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Lie symmetries and non-Noether conserved quantities for Hamiltonian canonical equations*

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This paper focuses on studying Lie symmetries and non-Noether conserved quantities of Hamiltonian dynamical systems in phase space. Based on the infinitesimal transformations with respect to the generalized coordinates and generalized momenta, we obtain the determining equations and structure equation of the Lie symmetry for Hamiltonian dynamical systems. This work extends the research of non-Noether conserved quantity for Hamilton canonical equations, and leads directly to a new type of non-Noether conserved quantities of the systems. Finally, an example is given to illustrate these results.

Keywords: Hamiltonian system, Lie symmetry, non-Noether conserved quantity, Lie groups
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1. Introduction

It is well known that symmetry theorems are extremely useful tools for resolving problems in mathematics, physics and mechanics. For Lagrangian systems, the best known type of symmetries are of course the Noether symmetries, which directly shows that there is a close connection between the conserved quantities of a Lagrangian dynamical system and those transformations of coordinates and time that preserve the action integral.[1] Djukić and Vujanović gave the Noether symmetries of the non-conservative mechanical systems.[2,3] Li extended Noether theory to the classical and quantum constraint systems.[4,5] Mei extended Noether theorem from Hamiltonian systems to Birkhoffian systems.[6] Sarlet *et al* studied the pseudo-symmetries, Noether's theorem and the adjoint equation.[7] Now Noether symmetrical theory has been widely applied in many research fields.[8−10] Another method to obtain the conserved quantities of mechanical and physical systems is the Lie symmetry method (also called the dynamical symmetry method) which is based on the form of equations of motion for the dynamical systems that remain invariant under the infinitesimal transformations of the coordinates and time.[11−17] The symmetries, however, only give the conserved quantities corresponding to Noether symmetries.

A theorem regarding the conserved quantities for second-order dynamical systems has recently been given by Hojman[18] and Luo *et al*;[19] and generalized by González-Gascón,[20] using geometric techniques; subsequently the generalized theorem in generalized coordinates is expressed in the presence of a velocity-dependent symmetry by Lutzky and others.[21,22] Recently, study of symmetries of Hamiltonian systems is very active.[17,23−27] Lately Mei derived a certain conserved quantity for Hamiltonian systems in phase space, and showed that the result may be stated as follows: for the equation of motion in phase space $\dot{q}_s = Q_s(t, \boldsymbol{q}, \boldsymbol{p})$, $\dot{p}_s = P_s(t, \boldsymbol{q}, \boldsymbol{p})$, if a function $\mu(t, \boldsymbol{q}, \boldsymbol{p})$ can be found such that the quantity[28,29]

$$\psi = \frac{\partial Q_s}{\partial q_s} + \frac{\partial P_s}{\partial p_s} + \frac{\mathrm{d}}{\mathrm{d}t} \ln \mu = 0, \tag{1}$$

the system possesses the following non-Noether conserved quantity

$$\phi = \frac{1}{\mu} \frac{\partial (\mu \xi_s)}{\partial q_s} + \frac{1}{\mu} \frac{\partial (\mu \eta_s)}{\partial p_s} = \text{const.}, \tag{2}$$

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where ξ_s, η_s are infinitesimal generator functions, and the above non-Noether conserved quantities are all of Hojman type. Recently we have studied non-Noether symmetries and conserved quantities of nonconservative dynamical systems, and obtained some useful results.[30] In this paper, first, we generalize the theorem by showing that it is sufficient for ψ to be an invariant of the symmetry group in order for ϕ to be a conserved quantity. Second, we derive directly a new non-Noether conserved quantity for Hamiltonian systems, and the conserved quantity is associated with the presence of a Lie symmetry.

