m-1)(N-1)}&\cdots&s_{N-1}\end{pmatrix}, \quad (20)$$

where $N=2m$, $c_i=\cos\left(\frac{2\pi i}{N}\right)$, $s_i=\sin\left(\frac{2\pi i}{N}\right)$. In addition, there exists

$$\Phi\Phi^t=I. \quad (21)$$

Equation (15) is in the same form as that derived from the Poisson equation by Qian and Weiss except for its diagonal elements resulting from the divergence term ($\lambda_0\neq0$) of the barotropic potential vorticity equation. Combining Eqs.(17), (18), (20) and (21), we obtain the solution of the Helmholtz equation (15) in the matrix form

$$\Psi=\Phi(B_\Omega.\backslash(\Phi^tC\Phi))\Phi^t, \quad (22)$$

where

$$B_\Omega=D_\Omega\begin{pmatrix}1&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&1\end{pmatrix}+\begin{pmatrix}1&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&1\end{pmatrix}D_\Omega, \quad (23)$$

and '. \' in Eq.(23) is the matrix element-to-element division operator.

4. Matrix decomposition errors

4.1. Poisson equation

Let λ_0=0, Eq.(12) then becomes a Poisson equation. There are two important factors that determine the solution accuracy in Eq.(22): one is the orthogonality of matrix Φ and its transition Φ^t, and the other is the singularity of matrix B_Ω, which is associated with the singularity of matrix Ω. We, firstly, examine the orthogonality of matrix Φ. In fact, Eq.(21) does not strictly hold true if Eq.(20) is used as matrix Φ. The departures of $\Phi\Phi^t$ from unity matrix I reaches the order of 10^{-1}, with a maximum 0.12 and a minimum –0.12 (Table 1). The reconstruction error of matrix Ω using Eq.(17) is of the order of 10^0, equal to the order of Ω's elements (–2, 2). Such errors increase so rapidly during the time integration in the use of the barotropic model that the computation quickly stops resulting from the overflow. This may be due to the singularity of the matrix Ω, for Eqs.(17) and (21) hold true if and only if the eigenvectors are linearly independent of each other according to the linear algebraic theory.[27,28] This condition, nevertheless, cannot hold

true if matrix Ω has a very small eigenvalue. The smallest eigenvalues calculated by FFT, Jacobian and QR methods are $-.6811988\times10^{-7}$, $-.9849391\times10^{-7}$ and $-.5758179\times10^{-6}$, respectively. The latter two are bigger than the first one, which may significantly influence their orthogonal and reconstruction accuracy. Their errors and total variance are listed in Table 1.

Table 1. Reconstruction and orthogonal errors from three kinds of matrix decomposition methods (λ_0=0).

	Reconstruction error extremes		Orthogonal error extremes		Total variance	
	maximum	minimum	maximum	minimum	reconstruction	orthogonal
FFT	2.0	–2.0	0.12	–0.12	440.01	1.97
Jacobian	1.2×10^{-6}	-1.8×10^{-6}	2.5×10^{-7}	-3.0×10^{-7}	9.42×10^{-11}	1.85×10^{-12}
QR	3.5×10^{-6}	-2.5×10^{-6}	1.5×10^{-7}	-2.0×10^{-7}	3.04×10^{-10}	2.08×10^{-12}

Table 1 shows that the highest accuracy for both orthogonality and reconstruction is offered by the Jacobian method, while the lowest one is by FFT. The accuracy of the QR method is appreciably lower than that of the Jacobian method. The errors in Table 1 are obtained from the calculation in single precision. If double precision is used, their accuracy will increase further. For example, the reconstruction and orthogonal errors are much less than that in single precision for the Jacobian decomposition (Table 2).

Table 2. Reconstruction and orthogonal errors from the Jacobian decomposition method in double precision (λ_0=0).

