Vol 13 No 10, October 2004
1009-1963/2004/13(10)/1644-05

Chinese Physics

© 2004 Chin. Phys. Soc.
and IOP Publishing Ltd

Preparation of arbitrary n-particle d-dimensional superposition states using only single qubit operations and CNOT gates*

Wang Yan-Hui(王艳辉) and Fang Mao-Fa(方卯发)†

College of Physics and Information Science, Hunan Normal University, Changsha 410081, *China*

(Received 20 November 2003; revised manuscript received 1 June 2004)

In this article, using only single qubit operation and a CNOT gate, we propose a scheme for creating arbitrary n-particle d-dimensional superposition states including entangled states and give the relevant circuits for realizing this scheme.

Keywords: single qubit operation, CNOT gates, C-u operation, two-level unitary matrix
PACC: 0365, 0758, 4250

1. Introduction

No one can deny the fact that all kinds of states play an important role in the quantum realm. State preparation is such a prerequisite procedure for studying fundamental aspects of quantum computation,[1,2] quantum information,[3,4] quantum lithography,[5,6] etc., that it has been widely studied. References [7] and [8] presented schemes for preparation of entangled states in ion traps. References [9] and [10] introduced methods to prepare remote states. A strategy for optically created entangled squeezed vacuum states was proposed in Ref.[11]. Reference [12] illuminates how to produce a two-mode field entangled states via Raman interaction. Reference [13] demonstrated how to generate an arbitrary quantum state via atom feedback. From the above, we see that much attention is focused on the preparation of entangled states while few papers are concerned with the generation of an arbitrary n-particle d-dimensional superposition states which are used in quantum computation and quantum information. In this paper, we give an explicit strategy to generate such states. On the other hand, quantum circuits have been recognized as a necessary infrastructure to support secure quantum communication, quantum cryptography, and precise measurement;[14] they also have been studied in quantum teleportation,[15] quantum coding,[16] concentrated entanglement,[17] quantum algorithm,[18] etc. In this paper, we also give the circuit for realizing this strategy.

We know that the state evolution of a closed quantum system is denoted by a unitary operation U which can be realized by single qubit operations and control-not (CNOT) operations.[1,18] Therefore, it is reasonable to perform a transformation of states with single qubit operation and CNOT gates. In this paper, we introduce a practicable scheme for the preparation of an arbitrary n-particle d-dimensional superposition states including entangled states, from relevant classical states by using single qubit operations and CNOT gates only, and present their relevant circuits.

2. The number needed of two-level unitary operations to prepare n-particle d-dimensional superposition states

Firstly, we show that the required number of two-level unitary gates to prepare n-particle d-dimensional

*Project supported by the National Nature Science Foundation of China (Grant No 10374025), by the Natural Science Foundation of Hunan Province, China (Grant No 01JJY3030) and by the Scientific Research Fund of Hunan Provincial Education Department, China (Grant No 02C2 05).

†Corresponding author Email: mffang@hunnu.edu.cn

superposition states from n-particle classical states is $d-1$. Two-level unitary matrices are defined to act non-trivially only on two-or-fewer vector components, therefore they have the form in which all the diagonal entries are 1 except for the ith and jth ones; meanwhile all the nondiagonal entries are 0 except for the jth element in the ith row and the ith element in the jth row. Suppose that we want to prepare an arbitrary n-particle 2-dimensional state $|\Psi\rangle$ which can be written in an $n \times 1$ matrix with only the ith and jth entries not 0, say, a and b respectively. Then let

$$|\Psi\rangle = a|\Psi_1\rangle + b|\Psi_2\rangle, \tag{1}$$

where $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are n-particle classical states whose nonzero elements are the ith and jth elements respectively. There exists an n-particle two-level unitary matrix U:

$$U = \begin{bmatrix} I_{i-1} & 0 & 0 & 0 & 0 \\ 0 & a & 0 & c & 0 \\ 0 & 0 & I_{j-i-1} & 0 & 0 \\ 0 & b & 0 & d & 0 \\ 0 & 0 & 0 & 0 & I_{2^n-j-1} \end{bmatrix}, \tag{2}$$

where a, b, c and d are some certain complex numbers to make U unitary. As a result, its non-trivial 2×2 submatrix u:

$$u = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \tag{3}$$

is a unitary matrix too, which is regarded as a 'single qubit operator' on a single qubit. The actions of U on $|\Psi_1\rangle$, $|\Psi_2\rangle$ are as follows:

$$U|\Psi_1\rangle = a|\Psi_1\rangle + b|\Psi_2\rangle, \tag{4}$$

$$U|\Psi_2\rangle = c|\Psi_1\rangle + d|\Psi_2\rangle, \tag{5}$$

from the above, we see that a two-level unitary operation U changes an n-particle classical state into an n-particle 2-dimensional superposition state. If we want an n-particle d-dimensional superposition state from an n-particle classical state, how many two-level unitary operations U are required? Figure 1 shows the total number needed of two-level unitary operations U.