2. Lie symmetries and conserved quantities of Hamiltonian systems

Let us study a Hamiltonian system whose configuration is determined by the n generalized coordinates $\boldsymbol{q} = \{q_1, \cdots, q_n\}$ and n generalized momentums $\boldsymbol{p} = \{p_1, \cdots, p_n\}$. The Hamiltonian $H(t, \boldsymbol{q}, \boldsymbol{p})$ has the Hamilton canonical equations in the form

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \quad \dot{p}_s = -\frac{\partial H}{\partial q_s} \quad (s = 1, \cdots, n). \tag{3}$$

Equation (3) can be transformed into the form

$$\dot{q}_s = h_s(t, \boldsymbol{q}, \boldsymbol{p}), \quad \dot{p}_s = g_s(t, \boldsymbol{q}, \boldsymbol{p}). \tag{4}$$

Introduce an infinitesimal transformation with respect to the generalized coordinates and the generalized momenta,

$$\begin{aligned} q_s^{\bullet}(t) &= q_s(t) + \varepsilon \xi_s(t, \boldsymbol{q}, \boldsymbol{p}), \\ p_s^{\bullet}(t) &= p_s(t) + \varepsilon \eta_s(t, \boldsymbol{q}, \boldsymbol{p}), \end{aligned} \tag{5}$$

where ε is a small parameter, and ξ_s, η_s are the generators of the infinitesimal transformation.

If (4) remains invariant under the infinitesimal transformation (5) of the one-parameter Lie group, the infinitesimal generators $\xi_s(t, \boldsymbol{q}, \boldsymbol{p})$, $\eta_s(t, \boldsymbol{q}, \boldsymbol{p})$ satisfy the determining equations

$$\dot{\xi}_s = E^0(h_s), \quad \dot{\eta}_s = E^0(g_s), \tag{6}$$

where the operator E^0 is the generator of the symmetrical group,[28,29,31] given by

$$E^0 = \xi_s \frac{\partial}{\partial q_s} + \eta_s \frac{\partial}{\partial p_s}. \tag{7}$$

Its first extended form

$$E^1 = E^0 + \dot{\xi}_s \frac{\partial}{\partial \dot{q}_s} + \dot{\eta}_s \frac{\partial}{\partial \dot{p}_s}, \tag{8}$$

and the vector field

$$\frac{d}{dt} = \frac{\partial}{\partial t} + h_s \frac{\partial}{\partial q_s} + g_s \frac{\partial}{\partial p_s} \tag{9}$$

represents the total time derivative along the trajectories of Equation (3). Then, for any function ω,

$$\dot{\omega} = \frac{\partial \omega}{\partial t} + h_s \frac{\partial \omega}{\partial q_s} + g_s \frac{\partial \omega}{\partial p_s}. \tag{10}$$

The invariance is called Lie symmetry, associated with Hamiltonian systems in phase space. Equation (6) may be regarded as the criterion of the symmetry.

Equation (6) is proven with the following process. Based on the invariance of the differential equations under the infinitesimal transformation, the invariance of Eq.(4) under the infinitesimal transformation (5) is represented as

$$\begin{aligned} E^1(\dot{q}_s - h_s(t, \boldsymbol{q}, \boldsymbol{p})) = 0, \\ E^1(\dot{p}_s - g_s(t, \boldsymbol{q}, \boldsymbol{p})) = 0. \end{aligned} \tag{11}$$

Substituting formulae (4), (7) and (8) into (11) leads to Eq.(6).

Then we have the following theorem.

Theorem 1 For the Hamiltonian system (4), the gauge function G can be found such that the quantity

$$\frac{\partial h_s}{\partial q_s} + \frac{\partial g_s}{\partial p_s} + \dot{G} = \psi, \tag{12}$$

and a constant of the motion is given by

$$\phi = E^0[G] + \frac{\partial \xi_s}{\partial q_s} + \frac{\partial \eta_s}{\partial p_s}, \tag{13}$$

if the infinitesimal transformation generations $\xi_s = \xi_s(t, \boldsymbol{q}, \boldsymbol{p})$ and $\eta_s = \eta_s(t, \boldsymbol{q}, \boldsymbol{p})$ are the Lie symmetry of (4). We call Eq.(12) the structural equation of the Lie symmetry for Hamiltonian systems, and there exists a non-Noether conserved quantity (13) of the system. We should indicate that the conserved quantity (13) be equal to (2) can be proved by taking $G = \ln \mu$ and $\psi = 0$.

Here we point out that it is sufficient that ψ is an invariant of the symmetry group in order for ϕ to be a conserved quantity; that is, we show that $E(\psi) = 0$ implies $\dot{\phi} = 0$. This is accomplished by explicitly differentiating ϕ to demonstrate that $\dot{\phi} = E(\psi)$. The calculation is greatly simplified by the following easily proven identities.