	Reconstruction error extremes		Orthogonal error extremes		Total variance	
	maximum	minimum	maximum	minimum	reconstruction	orthogonal
Jacobian	1.0×10^{-13}	-2.0×10^{-13}	6.0×10^{-15}	-8.0×10^{-15}	0.46×10^{-24}	3.90×10^{-27}

4.2. Helmholtz equation

As for the Helmholtz equation (12), matrix $\Omega(\lambda_0 \neq 0)$ is also decomposed by the same methods as the above ones. The smallest eigenvalues of matrix Ω become much bigger than those of the Poisson equation because λ_0 decreases the singularity of matrix Ω. The smallest eigenvalues for the FFT, Jacobian and QR methods are very similar, and they are -0.9964923×10^{-4}, -0.9989738×10^{-4} and -0.9952164×10^{-4}, respectively. Their departures or errors calculated (Table 3) are qualitatively similar to those in Table 1. The Jacobian method is also the best one while the FFT method has the biggest error among them.

Table 3. Reconstruction and orthogonal errors from the three kinds of matrix decomposition methods ($\lambda_0 \neq$0).

	Reconstruction error extremes		Orthogonal error extremes		Total variance	
	maximum	minimum	maximum	minimum	reconstruction	orthogonal
FFT	2.0	–2.0	0.12	–0.12	440.02	1.97
Jacobian	1.5×10^{-6}	-1.5×10^{-6}	3.0×10^{-7}	-2.0×10^{-7}	1.16×10^{-10}	1.76×10^{-12}
QR	5.0×10^{-6}	-3.0×10^{-6}	2.0×10^{-7}	-3.5×10^{-7}	3.13×10^{-10}	2.23×10^{-12}

5. Numerical examples

5.1. Ideal solution of Helmholtz equation

The solution accuracy of Helmholtz equation with different decompositions for matrix Ω may be tested through a simple example. Under a given periodic boundary condition a formally ideal solution to stream function is expressed as

$$\psi_0(x,y) = \cos(2\pi x)\sin(2\pi y). \tag{24}$$

Substituting Eq.(24) into the following Helmholtz equation:

$$\nabla^2\psi - \lambda_0^2\psi = -F(x,y), \tag{25}$$

we have

$$F(x,y) = (8\pi^2 + \lambda_0^2)\psi_0. \tag{26}$$

Substituting Eq.(26) into Eq.(25) yields

$$\nabla^2\psi - \lambda_0^2\psi = -(8\pi^2 + \lambda_0^2)\psi_0. \tag{27}$$

5.2. Numerical solution of Poisson equation

Let λ_0=0, then Eq. (27) becomes a Poisson equation

$$\nabla^2\psi = -8\pi^2\psi_0. \tag{28}$$

Using Eq.(13), Eq.(28) can be transformed into a wavelet-spectrum form

$$\sum_{l=2-ND}^{ND-2} \psi_{r,l+p,q}\Omega_l^2 + \sum_{m=2-ND}^{ND-2} \psi_{r,p,m+q}\Omega_m^2 = -2^{-2r}8\pi^2\psi_{0r}(p,q), \tag{29}$$

where resolution r is 4, i.e. 16×16 grids in a square $\boldsymbol{G}$. The solution of Eq.(29) can be easily obtained by Eq.(22). When the FFT decomposition is used, i.e. matrix Φ is calculated with Eq.(20), the difference between the solutions of Eqs.(28) and (24) is very large (Fig.1(a)), reaching the order of 10^6. Thus, we have to give up FFT decomposition and try some others. If QR and Jacobian decompositions are employed the solution accuracy will increase greatly, with errors decreasing to the order of 10^{-3} (Fig.1(b), (c)) .

