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The Lie symmetrical non-Noether conserved quantity of holonomic Hamiltonian system

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In this paper, we study the Lie symmetrical non-Noether conserved quantity of a holonomic Hamiltonian system under the general infinitesimal transformations of groups. Firstly, we establish the determining equations of Lie symmetry of the system. Secondly, the Lie symmetrical non-Noether conserved quantity of the system is deduced. Finally, an example is given to illustrate the application of the result.

Keywords: Hamiltonian system, Lie symmetry, non-Noether conserved quantity
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1. Introduction

Conservation laws play an important role in theoretical physics and analytical mechanics. To study the conserved quantity of a mechanical system by using the symmetry theory is a recent development direction in mathematical physics. The symmetry theory mainly covers Noether symmetry, Lie symmetry, and Mei symmetry (or Mei's form invariance). Noether symmetry is an invariance of the Hamiltonian's action under the infinitesimal transformations of groups.[1] Lie symmetry is an invariance of differential equations under the infinitesimal transformations of groups.[2] Mei symmetry means that the form of the equations of motion remains invariant under the infinitesimal transformations of groups.[3,4] In 1979, Lutzky applied Lie's theory to the differential equations of motion for a mechanical system, and studied Lie symmetry and conserved quantity of a mechanical system.[2] In the last 20 years, the study about the theory of Lie symmetry has acquired a series of important results.[5−19] Traditionally, finding a conserved quantity by using Lie symmetry has utilized the Noether identity. In 1992, a direct method of finding conserved quantity by using Lie symmetry was given by Hojman.[20] This direct method has been generalized by González-Gascón[21] and Lutzky.[22,23] Although Pillay and Leach[24] demonstrated that the invariant derived using Hojman's conservation law is trivial for all Noether symmetries, it may be meaningful using this direct method to find non-Noether conserved quantity of a mechanical system. In Refs.[25] and [26], Mei used this direct method to study the non-Noether conserved quantities for the differential equations of motion in phase space and also for the generalized Hamiltonian system. The non-Noether conserved quantities of some mechanical systems were studied by using this direct method in Refs.[27–29].

The direct method of finding the conserved quantity presented above is all utilized only under the special infinitesimal transformations of groups in which the time variable is not subject to variation. In this paper, we study the Lie symmetrical non-Noether conserved quantity for a holonomic Hamiltonian system under the general infinitesimal transformations of groups. Firstly, we establish the determining equations of Lie symmetry for a holonomic Hamiltonian system. Secondly, Lie symmetrical non-Noether conserved quantity is deduced. Finally, an example is given to illustrate the application of the result.

2. Lie symmetrical determining equation

The differential equations of motion for the holonomic Hamiltonian system can be written as

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \quad \dot{p}_s = -\frac{\partial H}{\partial q_s}, \quad (s = 1, \ldots, n), \qquad (1)$$

where H is the Hamiltonian.

Equation (1) can be expressed as follows:

$$\begin{aligned}&\dot{q}_s = g_s(t, q_k, p_k),\\ &\dot{p}_s = h_s(t, q_k, p_k), \quad (s, k = 1, \ldots, n).\end{aligned} \tag{2}$$

Introducing the general infinitesimal transformations

$$\begin{aligned}&t^* = t + \Delta t,\\ &q_s^*(t^*) = q_s(t) + \Delta q_s,\\ &p_s^*(t^*) = p_s(t) + \Delta p_s,\end{aligned} \tag{3}$$

yields the following expanding forms:

$$\begin{aligned}&t^* = t + \varepsilon\xi_0(t, \boldsymbol{q}, \boldsymbol{p}),\\ &q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \boldsymbol{p}),\\ &p_s^*(t^*) = p_s(t) + \varepsilon\eta_s(t, \boldsymbol{q}, \boldsymbol{p}),\end{aligned} \tag{4}$$

where ε is an infinitesimal parameter, ξ_0, ξ_s and η_s are the generators of infinitesimal transformations.

From the theory of invariance of the differential equations under the infinitesimal transformations, the invariance of Eq.(2) under the infinitesimal transformations (4) leads to the following determining equations:

$$\begin{aligned}&\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t} - g_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} = \xi_0\frac{\partial g_s}{\partial t} + \xi_k\frac{\partial g_s}{\partial q_k} + \eta_k\frac{\partial g_s}{\partial p_k},\\ &\frac{\bar{\mathrm{d}}\eta_s}{\mathrm{d}t} - h_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} = \xi_0\frac{\partial h_s}{\partial t} + \xi_k\frac{\partial h_s}{\partial q_k} + \eta_k\frac{\partial h_s}{\partial p_k},\\ &\quad (s, k = 1, \ldots, n),\end{aligned} \tag{5}$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + g_s\frac{\partial}{\partial q_s} + h_s\frac{\partial}{\partial p_s}. \tag{6}$$

If the generators ξ_0, ξ_s and η_s of infinitesimal transformations (4) satisfy the determining equation (5), then the corresponding transformations are called Lie symmetrical transformations of the holonomic Hamiltonian system (1).

3. Lie symmetrical non-Noether conserved quantity

The direct method of finding the conserved quantity by using Lie symmetry under the general infinitesimal transformations is described as follows.