Proof For the theorem, we show that $E^0(\psi) = 0$ implies $\dot{\phi} = 0$. This is accomplished by explicitly differentiating ϕ to demonstrate that

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[E^0(G)\right] + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \xi_s}{\partial q_s} + \frac{\partial \eta_s}{\partial p_s}\right]. \tag{14}$$

It is straightforward to show that for any function $f = f(t, \boldsymbol{q}, \boldsymbol{p})$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial q_s} = \frac{\partial \dot{f}}{\partial q_s} - \frac{\partial f}{\partial q_k}\frac{\partial h_k}{\partial q_s} - \frac{\partial f}{\partial p_k}\frac{\partial g_k}{\partial q_s}, \tag{15}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial p_s} = \frac{\partial \dot{f}}{\partial p_s} - \frac{\partial f}{\partial q_k}\frac{\partial h_k}{\partial p_s} - \frac{\partial f}{\partial p_k}\frac{\partial g_k}{\partial p_s}, \tag{16}$$

$$E^0\left(\frac{\partial f}{\partial q_s}\right) = \frac{\partial}{\partial q_s}E^0(f) - \frac{\partial f}{\partial q_k}\frac{\partial \xi_k}{\partial q_s} - \frac{\partial f}{\partial p_k}\frac{\partial \eta_k}{\partial q_s}, \tag{17}$$

$$E^0\left(\frac{\partial f}{\partial p_s}\right) = \frac{\partial}{\partial p_s}E^0(f) - \frac{\partial f}{\partial q_k}\frac{\partial \xi_k}{\partial p_s} - \frac{\partial f}{\partial p_k}\frac{\partial \eta_k}{\partial p_s}. \tag{18}$$

The condition that ξ_s, η_s define a symmetry (11) for $\dot{q}_s = h_s$, $\dot{p}_s = g_s$ is given by $\dot{\xi}_s = E^0(h_s)$, $\dot{\eta}_s = E^0(g_s)$; using this we can show that if E^0 is symmetric, then

$$\frac{\mathrm{d}}{\mathrm{d}t}E^0(f) = E^0\left(\dot{f}\right). \tag{19}$$

Differentiating ϕ one obtains

$$\dot{\phi} = E^0(\dot{G}) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \xi_s}{\partial q_s} + \frac{\partial \eta_s}{\partial p_s}\right), \tag{20}$$

where we have used (19).

Using (15) and (16) we can show that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \xi_s}{\partial q_s} + \frac{\partial \eta_s}{\partial p_s}\right)\\ =&\frac{\partial \dot{\xi}_s}{\partial q_s} - \frac{\partial \xi_s}{\partial q_k}\frac{\partial h_k}{\partial q_s} - \frac{\partial \xi_s}{\partial p_k}\frac{\partial g_k}{\partial q_s} + \frac{\partial \dot{\eta}_s}{\partial p_s}\\ &- \frac{\partial \eta_s}{\partial q_k}\frac{\partial h_k}{\partial p_s} - \frac{\partial \eta_s}{\partial p_k}\frac{\partial g_k}{\partial p_s}.\end{aligned} \tag{21}$$

Furthermore, (17) and (18) allow us to show that

$$\begin{aligned}&E^0\left(\frac{\partial h_s}{\partial q_s} + \frac{\partial g_s}{\partial p_s}\right)\\ =&\frac{\partial E^0(h_s)}{\partial q_s} - \frac{\partial \xi_s}{\partial q_k}\frac{\partial h_k}{\partial q_s} - \frac{\partial \eta_s}{\partial q_k}\frac{\partial h_k}{\partial p_s}\\ &+ \frac{\partial E^0(g_s)}{\partial p_s} - \frac{\partial \xi_s}{\partial p_k}\frac{\partial g_k}{\partial q_s} - \frac{\partial \eta_s}{\partial p_k}\frac{\partial g_k}{\partial p_s}.\end{aligned} \tag{22}$$

Inserting determining equation (6) into Eqs.(21) and (22) one derives that the right-hand sides of them are equal, so that we may put

$$E^0\left(\frac{\partial h_s}{\partial q_s} + \frac{\partial g_s}{\partial p_s}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \xi_s}{\partial q_s} + \frac{\partial \eta_s}{\partial p_s}\right]. \tag{23}$$

Using (23) in (16) yields

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = E^0\left(\frac{\partial h_s}{\partial q_s} + \frac{\partial g_s}{\partial p_s} + \dot{G}\right) = E^0(\psi). \tag{24}$$

It is sufficient that ψ is an invariant for the existence of the conserved quantity ϕ, which is the expected result.