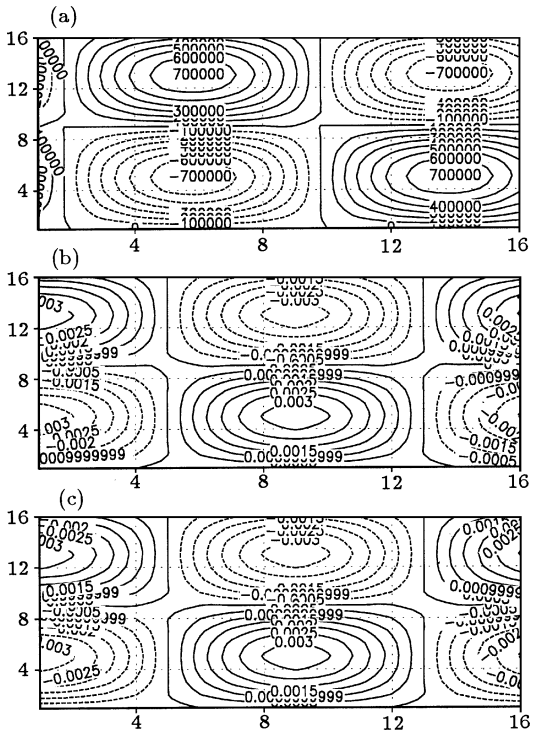


Fig.1. Differences between the ideal solution and numerical solution of the Poisson equation with three matrix decomposition methods. (a) FFT; (b) QR; (c) Jacobian.

5.3. Numerical solution of Helmholtz equation

The numerical solution of the Helmholtz equation is tested with the same ideal solution, Eq.(24). The solution with the FFT decomposition shows that the errors are of the order of 10^3 (Fig.2(a)), much smaller than that of the Poisson equation. But other two solutions (Fig.2(b),(c)) are almost the same as that of Poisson equation. This manifests that the numerical solution is very sensitive to the singularity of matrix Ω if the FFT decomposition is employed. Consequently, the FFT decomposition is not suitable for numerical calculation.

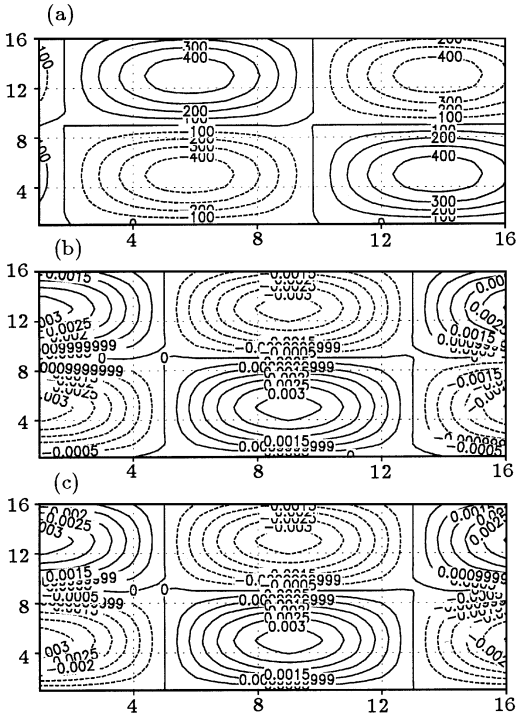


Fig.2. Differences between the ideal solution and numerical solution of the Helmholtz equation with three matrix decomposition methods. (a) FFT; (b) QR; (c) Jacobian.

5.4. Iteration solution

We also employ the iteration method[6] to solve Eq.(28) for comparison. The numerical iteration solution shows that the accuracy keeps almost the same as that of the wavelet-Galerkin solution when the threshold is within $10^{-6} - 10^{-10}$. The distribution of the errors corresponding to a threshold 10^{-10} is shown in Fig.3.

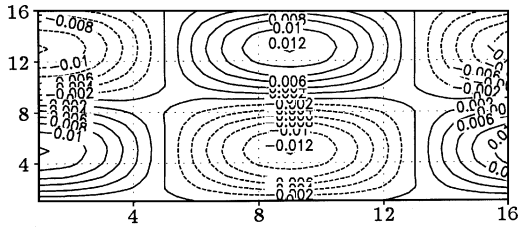


Fig.3. Solution errors of Poisson equation using iteration at a threshold 10^{-10}.