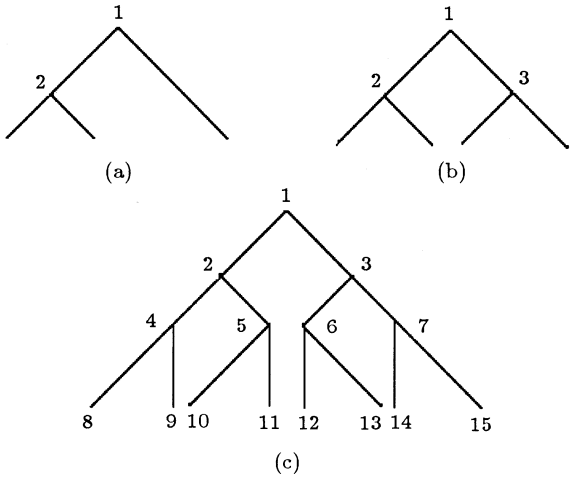


Fig.1. A graph to show the needed number of two-level unitary operations.

In this figure, each crossing point stands for a two-level operation, whereas each line stretching out of every such point presents an n-particle classical state. It can be seen that after some two-level unitary operations have acted on the initial state, the output state is an s-dimensional state. Then if we apply again an appropriate two-level unitary operations to one of the s n-particle-classical-state components, we can obtain an n-particle $(s+1)$-dimensional superposition state. So by repeating this procedure, we can generate an n-particle d-dimensional superposition state by applying $d-1$ proper different two-level unitary operations to the n-particle classical state. For example, the state to be prepared is

$$|\Psi\rangle = a|00\rangle + b|10\rangle + c|01\rangle + d|11\rangle. \tag{6}$$

Using three two-level unitary operations, namely U_1, U_2, U_3, with the following action:

$$U_1|00\rangle = a|00\rangle + d'|11\rangle, \tag{7}$$

$$U_2|11\rangle = b'|10\rangle + d''|11\rangle, \tag{8}$$

$$U_3|10\rangle = b''|10\rangle + c'|01\rangle, \tag{9}$$

where the coefficients b', b'', c', d', d'' must satisfy: $b'b''d' = b$, $b'c'd' = c$, $d'd'' = d$. We obtain $|\Psi\rangle$ in Eq.(6) as expected.

3. The decomposition of a two-level unitary matrix

After showing that $d-1$ two-level unitary operations are enough for preparation of the n-particle

d-dimensional superposition states, we discuss the decomposition of a two-level unitary matrix into a single qubit operation and CNOT gates.

As we know that U nontrivially acts on an n-particle state, say $|\Psi\rangle$, and the output state is the n-particle 2-dimensional state which can be regarded as a superposition of two classical states: $|s\rangle$ and $|t\rangle$ where $s = s_1 \cdots s_n$ and $t = t_1 \cdots t_n$ are the binary expansions for s and t. For simplicity, we take the initial state to be $|s\rangle$ or $|t\rangle$, then U has the same form as that in Eq.(2). We discuss the decomposition of U in the following two cases.

Case 1 In the computational basis, $|s\rangle$ and $|t\rangle$ only differ in one qubit. In this case, we only need to perform a 'single qubit operation' u just on this qubit and obtain the target state $|\Psi\rangle$, but no CNOT gate is needed. A simple instance is

$$\begin{aligned} U|000000\rangle =& I|000\rangle \otimes u|0\rangle \otimes I|00\rangle \\ =& a|000000\rangle + b|000100\rangle. \end{aligned} \tag{10}$$

Case 2 When $|s\rangle$ and $|t\rangle$ differ in two or larger than two qubits, say m qubits. Let g_1 through g_m be the location where $|s\rangle$ and $|t\rangle$ are different. Here $m \leq n-2$ is satisfied. If we select the state with g_1 qubit being '0' to be the initial state and perform a u operation on g_1 qubit and obtain an n-particle 2-dimensional state. Then we perform NOT operations on g_2 through g_m qubits, which are conditional on the g_1 qubit being in the state $|1\rangle$, and obtain the ideal final state $a|s\rangle + b|t\rangle$. In this case, one single qubit operation and $m-1$ CNOT gates are needed.

A simple example illuminates the procedure further. Suppose we wish to prepare a state $a|s\rangle + b|t\rangle$ with $|s\rangle = |00101101001\rangle$, $|t\rangle = |01110100101\rangle$. The state $|s\rangle$ is chosen to be the initial state, the two-level unitary operation U acts on it and gives:

$$U|00101101001\rangle = a|00101101001\rangle + b|01110100101\rangle. \tag{11}$$

To implement U, the first step is to apply a single qubit operation u to the second qubit of $|s\rangle$. Next we apply one CNOT gate on every one of the fourth, fifth, eighth and ninth qubits; each is conditional on the second qubit being $|1\rangle$ and has the target state $a|s\rangle + b|t\rangle$ as a result.

