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Lie algebraic analysis for the nonlinear transport of intense pulsed beams in electrostatics lenses*

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The Lie algebraic method is applied to the analysis of the nonlinear transport of an intense pulsed beam in cylindrically symmetrical electrostatic lenses, and particle orbits in a six-dimensional phase space (x, p_x, y, p_y, τ, p_τ) are obtained in the second order approximation. They can also be acquired in the third or higher order approximation if needed. In the analysis, we divide the electrostatic lenses into several segments. Each segment is considered as a uniform accelerating field, and each dividing point is treated as a thin lens. The particle distribution in a three-dimensional ellipsoid is of Gaussian type.

Keywords: electrostatic lenses, intense pulsed beams, Gauss distribution, Lie map, nonlinear
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1. Introduction

The nonlinear transport of particles without action by the space charge forces in electrostatic lenses has been described in Ref.[1]. When the space charge forces are taken into account, the calculations become very complicated, especially for the unsymmetrical pulsed beams, because different particle distributions have different electrical potentials, and the particle motions depend on the space charge potentials also. Therefore, it is needed to obtain self-consistent solutions. This paper describes the use of the Lie algebraic method to analyse the intense pulsed beam transport in electrostatic lenses. The Gaussian type of beam distribution in a three-dimensional ellipsoid is used in the work.

2. Hamiltonian and Lie map

According to the principle of classical mechanics, when a charged particle moves in an axial-symmetrical electrostatic field, the Hamiltonian with time t being an independent variable is

$$H_t = (m_0^2c^4 + p_x^2c^2 + p_y^2c^2 + p_z^2c^2)^{1/2} + q\psi, \qquad (1)$$

where, m_0 and q are the rest mass and the charge of the particle respectively; p_x, p_y and p_z are the particle momenta respectively in x, y and z directions; ψ is the electrostatic potential, which consists of two parts: the external potential ψ_e and the space charge potential ψ_s, and expressed as follows:

$$\begin{aligned}\psi_e =&\varphi(z) - \frac{1}{4}\varphi''(z)(x^2+y^2)\\ &+\frac{1}{64}\varphi^{(4)}(z)(x^4+2x^2y^2+y^4)\\ &-\frac{1}{2304}\varphi^{(6)}(z)(x^6+3x^4y^2+3x^2y^4+y^6)+\cdots,\end{aligned} \qquad (2)$$

$$\begin{aligned}\psi_s=&\frac{IT_{\rm rf}}{8\pi^{3/2}\varepsilon_0}\\ &\times\int_0^\infty \frac{\exp\left(-\dfrac{x^2}{2X^2+\xi}-\dfrac{x^2}{2Y^2+\xi}-\dfrac{x^2}{2Z^2\gamma^2+\xi}\right)}{\sqrt{(2X^2+\xi)(2Y^2+\xi)(2Z^2\gamma^2+\xi)}}{\rm d}\xi,\end{aligned} \qquad (3)$$

where I is the average current of the pulsed beams; $T_{\rm rf}$ is the pulse period; X, Y and Z are the three dimensions of beam pulses; φ is the axial potential distribution of the external field.

Now, introducing a new variable p_t

$$p_t = -H_t(x, p_x, y, p_y, z, p_z), \qquad (4)$$

and solving p_z from Eq.(4), one obtains

$$p_z = -K(t, x, p_x, y, p_y, z, p_z), \qquad (5)$$

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where

$$K = -p_z = -[(p_t+q\psi)^2/c^2 - p_x^2 - p_y^2 - m_0^2c^2]^{1/2}. \quad (6)$$

Equation (6) is the Hamiltonian with z being an independent variable and (x, p_x, y, p_y, t, p_t) being canonical variables. We perform the following canonical transformation:

$$\begin{aligned} &x = x,\ p_x = p_x,\ y = y,\ p_y = p_y,\\ &\tau = t - z/v_0,\ p_\tau = p_t - p_t^0, \end{aligned} \quad (7)$$

where v_0 is the velocity of the reference particle, p_t^0 is the value of p_t for the reference particle. We can see from Eq.(7) that the coordinates of reference particle in the phase space $\boldsymbol{\xi} = (x, p_x, y, p_y, \tau, p_\tau)$ are always zero, i.e. the reference particle is always at a fixed point. Taking the value of p_t for the reference particle, one obtains

$$p_\tau^0 = -H_t|_{\text{reference orbit}} = -m_0\gamma_0c^2 - q\varphi, \quad (8)$$

where $\gamma_0 = [1-(v_0/c)^2]^{-1/2}$.