As for the Helmholtz equation, the errors calculated by the iteration method are of the order of 10^{-2} (Fig.4(a)), the same as the Poisson equation's (Fig.3(a)). When the resolution increases from 16×16 to 32×32 the order of the errors becomes much lower than the former one, being 10^{-3}, the same as that of the wavelet-Galerkin solution with a resolution 16×16. Thus, with the wavelet-Galerkin method we can obtain a more accurate solution than with the iteration method under the same resolution.

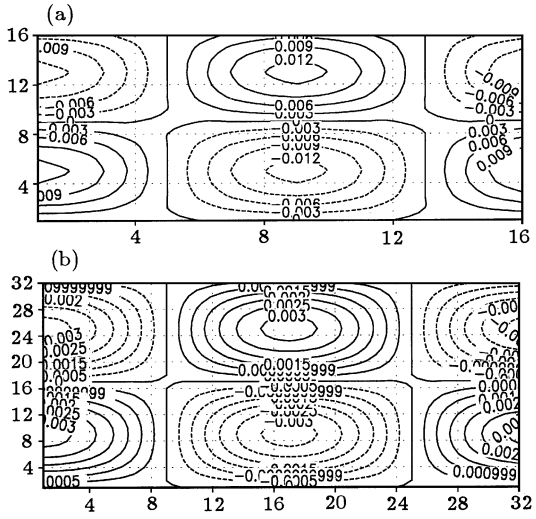


Fig.4. Solution errors of Helmholtz equation using iteration method at a threshold 10^{-10} with resolutions 16×16 (a) and 32×32 (b), respectively.

6. Influence of the order of wavelet

The order of wavelet bases has an influence on the numerical solution with wavelet-Galerkin technique. A numerical test is performed with Daubechies wavelet (DaubND, ND denotes the order of the wavelet)[23] in a resolution 32×32. There are big differences among the solutions of the Helmholtz equations calculated by the bases with different orders. The higher the order of the bases, the more accurate the solution is. The errors corresponding to DaubND (ND=6, 8, 10) are of the order of 10^{-4}, 10^{-6} and 10^{-7}, respectively (Fig.5).

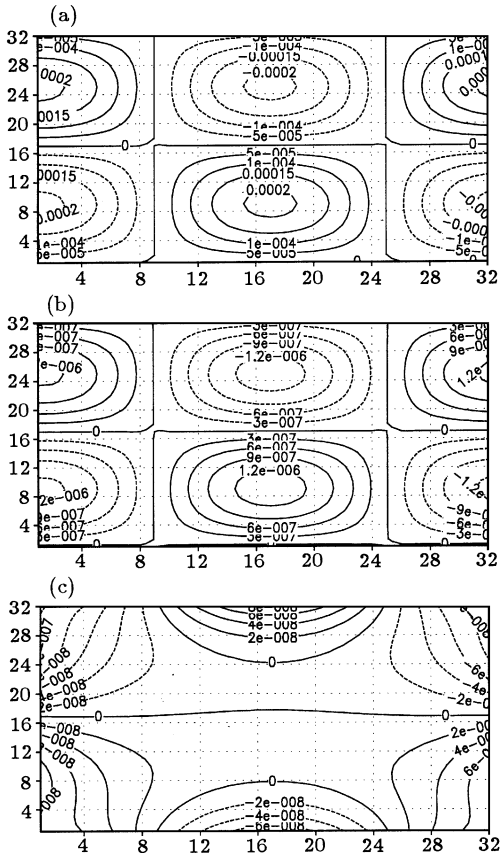


Fig.5. Differences between the ideal solution and the numerical solution of Helmholtz equation with different orders of Daubechies wavelets. (a) Daub6; (b) Daub8; (c) Daub10.