3. One example

Consider a dynamical system with Lagrangian

$$L = \frac{1}{2}m\left(\dot{q}_1^2 + \dot{q}_2^2\right) + \frac{1}{2}J\dot{q}_3^2 - \frac{1}{2}kq_3^2, \tag{25}$$

where m, J and k are constants. The Hamiltonian of the system can be written as

$$\begin{aligned}H(t, \boldsymbol{q}, \boldsymbol{p}) =& p_s\dot{q}_s - L\\ =&\frac{1}{2m}(p_1^2 + p_2^2) + \frac{1}{2J}p_3^2 + \frac{1}{2}kq_3^2.\end{aligned} \tag{26}$$

Using the formulae (3) and (4) one can obtain the Hamiltonian canonical equations in the form

$$\begin{aligned}\dot{q}_1 =&\frac{1}{m}p_1 = h_1,\\ \dot{q}_2 =&\frac{1}{m}p_2 = h_2,\\ \dot{q}_3 =&\frac{1}{J}p_3 = h_3,\\ \dot{p}_3 =&-kq_3 = g_3.\end{aligned} \tag{27}$$

The determining equations of the momentum-department symmetry for the formula (27) under the infinitesimal generators $\xi_s = \xi_s(t, \boldsymbol{q}, \boldsymbol{p})$, $\eta_s = \eta_s(t, \boldsymbol{q}, \boldsymbol{p})$ is

$$\begin{aligned}&\dot{\xi}_1 = \frac{1}{m}\eta_1, \quad \dot{\xi}_2 = \frac{1}{m}\eta_2, \quad \dot{\xi}_3 = \frac{1}{J}\eta_3,\\ &\dot{\eta}_1 = 0, \quad \dot{\eta}_2 = 0, \quad \dot{\eta}_3 = -k\xi_3.\end{aligned} \tag{28}$$

For Eq.(28) there is solution as follows:

$$\begin{aligned}\xi_1 =&\frac{c_1}{2m}t^2 + \frac{c_2}{m}t + c_5,\\ \xi_2 =&\frac{c_3}{2m}t^2 + \frac{c_4}{m}t + c_6, \quad \xi_3 = 0,\\ \eta_1 =&c_1t + c_2, \quad \eta_2 = c_3t + c_4, \quad \eta_3 = 0.\end{aligned} \tag{29}$$

Substituting Eq.(27) into Eq.(12) yields

$$\dot{G} = \psi. \tag{30}$$

In Eq.(30), when taking

$$\psi = \frac{\mathrm{d}}{\mathrm{d}t}(q_1p_2) - \frac{\mathrm{d}}{\mathrm{d}t}(q_2p_1), \tag{31}$$

then $E^1(\psi) = 0$, and one can find a gauge function such that

$$G = q_1p_2 - q_2p_1. \tag{32}$$

Inserting generators (29) and gauge function (32) into Eq.(13) leads to a non-Noether conserved quantity in the form

$$\begin{aligned}\phi =&\frac{c_1}{2m}t^2p_2 + \frac{c_2}{m}tp_2 + c_5p_2\\ &- \frac{c_3}{2m}t^2p_1 - \frac{c_4}{m}tp_1 - c_6p_1\\ &+ c_1tq_1 + c_2q_1 - c_3tq_2 + c_4q_2 = \text{const}.\end{aligned} \tag{33}$$

When one takes $c_1 = c_2 = c_3 = c_4 = 0$, generators (29) give

$$\xi_1 = c_5, \quad \xi_2 = c_6, \quad \xi_3 = 0; \quad \eta_1 = \eta_2 = \eta_3 = 0, \quad (34)$$

then conserved quantity (33) has the following form:

$$\phi = c_5 p_1 + c_6 p_2 = \text{const.} \quad (35)$$

Equation (35) represents the momentum conservation of the system along the tract of the q_1 and q_2 respectively. Therefore, we obtain a conserved quantity associated with the Lie symmetry of the Hamiltonian system (27). It is a non-Noether conserved quantity.