The new Hamiltonian corresponding to the canonical transformation (7) is

$$\begin{aligned} H = &-[(p_\tau + q(\psi_e - \psi_s) - m_0\gamma_0c^2)^2/c^2\\ &- p_x^2 - p_y^2 - (m_0c)^2]^{1/2}\\ &- (p_\tau - m_0\lambda_0c^2)/v_0. \end{aligned} \quad (9)$$

Expanding the Hamiltonian into power series about the reference orbit, we have

$$H = \sum_{n=0}^{\infty} H_n, \quad (10)$$

where

$$\begin{aligned} H_0 =& \frac{m_0c}{\beta_0\gamma_0} + \frac{q\varphi}{\beta_0c},\\ H_1 =& 0,\\ H_2 =& -x^2\frac{q\varphi'' + 4Q\mu_x}{4v_0} - y^2\frac{q\varphi'' + 4Q\mu_y}{4v_0}\\ &+ \frac{(p_x^2+p_y^2)}{2p_0} - \tau^2Q\mu_zv_0\gamma_0^2 - p_\tau^2\frac{1}{2p_0v_0^2\gamma_0^2},\\ H_3 =& -x^2p_\tau\frac{q\varphi'' + 4Q\mu_x}{4p_0v_0^2\gamma_0^2} - y^2p_\tau\frac{q\varphi'' + 4Q\mu_y}{4p_0v_0^2\gamma_0^2}\\ &+ (p_x^2+p_y^2)p_\tau\frac{1}{2p_0^2v_0} - \tau^2p_\tau\frac{Q\mu_z}{p_0} + p_\tau^3\frac{1}{2p_0^2v_0^3\gamma_0^2},\\ &\cdots\cdots \end{aligned} \quad (11)$$

and

$$Q = \frac{qIT_{\text{rf}}}{8\pi^{3/2}\varepsilon_0}, \quad (12)$$

$$\begin{aligned} \mu_x =& \int_0^\infty \frac{\mathrm{d}\xi}{(2X^2+\xi)\sqrt{(2X^2+\xi)(2Y^2+\xi)(2Z^2\gamma^2+\xi)}},\\ \mu_y =& \int_0^\infty \frac{\mathrm{d}\xi}{(2Y^2+\xi)\sqrt{(2X^2+\xi)(2Y^2+\xi)(2Z^2\gamma^2+\xi)}},\\ \mu_z =& \int_0^\infty \frac{\mathrm{d}\xi}{(2Z^2\gamma^2+\xi)\sqrt{(2X^2+\xi)(2Y^2+\xi)(2Z^2\gamma^2+\xi)}}. \end{aligned} \quad (13)$$

3. Lie map

The Lie map associated with the function H is[2]

$$\boldsymbol{M} = \exp\left[-:\int_{z_0}^{z_f} H\mathrm{d}z:\right] = \cdots\boldsymbol{M}_4\boldsymbol{M}_3\boldsymbol{M}_2, \quad (14)$$

where

$$\begin{aligned} \boldsymbol{M}_2 &= \exp(:f_2:),\\ \boldsymbol{M}_3 &= \exp(:f_3:),\\ \boldsymbol{M}_4 &= \exp(:f_4:)\ldots, \end{aligned} \quad (15)$$

and

$$f_2 = -\int_{z_0}^{z_f} H_2\mathrm{d}z,\ f_3 = -\int_{z_0}^{z_f} h_3^{int}\mathrm{d}z, \ldots, \quad (16)$$

$$h_n^{\text{int}}(z) = \boldsymbol{M}_2H_n. \quad (17)$$

Applying the Lie map $\boldsymbol{M}$ onto the canonical variable $\boldsymbol{\xi}$, and denoting the approximation order with the subscript 'i', we have

$$\begin{aligned} \boldsymbol{\xi}_1 &= \exp(:f_2:)\boldsymbol{\xi}, \quad \text{first order}\\ \boldsymbol{\xi}_2 &=: f_3 : \boldsymbol{\xi}_1, \quad \text{second order } \ldots \end{aligned} \quad (18)$$

In the above equations, the symbol : : stands for the Poisson bracket operation. For example, for two arbitrary functions f and g, $:f: g$ means $[f, g]$. The power of $:f:$ is defined as the repetition operation of Poisson bracket. For example, $:f:^2$ means

$$:f:^2 g = [f,[f,g]].$$

The zero power of :f: is referred to as an identical operation:

$$:f:^0 g = g.$$

4. First order approximation

As mentioned in the above introduction, in the analysis, the electrostatic lens is divided into several small segments. Each segment is considered as a uniform accelerating field, and each dividing point as a

thin focusing (defocusing) lens. Applying the Lie map onto the small uniform accelerating fields and the dividing points step by step, we can obtain nonlinear particle orbits. The details are shown as follows.