7. Conclusion

One of the difficulties in numerical solution with the wavelet spectrum model for the barotropic atmosphere is how to solve a two-dimensional Helmholtz equation with high accuracy. The equation can be transformed into an algebraic equation by wavelet transform and a series of matrix transformations. Its solution can be analytically presented by the eigenvalues and eigenvectors of matrix Ω and the right-hand side of the equation. Our numerical examination reveals that the solution accuracy is greatly related to the decomposition method of matrix Ω. If the FFT decomposition is employed under the periodic boundary condition, its reconstruction and orthogonal errors are relatively large, and the solution accuracy of the Helmholtz or Poisson equation is very low. As for QR and Jacobian decompositions, their reconstruction and orthogonal accuracy are very high, and we can obtain a numerical solution of the equation with higher

accuracy than by the use of classical iteration. In addition, the solution accuracy may further increase with the order of wavelet bases.

As is well known, the climate model, especially the paleoclimate model, has much lower resolution than the weather forecasting model due to the limited capacity of computation. By employing the wavelet-Galerkin technique the model accuracy and running speed may be raised significantly.

References

[1] Chou J F and Gao J D 1995 *Long-Term Numerical Weather Prediction* (Beijing: Meteorological Press) (in Chinese)

[2] Li J P and Chou J F 1998 *Chinese Journal of Atmospheric Sciences* **22** 443 (in Chinese)

[3] Feng G L and Dong W J 2004 *Chin. Phys.* **13** 413

[4] Feng G L and Dong W J 2003 *Acta Phys. Sin.* **52** 2347 (in Chinese)

[5] Feng G L and Dong W J and Chou J F 2004 *Acta Phys. Sin.* **53** 1 (in Chinese)

[6] Liao D X and Wang L M 1986 *The Principle and Application of Numerical Weather Prediction* (Beijing: Meteorological Press) (in Chinese)

[7] Beylkin and Keiser J M 1997 *J. Comput. Phys.* **132** 233

[8] Dai X G, Wang P and Chou J F 2003 *Chin. Sci. Bull.* **48** 2730

[9] Daubechies I 1988 *Pure Appl. Math.* 41 906

[10] Feng G L, Dai X G and Wang A H 2001 *Acta Phys. Sin.* **50** 605 (in Chinese)

[11] Mallat S 1989 *IEEE Trans. on Pattern Analysis and Machine Intelligence* **11** 674

[12] Dai X G, Wang G J and Wang P 2003 *J. Comput. Phys.* 20 245 (in Chinese)

[13] Dai X G, Wang G J and Wang P 2003 *J. Comput. Phys.* **20** 529 (in Chinese)

[14] Song F J, Zhao W J and Jutamulia S 2003 *Acta Phys. Sin.* 52 3055 (in Chinese)

[15] Qian S and Weiss J 1995 *J. Comput. Phys.* **122** 155

[16] Restrepo J M and Leaf G K 1995 *J. Comput. Phys.* **122** 118

[17] Vasilyev O V, Podladchikov Y Y and Yuen D A 2001 *Geophys J. Int.* **147** 579

[18] Vasilyev O V, Samuel P and Mihir S 1995 *J Comput. Phys.* **120** 33

[19] Vasilyev O V and Samuel P 1996 *J. Comput. Phys.* **125** 498

[20] Proskurowski W and Vassilevski P S 1994 *SIAM J. Sci. Comput.* **15** 77

[21] Proskurowski W and Vassilevski P S 1995 *SIAM J. Sci. Comput.* **16** 411

[22] Cohen A, Daubechies I and Vial P 1993 *Appl. Comput. Harmonic Anal.* **1** 54

[23] Daubechies I 1992 *Ten Lectures on Wavelets* CBMS-NSF Series in Applied Mathematics **61** (Philadelphia, PA: SIAM)

[24] Dahmen W and Micchelli C A 1993 *J. Numer. Anal.* **30** 507

[25] Charney J G and Straus D M 1980 *J. Atmos. Sci.* **37** 1157

[26] Davis P J 1979 *Circulant Matrices* (New York: Wiley)

[27] Wilkinson J H 2001 *Algebraic eigenvalue problem* (Beijing: Academic) (in Chinese)

[28] Li Q Y, Wang N C and Yi D Y 2001 *Numerical Analysis* (Beijing: Tsinghua University Press) p292–295 (in Chinese)