We have shown that the Lie symmetries can lead directly to the non-Noether conserved quantities; these symmetries are different from Noether symmetries of the dynamical systems, and the gauge function of the Lie symmetries is determined both by Eq.(12) and choosing invariant ψ of the symmetry group before the conserved quantities are given with (13); nevertheless, the conserved quantities of the Noether symmetries are obtained with the help of Noether theorem. The conserved quantities of Lie symmetries are also different from the usual one of Lie symmetries of the dynamical systems; the gauge function and the conserved quantities of the Lie symmetries are obtained using the structural equation and the formula of conserved quantities associated with Noether symmetry, respectively.

References

[1] Noether A E 1918 *Nachr. Akad. Wiss. Göttingen Math. Phys.* KI, **II** 235

[2] Djukić Dj S and Vujanović B 1975 *Acta Mech.* **23** 17

[3] Vujanović B 1978 *Int. J. Non-Linear Mech.* **13** 185

[4] Li Z P 1981 *Acta Phys. Sin.* **30** 1659 (in Chinese)

[5] Li Z P 1993 *Classical and Quantum of Dynamical Constraint Systems and Their Symmetric Properties* (Beijing: Beijing Polytechnic University Press) (in Chinese)

[6] Mei F X 1993 *Sci. Chin.* **23** 1456

[7] Sarlet W, Cantrijn F and Crampint M 1987 *J. Phys. A: Math Gen.* **20** 1365

[8] Wafo Soh C and Mahomed F M 1999 *Class. Quantum Grav.* **16** 3553

[9] Sléphane Fay 2001 *Class. Quantum Grav.* **18** 4863

[10] Li Z P and Long Z W 1999 *J. Phys. A: Math. Gen.* **32** 6391

[11] Lutzky M 1979 *J. Phys. A: Math. Gen.* **12** 973

[12] Abraham-Shranner B, Govinder K S and Leach P G L 1995 *Phys. Lett.* A **203** 169

[13] Lakshmanan M and Senthil Velan M 1992 *J. Math. Phys.* **33** 4068

[14] Muñoz Masqué J and Engenía Rasado María M 2002 *J. Phys. A: Math Gen.* **35** 2013

[15] Me F X 1999 *Applications of Lie Groups and Lie Algebras to Constrained Mechanical System* (Beijing: Science Press) (in Chinese)

[16] Zhao Y Y and Mei F X 1993 *Adv. Mech.* **23** 360 (in Chinese)

[17] Fu J L and Chen L Q 2004 *Mech. Res. Commun.* **31** 9

[18] Hojman S A 1992 *J. Phys. A: Math Gen.* **25** L291

[19] Luo S K, Guo Y X and Mei F X 2004 *Acta Phys. Sin.* **53** 1271 (in Chinese)

[20] Gonzáler-Cascón F 1994 *J. Phys. A: Math. Gen.* **27** L59

[21] Lutzky M 1995 *J. Phys. A: Math. Gen.* **28** L637

[22] Fu J L and Chen L Q 2004 *Chin. Phys.* **13** 287

[23] Luo S K, Jia L Q and Cai J L2003 *Chin. Phys.* **12** 841

[24] Luo S K 2004 *Acta Phys. Sin.* **53** 5 (in Chinese)

[25] Zhang Y 2002 *Acta Phys. Sin.* **51** 2423 (in Chinese)

[26] Qiao Y F, Zhang Y L and Han G C 2003 *Acta Phys. Sin.* **52** 1051 (in Chinese)

[27] Zhang Y 2003 *Acta Phys. Sin.* **52** 1326 (in Chinese)

[28] Mei F X 2002 *Chin. Sci. Bull.* **47** 1544 (in Chinese)

[29] Mei F X 2003 *Acta Phys. Sin.* **52** 1048 (in Chinese)

[30] Fu J L and Chen L Q 2003 *Phys. Lett.* A **317** 255

[31] Lutzky M 1979 *Phys. Lett.* A **72** 86