4.1. In the open interval $(z_i,\ z_{i+1})$

In the interval $(z_i,\ z_{i+1})$, $i = 1, 2, \ldots n$, let $l_i = z_{i+1} - z_i$, and '$i+1$' be the final value of the ith interval, 'i' the initial value, '0' the value related to the reference particle. The electrical field is approximately uniform, i.e. $\phi'' \approx 0$. By using the first expressions of Eq.(15) , Eq.(16) and Eq.(18) one obtains

$$\begin{bmatrix} x_1 \\ p_{x1} \\ y_1 \\ p_{y1} \\ \tau_1 \\ p_{\tau 1} \end{bmatrix} = \begin{bmatrix} \cosh(k_x l_{ie}) & \frac{\sinh(k_x l_{ie})}{p_{0i}k_x} & 0 & 0 & 0 & 0 \\ p_{0i}k_x \sinh(k_x l_{ie}) & \cosh(k_x l_{ie}) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh(k_y l_{ie}) & \frac{\sinh(k_y l_{ie})}{p_{0i}k_y} & 0 & 0 \\ 0 & 0 & p_{0i}k_y \sinh(k_y l_{ie}) & \cosh(k_y l_{ie}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \cosh(k_z l_{ie}) & \frac{\sinh(k_z l_{ie})}{sk_z} \\ 0 & 0 & 0 & 0 & sk_z \sinh(k_z l_{ie}) & \cosh(k_z l_{ie}) \end{bmatrix} \begin{bmatrix} x \\ p_x \\ 1 \\ p_y \\ \tau \\ p_1 \end{bmatrix}, \tag{19}$$

where

$$k_x^2 = \frac{2Q\mu_x}{p_{0i}v_{0i}},$$

$$k_y^2 = \frac{2Q\mu_y}{p_{0i}v_{0i}},$$

$$k_z^2 = \frac{2Q\mu_z}{p_{0i}v_{0i}}\frac{\eta_i^2 + \eta_i + 1}{3\eta_i},$$

$$s = p_{0i}v_{0i}^2\tilde{\gamma}_{0i}^2\eta_i,$$

$l_{ie} = \dfrac{2l_i}{\eta_i + 1}$, ($l_{ie}$ is an equivalent length to the segment length l_i),

$\tilde{\gamma}_{0i} = \dfrac{c}{v_{0i}}\dfrac{\arcsin(v_{0i}\eta_i^2/c) - \arcsin(v_{0i}/c)}{\eta_i^2 - 1}$, ($\tilde{\gamma}_{0i}$ is an average over the accelerating segment l_i),

$$\eta_i^2 = V_{i+1}/V_i. \tag{20}$$

V_i and V_{i+1} are the energy gauged potentials at z_i and z_{i+1} respectively, v_{0i} the voltage of reference particle at z_i. In Eq.(19), if we expand each matrix element into power series, the first terms of each series will be the matrix elements without space charge forces.

4.2. At the dividing point z_i

Each dividing point z_i $(i = 1, \ldots, n)$ is considered as a thin lens. The first order orbits and the values at z_i are denoted respectively with the subscript '1' and the subscript 'i'. Having performed some calculations, we obtain

$$\begin{bmatrix} x_1 \\ p_{x1} \\ y_1 \\ p_{y1} \\ \tau_1 \\ p_{\tau 1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{p_{0i}(\eta_i^2\eta_{i-1}^2 - 2\eta_{i-1}^2 + 1)}{4l_i\eta_{i-1}^2} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{p_{0i}(\eta_i^2\eta_{i-1}^2 - 2\eta_{i-1}^2 + 1)}{4l_i\eta_{i-1}^2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ p_x \\ y \\ p_y \\ \tau \\ p_\tau \end{bmatrix}, \tag{21}$$

where p_{0i} is the momentum of the reference particle at z_i.

5. Second order approximation

5.1. In the interval $(z_i,\ z_{i+1})$

Denoting the second order terms of the orbit with the subscript '2' and using the second expressions of Eqs.(15), (16) and (18), we have