Theorem For the holonomic Hamiltonian system (1), if the generators ξ_0, ξ_s and η_s of the infinitesimal transformations (4) satisfy the determining equation (5), then the system has the Lie symmetrical non-Noether conserved quantities as follows:

$$I = \frac{\partial\xi_0}{\partial t} + \frac{\partial\xi_s}{\partial q_s} + \frac{\partial\eta_s}{\partial p_s} - \frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} = \text{const}. \tag{7}$$

Proof We have

$$\frac{\bar{\mathrm{d}}I}{\mathrm{d}t} = \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\xi_0}{\partial t} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\xi_s}{\partial q_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\eta_s}{\partial p_s} - \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right), \tag{8}$$

$$\begin{aligned}&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\xi_0}{\partial t} = \frac{\partial}{\partial t}\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} - \frac{\partial\xi_0}{\partial q_k}\frac{\partial g_k}{\partial t} - \frac{\partial\xi_0}{\partial p_k}\frac{\partial h_k}{\partial t},\\ &\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\xi_s}{\partial q_s} = \frac{\partial}{\partial q_s}\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t} - \frac{\partial\xi_s}{\partial q_k}\frac{\partial g_k}{\partial q_s} - \frac{\partial\xi_s}{\partial p_k}\frac{\partial h_k}{\partial q_s},\\ &\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\eta_s}{\partial p_s} = \frac{\partial}{\partial p_s}\frac{\bar{\mathrm{d}}\eta_s}{\mathrm{d}t} - \frac{\partial\eta_s}{\partial q_k}\frac{\partial g_k}{\partial p_s} - \frac{\partial\eta_s}{\partial p_k}\frac{\partial h_k}{\partial p_s}.\end{aligned} \tag{9}$$

Substituting Eq.(9) into Eq.(8) and using Eq.(5), we obtain

$$\begin{aligned}\frac{\bar{\mathrm{d}}I}{\mathrm{d}t} =& \frac{\partial}{\partial t}\frac{\partial\xi_0}{\partial t} - \frac{\partial\xi_0}{\partial q_k}\frac{\partial g_k}{\partial t} - \frac{\partial\xi_0}{\partial p_k}\frac{\partial h_k}{\partial t} - \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right)\\ &+ \frac{\partial}{\partial q_s}\left(g_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} - \xi_0\frac{\partial g_s}{\partial t}\right)\\ &+ \xi_k\frac{\partial^2 g_s}{\partial q_s\partial q_k} + \eta_k\frac{\partial^2 g_s}{\partial q_s\partial p_k}\\ &+ \frac{\partial}{\partial p_s}\left(h_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} - \xi_0\frac{\partial h_s}{\partial t}\right)\\ &+ \xi_k\frac{\partial^2 h_s}{\partial p_s\partial q_k} + \eta_k\frac{\partial^2 h_s}{\partial p_s\partial p_k}.\end{aligned} \tag{10}$$

From Eqs.(1) and (2), we know

$$\frac{\partial g_s}{\partial q_s} = -\frac{\partial h_s}{\partial p_s}. \tag{11}$$

Substituting Eq.(11) into Eq.(10), we can obtain

$$\frac{\bar{\mathrm{d}}I}{\mathrm{d}t} = 0. \tag{12}$$

4. Illustrated example

The Lagrangian of the mechanical system is

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - q_2. \tag{13}$$

Now we study the Lie symmetrical non-Noether conserved quantity. One has

$$p_1 = \frac{\partial L}{\partial\dot{q}_1} = \dot{q}_1, \quad p_2 = \frac{\partial L}{\partial\dot{q}_2} = \dot{q}_2,$$

$$H = p_1\dot{q}_1 + p_2\dot{q}_2 - L = \frac{1}{2}(p_1^2 + p_2^2) + q_2. \tag{14}$$

From Eqs.(2) and (1) we can obtain

$$\begin{aligned}&g_1 = \dot{q}_1 = \frac{\partial H}{\partial p_1} = p_1,\\ &g_2 = \dot{q}_2 = \frac{\partial H}{\partial p_2} = p_2,\\ &h_1 = \dot{p}_1 = -\frac{\partial H}{\partial q_1} = 0,\\ &h_2 = \dot{p}_2 = -\frac{\partial H}{\partial q_2} = -1.\end{aligned} \tag{15}$$

The determining equation (5) gives

$$\frac{\bar{\mathrm{d}}\xi_1}{\mathrm{d}t} - p_1\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} = \eta_1, \quad \frac{\bar{\mathrm{d}}\xi_2}{\mathrm{d}t} - p_2\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} = \eta_2,$$
$$\frac{\bar{\mathrm{d}}\eta_1}{\mathrm{d}t} = 0, \quad \frac{\bar{\mathrm{d}}\eta_2}{\mathrm{d}t} + \frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} = 0. \tag{16}$$

Equation (16) has the following solutions:

$$\xi_0 = t,$$
$$\xi_1 = q_1 + p_1 q_1 + p_1^2 p_2,$$
$$\xi_2 = q_2 - \frac{1}{2}t^2,$$
$$\eta_1 = 0, \quad \eta_2 = -t. \tag{17}$$

From Eq.(7) we obtain the following conserved quantity:

$$I = p_1 = \text{const.} \tag{18}$$

The conserved quantity (18) is a non-Noether conserved quantity. In terms of Noether's inverse theorem,[10,30] and because of Noether symmetry, the infinitesimal generators corresponding to conserved quantity (18) are as follows:

$$\xi_0 = 0, \quad \xi_1 = 1, \quad \xi_2 = 0,$$
$$\eta_1 = 0, \quad \eta_2 = 0. \tag{19}$$

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