But what we plan to prepare is not a 2-dimensional state. How many operations needed to prepare an n-particle d-dimensional superposition state? Note that there are more than two components in the superposition state now, so in order to protect the single qubit operation and CNOT gates from being disturbed by other components, we should change the single qubit operation into control-u (C-u) operation which is conditional on other qubits being correspondingly valued; meanwhile we condition some of the CNOT gates, which were conditional only on one qubit earlier, on all the other $n-2$ qubits being correspondingly valued. For instance, if we hope to change $a|001\rangle + b|101\rangle$ into $a|001\rangle + b|111\rangle$, then we will carry out a CNOT operation on the second qubit which is conditional only on the first qubit being valued; but if we want to change $a|001\rangle + b|101\rangle + c|110\rangle$ into $a|001\rangle + b|111\rangle + c|110\rangle$, a CNOT operation on the second qubit must be conditioned on the first qubit being $|1\rangle$ and the third being $|1\rangle$ too, in order that the CNOT operation acts not on $|110\rangle$ but on $|101\rangle$. Therefore, a two-level unitary operation can be replaced by a C-u operation and $m-1$ CNOT gates which are conditional on a certain number of qubits. Each of these controlled operations can be realized using $O(n)$ single qubit and CNOT gates; meanwhile the C-u operation also requires $O(n)$ gates. Thus in order to implement a two-level unitary operation U on an n-particle multiple dimensional state, $O(n^2)$ single qubits and CNOT gates are required.[18] Combining this result with the previous discussion, we see that the number $(d-1) \times O(n^2)$ of single qubit operations and CNOT gates, which is much smaller than $O(n^2 * d^2)$ implied in Ref.[18], is sufficient to prepare an n-particle d-dimensional superposition state from an n-particle classical state. Our method is different from the one shown in Ref.[19] where rotation are used.

4. The scheme and its circuit realization

Now, we illustrate the scheme more explicitly. Suppose we want to prepare an arbitrary state, say

$$|\Psi\rangle = a|\Psi_8\rangle + b|\Psi_9\rangle + c|\Psi_{10}\rangle + d|\Psi_{11}\rangle + e|\Psi_6\rangle + f|\Psi_7\rangle; \tag{12}$$

each component in $|\Psi\rangle$ is denoted by a radiation from the corresponding point in Fig.1. Then five two-level unitary operators are enough. Firstly we select one of these components as the initial state, say $|\Psi_8\rangle$. Then we apply a two-level unitary operator U_1 to it so that $a'|\Psi_8\rangle + b'|\Psi_3\rangle$ is obtained. Next we apply two two-level unitary operators U_2 and U_3 to $|\Psi_8\rangle$ and $|\Psi_3\rangle$ respectively, and we obtain

$a''|\Psi_8\rangle + c'|\Psi_5\rangle + e'|\Psi_6\rangle + f'|\Psi_3\rangle$. After U_4 and U_5 have been applied to $|\Psi_8\rangle$ and $|\Psi_5\rangle$ respectively, we obtain $a|\Psi_8\rangle + b|\Psi_9\rangle + c|\Psi_{10}\rangle + d|\Psi_{11}\rangle + e|\Psi_6\rangle + f|\Psi_3\rangle$. If we appropriately select U_1 so as to let $U_1|\Psi_8\rangle = a'|\Psi_8\rangle + b'|\Psi_7\rangle$, as a result $|\Psi\rangle$ is prepared.

We show two examples here. One is to generate the 3-particle entangled Greenberger–Horne–Zeilinger (GHZ) state: $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. According to Case 2, we choose $|000\rangle$ as the initial state and perform a u operation on the first qubit followed by two CNOT gates on the second and third qubits respectively, conditionally on the first qubit being $|1\rangle$, and complete the preparation. The procedure can be expressed with a circuit shown in Fig.2.

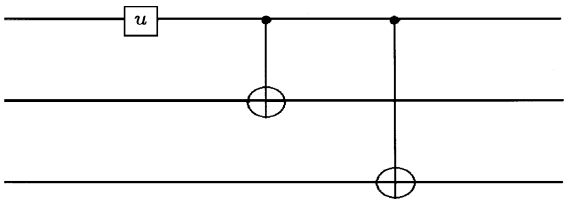


Fig.2. A circuit to prepare GHZ state.

The second example is to create an arbitrary n-particle d-dimensional superposition state. Without losing generality, suppose we wish to prepare a 3-particle 6-dimensional state: $a|000\rangle + b|100\rangle + c|110\rangle + d|111\rangle + e|011\rangle + f|010\rangle$.

