Vol 13 No 10, October 2004
1009-1963/2004/13(10)/1615-05

Chinese Physics

© 2004 Chin. Phys. Soc.
and IOP Publishing Ltd

Lie symmetries and invariants of constrained Hamiltonian systems*

Liu Rong-Wan(刘荣万)[a)b)†] and Chen Li-Qun(陈立群)[b)]

[a)] *Department of Physics, Shaoguan University, Shaoguan* 512005, *China*

[b)] *Shanghai Institute of Applied Mathematics and Mechanics, Shanghai University, Shanghai* 200072, *China*

(Received 30 December 2003; revised manuscript received 31 May 2004)

According to the theory of the invariance of ordinary differential equations under the infinitesimal transformations of group, the relations between Lie symmetries and invariants of the mechanical system with a singular Lagrangian are investigated in phase space. New dynamical equations of the system are given in canonical form and the determining equations of Lie symmetry transformations are derived. The proposition about the Lie symmetries and invariants are presented. An example is given to illustrate the application of the result in this paper.

Keywords: analytical mechanics, constrained Hamilton system, Lie symmetry, invariant
PACC: 0320

1. Introduction

A dynamical system with a singular Lagrangian is subject to some constraints in phase space and is called a constrained Hamiltonian system (or constrained canonical system). The theories of constrained Hamiltonian systems play an important role in modern physics. For example, it is widely applied to gauge field theories (quantum electrodynamics, quantum flavourdynamics, quantum chromodynamics, supergravity, superstring), particle physics, condensed matter and some other research areas. On the other hand, symmetry is now a fundamental concept in modern physics. As is well known, the discussion of symmetry properties of a system is usually based on the examination of the Lagrangian in configuration space. In quantum theory, however, the phase-space path integral is more fundamental than the configuration-space path integral. Therefore, the investigation of symmetry properties of a system in phase space has a basic sense in quantum theories, which can be applied to more general cases.

The first systematic study of a dynamical system with a singular Lagrangian (including field theories) was by Dirac.[1] The canonical formalism for the system is characterized in the presence of certain constraints among the canonical variables. The relations between constraints and invariance were clarified by Bergmann and co-workers.[2] Shanmugadhasan analysed the influence of singularity on the Lagrange equations and formulated the corresponding Hamiltonian formalism.[3] Kamimura established the interrelation between Lagrangian and Hamiltonian constraints.[4] Li has investigated the Noether symmetries of canonical Hamiltonian systems.[5] However, the relations between Lie symmetries and invariants have not been discussed in detail yet.

This paper discusses the relations between Lie symmetries and invariants of the classic singular constrained Hamiltonian systems. The results can be extended and applied to quantal systems. In section 2, the concept of constrained Hamiltonian system is described briefly, and the canonical equations of the system are derived. In section 3, the invariance of the dynamical equations under the infinitesimal group of transformations is discussed. The determining equations of Lie symmetry are derived according to the invariance criterion for ordinary differential equations under infinitesimal transformations in this section. The proposition about symmetries and invariants of the systems in phase space is presented in section 4. According to the proposition, if the generators which satisfy the determining equations of Lie symmetry are

*Project supported by the National Natural Science Foundation of China (Grant No 19972010).
†Corresponding author. E-mail:lrw@sgu.edu.cn

found, the conserved quantities corresponding to Lie symmetries can be obtained. Finally, an example is provided to illustrate the application of the result.

2. Dynamical equations of constrained Hamilton systems

Consider a mechanical system with n degrees of freedom whose Lagrangian $L(q_s, \dot{q}_s)$ does not depend on time explicitly. The Euler–Lagrange equations are written as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = 0 \quad (s = 1, \ldots, n). \tag{1}$$

The canonical momenta are defined by

$$p_s = \frac{\partial L}{\partial \dot{q}_s} \quad (s = 1, \ldots, n), \tag{2}$$

and the canonical Hamiltonian of the system is written in the form (the repeated subscripts denote summation)

$$H = p_s \dot{q}_s - L(q_s, \dot{q}_s) = H(q_s, p_s), \tag{3}$$

which can be formed by eliminating the generalized velocities from Eq.(2).