$$
\begin{aligned}
x_2 =& \frac{x\tau s(1+\eta_i)}{2m_0^2c^4\beta_{0i}^4\tilde{\gamma}_{0i}^4k_x(4k_x^2-k_z^2)}\{[-2c^2m_0\beta_{0i}^2\tilde{\gamma}_{0i}^3k_x^4-2(2k_x^2-k_z^2)Q\mu_x \\
&+k_x^2(m_0c^2\beta_{0i}^2\tilde{\gamma}_{0i}^3(2k_x^2-k_z^2)+4Q\mu_x)\cosh(k_zl_{i\mathrm{e}})]\sinh(k_xl_{i\mathrm{e}}) \\
&+k_xk_z(m_0c^2\beta_{0i}^2\tilde{\gamma}_{0i}^3k_x^2-2Q\mu_x)\cosh(k_xl_{i\mathrm{e}})\sinh(k_zl_{i\mathrm{e}})\} \\
&+\frac{xp_\tau(1+\eta_i)}{2m_0^2c^4\beta_{0i}^4\tilde{\gamma}_{0i}^4k_z(4k_x^2-k_z^2)}\{k_z(m_0c^2\beta_{0i}^2\tilde{\gamma}_{0i}^3k_x^2-2Q\mu_x)(\cosh(k_zl_{i\mathrm{e}})-1)\cosh(k_xl_{i\mathrm{e}}) \\
&+k_x[m_0c^2\beta_{0i}^3\tilde{\gamma}_{0i}^3(2k_x^2-k_z^2)+4Q\mu_x]\sinh(k_xl_{i\mathrm{e}})\sinh(k_zl_{i\mathrm{e}})\} \\
&+\frac{p_x\tau s(1+\eta_i)}{2m_0^3c^5\beta_{0i}^5\tilde{\gamma}_{0i}^5k_x(4k_x^2-k_z^2)}\{k_x(m_0c^2\beta_{0i}^2\tilde{\gamma}_{0i}^3(2k_x^2-k_z^2)+4Q\mu_x)(\cosh(k_zl_{i\mathrm{e}})-1)\cosh(k_xl_{i\mathrm{e}}) \\
&+k_z(m_0c^2\beta_{0i}^2\tilde{\gamma}_{0i}^3k_x^2-2Q\mu_x)\sinh(k_xl_{i\mathrm{e}})\sinh(k_zl_{i\mathrm{e}})\} \\
&+\frac{p_xp_\tau(1+\eta_i)}{2m_0^3c^5\beta_{0i}^5\tilde{\gamma}_{0i}^5k_xk_z(4k_x^2-k_z^2)}\{k_z(m_0c^2\beta_{0i}^2\tilde{\gamma}_{0i}^3k_x^2-2Q\mu_x)(\cosh(k_zl_{i\mathrm{e}})-1)\sinh(k_xl_{i\mathrm{e}}) \\
&+k_x[(m_0c^2\beta_{0i}^2\tilde{\gamma}_{0i}^3(2k_x^2-k_z^2)+4Q\mu_x)\cosh(k_xl_{i\mathrm{e}})\sinh(k_zl_{i\mathrm{e}})]\}. \qquad (22)
\end{aligned}
$$

Due to the space limitation in this paper, the second terms of p_{x2}, y_2, p_{y2}, τ_2, and $p_{\tau 2}$ are not shown here.

5.2. At the dividing point z_i

The second terms of the orbit at the dividing point z_i are

$$
\begin{aligned}
&x_2=0,\quad p_{x2}=xp_\tau\frac{\eta_i^2\eta_{i-1}^2-2\eta_{i-1}^2+1}{4l_iv_{0i}\gamma_{0i}^2\eta_{i-1}^2}, \\
&y_2=0,\quad p_{y2}=yp_\tau\frac{\eta_i^2\eta_{i-1}^2-2\eta_{i-1}^2+1}{4l_iv_{0i}\gamma_{0i}^2\eta_{i-1}^2}, \\
&\tau_2=0,\quad p_{\tau 2}=y_0, \qquad (23)
\end{aligned}
$$

where γ_{0i} is the value of γ at z_i.

6. Discussion

It is a complicated procedure to calculate the nonlinear effect resulting from the space charge, for several reasons as indicated below. (1) The space charge potentials are related to the particle trajectory states, and the particle trajectories depend on the potentials of space charge and the external electromagnetic fields. In order to obtain self-consistent solutions, iterative calculations are needed. (2) Different particle distributions have different electrical potentials. Therefore, one cannot calculate particle trajectories for different particle distributions with one set of formulae. (3) In order to simulate the particle trajectories accurately, it is necessary to separate the lenses into small segments, and apply the obtained formulae to the intervals and dividing points step by step.

In our analysis, a 6D phase space (x, p_x, y, p_y, τ, p_τ) is used. In the practical calculations, however, one prefers to use the phase space (x, x', y, y', Δt, ΔE). The transformation from the former to the latter can be performed by using the definition (7).

References

[1] Lü J Q 1998 *Nucl. Instrum. Methods* A **416** 229

[2] Dragt A J and Finn J M 1976 *J. Math. Phys.* **17** 2215