We use the circuit in Fig.3 to demonstrate our general procedure to generate a superposition state using single qubit operations and CNOT gates only.

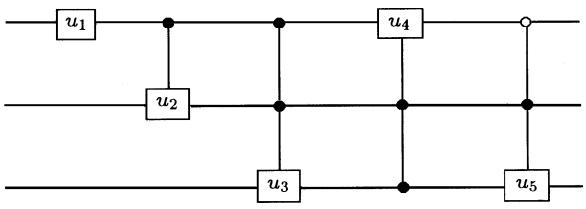


Fig.3. A circuit to prepare $a|000\rangle + b|100\rangle + c|110\rangle + d|111\rangle + e|011\rangle + f|010\rangle$ state.

We see that u_1 changes $|000\rangle$ into $a|000\rangle + b'|100\rangle$ which becomes $a|000\rangle + b|100\rangle + c'|110\rangle$ after the action of u_2. The state after operation u_3 is $a|000\rangle + b|100\rangle + c|110\rangle + d'|111\rangle$. After having performed u_4 on this state, we have $a|000\rangle + b|100\rangle + c|110\rangle + d|111\rangle + e'|011\rangle$. Finally, after the action of u_5 we obtain the state $a|000\rangle + b|100\rangle + c|110\rangle + d|111\rangle + e|011\rangle + f|010\rangle$ just as the state desired! It is worth noting that C-u and CNOT operations can be decomposed into individual qubit operations and CNOT gates, so we can regard the above circuit to consist of individual qubit operations and CNOT gates only. Furthermore, we can simplify the circuit by using some identical transformation, and the total number of the individual qubit operations and CNOT gates decreases greatly. In this way the circuit in Fig.3 is simplified into that in Fig.4.

It is clear that only 3 single qubit operations and two CNOT gates are needed!

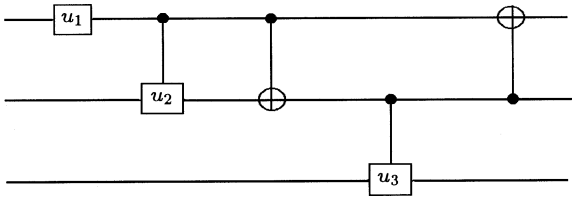


Fig.4. The simplified form of the circuit in Fig.3.

5. Conclusion

As a conclusion, in this paper, we have proposed a different practicable strategy from that demonstrated in Refs.[15] and [19] to prepare an n-particle d-dimensional superposition state by using single qubit operations and CNOT gates. Our results show that less than $(d-1) \times O(n^2)$ single qubit operations and CNOT gates are sufficient to prepare an n-particle d-dimensional superposition state from an n-particle classical state. The number of the total single qubit operations and CNOT gates needed in our protocol is much smaller than that in Ref.[18]. As a consequence, we can save systematic resources greatly and diminish noises generated in information processes which are of great use to quantum computation and quantum information.

References

[1] Wu Y 2001 *Phys. Rev.* A **63** 052303

[2] Wheeler J A and Zurek W H 1983 *Quantum Theory and Measurement* (Princeton, NJ: Princeton University Press)

[3] Caves C M and Drummond P D 1994 *Rev. Mod. Phys.* **66** 481

[4] Gong S Q *et al* 2000 *Chin. Phys.* **9** 0094

[5] Kok P, Boto A N, Abrams D S, Williams C P, Braunstein S L and Dowling J P 2001 *Phys. Rev.* A **63** 063407
Bjrk G and Snchez-Soto L L 2001 *Phys. Rev.* A **64** 013811

[6] Wang Y H and Kuang L M 2003 *J. Opt. B: Quantum Semiclass. Opt.* **5** 405
[7] Zheng S B 2001 *Phys. Rev.* A **63** 015801
[8] Kneer B and law C K 1998 *Phys. Rev.* A **57** 2096
[9] Berry D W and Sanders B C 2003 *Phys. Lett.* **90** 057901
[10] Devetak I and Berger T 2001 *Phys. Rev. Lett.* **87** 197901
[11] Zhou L and Kuang L M 2002 *Phys. Lett.* A **302** 273
[12] Fang M F, Zhou Q P and Zhou P 1999 *Acta Phys. Sin.* **48** 401 (in Chinese)
[13] Feng X L *et al* 1999 *Acta Phys. Sin.* **48** 826 (in Chinese)
[14] Shende V V, Prasad A K, Markov I L and Hayes J P 2002 e-print quant-ph/0207001
[15] Gorbachov V N and Trubilko A I 1999 e-print quant-/9906110
[16] Mermin N D 2002 e-print quant-ph/0204107
[17] Kaye P and Mosca M 2001 e-print quant-ph/0101009
[18] Nielsen M A and Chang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[19] Long G L and Yang S 2001 *Phys. Rev.* A **64** 014303