The Hessian matrix is defined by

$$[H_{ij}] = \left[\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right]. \tag{4}$$

If the Hessian matrix is nondegenerate, i.e. $\det|H_{ij}| \neq 0$, the Lagrangian is said to be regular. When $\det|\ H_{ij}\ | = 0$, the Lagrangian is called singular.

For a system with a singular Lagrangian, it is impossible to solve Eq.(2) for all the generalized velocities $\dot{q}_s$. Assuming the rank of the Hessian matrix $[H_{ij}]$ is $r (r < n)$, we can solve Eq.(2) for the r generalized velocities $\dot{q}_\sigma$ as a function of the generalized coordinates, the r canonical momenta and the remaining $n - r$ generalized velocities $\dot{q}_\rho$

$$\dot{q}_\sigma = f_\sigma(q_s, p_a, \dot{q}_\rho) \quad (\sigma, a = 1, 2, \ldots, r; \rho = r+1, \ldots, n). \tag{5}$$

Substituting Eq.(5) into Eq.(2) we obtain

$$\begin{aligned} p_s =& g_s(q_k, \dot{q}_\sigma, \dot{q}_\rho) \\ =& g_s[q_k, f_\sigma(q_k, p_a, \dot{q}_\rho), \dot{q}_\rho] = g_s(q_k, p_a, \dot{q}_\rho). \end{aligned} \tag{6}$$

It is clear that for $s = 1, 2, \ldots, r$, Eq.(6) are reduced to identities. On the other hand, for $s = \rho = r+1, \ldots, n$, we have

$$p_\rho = g_\rho(q_s, p_a), \tag{7}$$

or

$$\phi_\rho(q_s, p_s) = p_\rho - g_\rho(q_s, p_a) = 0 \quad (\rho = 1, 2, \ldots, n-r), \tag{8}$$

where $\phi_\rho(q_s, p_s)$ does not depend on the generalized velocities $\dot{q}_s$. Equation (8) represents $m = n - r$ constraints following the definitions of the canonical momenta p_s and the singularity of the Lagrangian. Hence, a system with a singular Lagrangian described in phase space has some inherent constraints, and is called a constrained Hamiltonian system.

From Eq.(2) we have

$$p_\rho - \frac{\partial L}{\partial \dot{q}_\rho} = 0. \tag{9}$$

Comparing Eq.(8) with (9) we can obtain

$$\phi_\rho(q_s, p_s) = p_\rho - \frac{\partial L}{\partial \dot{q}_\rho} = 0 \quad (\rho = r+1, \ldots, n). \tag{10}$$

The variation of the canonical Hamiltonian (3) is given by

$$\begin{aligned} \delta H =& p_s \delta \dot{q}_s + \dot{q}_s \delta p_s - \frac{\partial L}{\partial q_s}\delta q_s - \frac{\partial L}{\partial \dot{q}_s}\delta \dot{q}_s \\ =& \dot{q}_s \delta p_s - \frac{\partial L}{\partial q_s}\delta q_s \\ =& \frac{\partial H}{\partial q_s}\delta q_s + \frac{\partial H}{\partial p_s}\delta p_s. \end{aligned} \tag{11}$$

From Eqs.(11) and (2), we obtain

$$\left(\dot{q}_s - \frac{\partial H}{\partial p_s}\right)\delta p_s - \left(\dot{p}_s + \frac{\partial H}{\partial q_s}\right)\delta q_s = 0. \tag{12}$$

Since there are some inherent constraints among canonical variables for a constrained Hamiltonian system, canonical variables q_s and p_s are not independent, and we have

$$\delta \phi_\rho = \frac{\partial \phi_\rho}{\partial q_s}\delta q_s + \frac{\partial \phi_\rho}{\partial p_s}\delta p_s = 0. \tag{13}$$

Introducing Lagrangian constraint multipliers λ_ρ, from Eqs.(12) and (13) the canonical equations of motion for a constrained Hamiltonian system can be obtained as follows:

$$\begin{cases} \dot{q}_s = \dfrac{\partial H}{\partial p_s} + \Lambda_s, \\ \dot{p}_s = -\dfrac{\partial H}{\partial q_s} + \tilde{\Lambda}_s \end{cases} \quad (s = 1, \ldots, n; \rho = r+1, \ldots, n), \tag{14}$$

where $\Lambda_s = \lambda_\rho \dfrac{\partial \phi_\rho}{\partial p_s}$ and $\tilde{\Lambda}_s = -\lambda_\rho \dfrac{\partial \phi_\rho}{\partial q_s}$. Equations (14) can be written in the explicit form

$$\begin{cases} \dot{q}_s = h_s(q_k, p_k), \\ \dot{p}_s = g_s(q_k, p_k) \end{cases} \quad (s = 1, \ldots, n). \tag{15}$$

Canonical variables q_s and p_s can be solved from Eqs.(14) and the constraint equations (10).[6,7]

3. Lie symmetry and its determining equations

Introduce infinitesimal transformations as follows:[8−10]

$$\begin{cases} t^* = t + \Delta t, \\ q_s^* = q_s + \Delta q_s, \\ p_s^* = p_s + \Delta p_s, \end{cases} \quad (s = 1, \ldots, n). \tag{16}$$

Their expanding forms are[11,12]

$$\begin{cases} t^* = t + \varepsilon\xi_0(t, q, p), \\ q_s^* = q_s + \varepsilon\xi_s(t, q, p), \\ p_s^* = p_s + \varepsilon\eta_s(t, q, p), \end{cases} \quad (s = 1, \ldots, n), \tag{17}$$

where ε is a small parameter, and ξ_0, ξ_s and η_s are generators of the infinitesimal transformations of group. The generating function vector of the infinitesimal transformations of group is written as[13,14]

$$X^{(0)} = \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} + \eta_s\frac{\partial}{\partial p_s}. \tag{18}$$

Its first expansion is

$$X^{(1)} = X^{(0)} + (\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\frac{\partial}{\partial \dot{q}_s} + (\dot{\eta}_s - \dot{p}_s\dot{\xi}_0)\frac{\partial}{\partial \dot{p}_s}. \tag{19}$$

Definition If the infinitesimal transformations (17) leave Eqs.(15) and constraint equation (10) invariant, they are called Lie symmetry transformations of the constrained Hamiltonian system.

Criterion If the generators ξ_0, ξ_s and η_s satisfy the following conditions:

$$\begin{cases} \dot{\xi}_s - \dot{q}_s\dot{\xi}_0 = X^{(0)}(h_s), \\ \dot{\eta}_s - \dot{p}_s\dot{\xi}_0 = X^{(0)}(g_s), \end{cases} \quad (s = 1, \ldots, n), \tag{20}$$

the infinitesimal transformations (17) are Lie symmetry transformations of the constrained Hamiltonian system.

Proof According to the invariance criterion for ordinary differential equations under the infinitesimal transformations, the invariance of differential equations (15) under the transformations (17) can be represented in the form

$$\begin{cases} X^{(1)}[\dot{q}_s - h_s(q,p)] = 0, \\ X^{(1)}[\dot{p}_s - g_s(q,p)] = 0, \end{cases} \quad (s = 1, \ldots, n). \tag{21}$$

Using Eqs.(18)and (19), we can have

$$\begin{aligned} & X^{(1)}[\dot{q}_s - h_s(q,p)] \\ =& X^{(0)}(\dot{q}_s - h_s) + (\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\frac{\partial(\dot{q}_s - h_s)}{\partial \dot{q}_s} \\ & + (\dot{\eta}_s - \dot{p}_s\dot{\xi}_0)\frac{\partial(\dot{q}_s - h_s)}{\partial \dot{p}_s} \\ =& - X^{(0)}(h_s) + (\dot{\xi}_s - \dot{q}_s\dot{\xi}_0) = 0. \end{aligned} \tag{22}$$

Hence, we obtain

$$(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0) = X^{(0)}(h_s) \quad (s = 1, \ldots, n). \tag{23}$$

Similarly, we can have

$$(\dot{\eta}_s - \dot{p}_s\dot{\xi}_0) = X^{(0)}(g_s) \quad (s = 1, \ldots, n). \tag{24}$$

QED.

The invariance of constraint equations (10) under transformations (17) can be written in the form[15,16]

$$X^{(1)}(\phi_\rho(q_s, p_s)) = 0. \tag{25}$$

Equations (20) and (25) are called the determining equations of Lie symmetry transformations of the constrained Hamiltonian system.

4. Structure equation and invariants

Proposition If generators ξ_0, ξ_s and η_s satisfy the determining equations (20) and (25), and there is a gauge function $G = G(t, q_s, p_s)$ which satisfies the following equation:

$$L_p\dot{\xi}_0 + X^{(1)}(L_p) + \tilde{\Lambda}_s(\xi_s - \dot{q}_s\xi_0) - \Lambda_s(\eta_s - \dot{p}_s\xi_0) + \dot{G} = 0, \tag{26}$$

there exists the following invariant of the constrained Hamilton systems:

$$I = p_s\xi_s - H\xi_0 + G = \text{const}, \tag{27}$$

where $L_p(q_s, p_s)$ is the Lagrangian

$$L_p = p_s\dot{q}_s(q_k, p_k) - H. \tag{28}$$

Proof The derivative of Eq.(27) with respect to t can be written as

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \dot{p}_s\xi_s + p_s\dot{\xi}_s - \left(\frac{\partial H}{\partial t} + \frac{\partial H}{\partial q_s}\dot{q}_s + \frac{\partial H}{\partial p_s}\dot{p}_s\right)\xi_0 - H\dot{\xi}_0 + \dot{G}. \tag{29}$$

From Eq.(26) we can obtain

$$\begin{aligned} & (p_s\dot{q}_s - H)\dot{\xi}_0 - \frac{\partial H}{\partial t}\xi_0 - \frac{\partial H}{\partial q_s}\xi_s + \eta_s\Lambda_s + (\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)p_s \\ & + \tilde{\Lambda}_s(\xi_s - \dot{q}_s\xi_0) - \Lambda_s(\eta_s - \dot{p}_s\xi_0) + \dot{G} = 0, \end{aligned} \tag{30}$$

where $\Lambda_s = \dot{q}_s - \dfrac{\partial H}{\partial p_s}$. Using Eq.(30), Eq.(29) can be written as

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \left(\dot{p}_s + \frac{\partial H}{\partial q_s} - \tilde{\Lambda}_s\right)(\xi_s - \dot{q}_s\xi_0) - \Lambda_s(\eta_s - \dot{p}_s\xi_0) + \Lambda_s(\eta_s - \dot{p}_s\xi_0) = 0. \quad (31)$$

QED.

Equation (26) is called Lie symmetry structure equation of the constrained Hamiltonian systems. According to the proposition, if generators ξ_0, ξ_s and η_s, which satisfy determining equations (20) and (25), can be found and the gauge function G can be determined from Eq.(26), the corresponding invariants of the systems can be obtained from formula (27).

5. Illustrated example

Let us consider a mechanical system whose Lagrangian is given by

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \frac{1}{2}q_3^2(q_1^2 + q_2^2) - q_3(q_1\dot{q}_2 - q_2\dot{q}_1) - \frac{1}{2}(q_1^2 + q_2^2). \quad (32)$$

It is clear that this Lagrangian is singular. The canonical momenta are given by

$$p_1 = \dot{q}_1 + q_3q_2, p_2 = \dot{q}_2 - q_3q_1, p_3 = 0. \quad (33)$$

Hence, the Hamiltonian is given by

$$H = \frac{1}{2}(p_1^2 + p_2^2) + q_3(q_1p_2 - q_2p_1) + \frac{1}{2}(q_1^2 + q_2^2), \quad (34)$$

and the constraint equation is

$$\phi = p_3 - \frac{\partial L}{\partial \dot{q}_3} = p_3 = 0. \quad (35)$$

From Eqs.(14) and (35), we can obtain

$$\begin{cases} \Lambda_s = \tilde{\Lambda}_s = 0, \\ \dot{q}_1 = p_1 - q_3q_2 = h_1, \\ \dot{q}_2 = p_2 + q_3q_1 = h_2, \\ \dot{q}_3 = 0 = h_3, \dot{p}_1 = -q_3p_2 - q_1 = g_1, \\ \dot{p}_2 = q_3p_1 - q_2 = g_2, \\ \dot{p}_3 = 0 = g_3. \end{cases} \quad (36)$$

Substituting Eqs.(36) into determining equations (20) we obtain

$$\begin{cases} \dot{\xi}_1 - \dot{q}_1\dot{\xi}_0 = -\xi_2q_3 - \xi_3q_2 + \eta_1, \\ \dot{\xi}_2 - \dot{q}_2\dot{\xi}_0 = \xi_1q_3 + \xi_3q_1 + \eta_2, \\ \dot{\xi}_3 - \dot{q}_3\dot{\xi}_0 = 0, \dot{\eta}_1 - \dot{p}_1\dot{\xi}_0 = -\xi_1 - \xi_3p_2 - \eta_2q_3, \\ \dot{\eta}_2 - \dot{p}_2\dot{\xi}_0 = -\xi_2 + \xi_3p_1 + \eta_1q_3, \\ \dot{\eta}_3 - \dot{p}_3\dot{\xi}_0 = 0. \end{cases} \quad (37)$$

There are the following solutions for Eqs.(37):

(1)

$$\xi_0 = -1, \quad \xi_1 = \xi_2 = \xi_3 = 0, \quad \eta_1 = \eta_2 = \eta_3 = 0, \quad (38)$$

(2)

$$\xi_0 = 0, \quad \xi_1 = q_1, \quad \xi_2 = q_2, \quad \xi_3 = 0,$$

$$\eta_1 = p_1, \quad \eta_2 = p_2, \quad \eta_3 = 0. \quad (39)$$

Generators (38) and (39) satisfy the determining equations (25) and (20), hence they generate Lie symmetry transformations of the system.

Substituting the generators (38) into the structure equation (26) we obtain

$$\dot{G}_1 = 0, \quad \text{i.e.,} \quad G_1 = c. \quad (40)$$

The corresponding conserved quantity can be obtained from Eq.(27):

$$I = H + c = \frac{1}{2}(p_1^2 + p_2^2) + q_3(q_1p_2 - q_2p_1) + \frac{1}{2}(q_1^2 + q_2^2) + c = \text{const.} \quad (41)$$

Equation (41) represents that the mechanical energy of the system is conserved. Similarly, substituting the generators (39) into Eq.(26) we have

$$\dot{G}_2 = -\frac{\mathrm{d}}{\mathrm{d}t}(q_1p_1 + q_2p_2), \quad \text{i.e.} \quad G_2 = -(q_1p_1 + q_2p_2). \quad (42)$$

The corresponding invariant is written in the form

$$I = 0. \quad (43)$$

Solutions (39) are called trivial ones which do not result in invariant of the system. Hence, for the constrained Hamiltonian systems, Lie symmetries do not certainly result in invariants.

References

[1] Dirac P A M 1950 *Can. J. Math.* **2** 129
Dirac P A M 1964 *Lecture on Quantum Mechanics* (New York: Yeshiva University)

[2] Bergmann P G and Goldberg J 1955 *Phys. Rev.* **98** 531

[3] Shanmugadhasan S 1973 *J. Math. Phys.* **14** 677

[4] Kamimura K 1962 *Nuovo Cimento J. Phys.* B **68** 33

[5] Li Z P 2002 *Symmetries in Constrained Canonical Systems* (Beijing: Science Press)

[6] Mei F X 1999 *Applications of Lie Groups and Lie Algebras to Constrained Mechanical Systems* (Beijing: Science Press) (in Chinese)

[7] Mei F X, Zhang Y F and Shang M 1999 *Mech. Res. Commun.* **26** 7

[8] Wu H B 2004 *Chin. Phys.* **13** 589

[9] Fu J L and Chen L Q 2004 *Chin. Phys.* **13** 287

[10] Li A M, Jiang J H and Li Z B 2003 *Chin. Phys.* **12** 467

[11] Zhang Y C 2000 *Chin. Phys.* **9** 561

[12] Cheng X W 2000 *Chin. Phys.* **9** 721

[13] Lao S K 2003 *Chin. Phys.* **12** 841

[14] Zhang Y 2000 *Chin. Phys.* **9** 401

[15] Zhang H B 2002 *Chin. Phys.* **11** 765

[16] Fu J L and Chen L Q 2003 *Chin. Phys.* **12** 695