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The novel multi-solitary wave solution to the fifth-order KdV equation*

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By using Hirota's method, the novel multi-solitary wave solutions to the fifth-order KdV equation are obtained. Furthermore, various new solitary wave solutions are also derived by a reconstructed bilinear Bäcklund transformation.

Keywords: solitary wave, bilinear method, Bäcklund transformation, fifth-order KdV equation
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1. Introduction

Among the direct algebraic methods applicable to nonlinear evolution equations for the investigation of their integrability there is one which has proved particularly powerful: the bilinear method, which was first proposed by Hirota.[1] In this method, the equation under consideration is transformed into a bilinear equation by the dependent variable transformation. Then the multi-solitary wave solutions can be constructed, which should prove to be satisfactory to the bilinear equation. The use of the Hirota bilinear transformation in the search for exact solutions to soliton equation now has been well established.[2,3] Recently we have presented an extended bilinear method to obtain the novel N-solitary wave solution to some nonlinear evolution equations.[4,5] The method also can be applied to (2+1)-dimensional integrable systems like the KP and the breaking soliton equations.[6,7] In addition, other analytical methods combining with Hirota's method have been developed. Some special types of single solitary wave solutions and multi-soliton-like solutions are constructed in higher space-dimensional solitary equations.[8−13] Not only has the Hirota method produced most of the known families of closed-form solutions, but it has also related to the other major integrability features of soliton systems, such as Bäcklund transformation.[14] A key element is often the disclosure of a bilinear Bäcklund transformation acting at the level of a suitable Hirota representation of the nonlinear evolution equations. It provides a multi-solitary wave solution to a wide class of nonlinear equations in a unified way.

In dealing with the KdV equation by the inverse scattering method, one has

$$u_t + 120u^2u_x + 20(u_{xxx} + 2u_xu_{xx}) + u_{xxxxx} = 0, \quad (1)$$

which is also called the second member of the hierarchy of the KdV equation.[15]

Concerning Eq.(1), much research has been conducted.[16−20] In Ref.[16], Sawada *et al* applied Hirota's bilinear transformation method and presented the exact N-solitary wave solutions to Eq.(1).

In the present paper, we apply it to the fifth-order KdV equation and acquire novel N-solitary wave solutions. Further, we reconstruct a given bilinear Bäcklund transformation and find more exact novel multi-solitary wave solutions to the equation. It is noted that the solutions that we find have singularity. This case occurs in other equations too, such as KdV, KP, Boussineq and Toda Lattice equations etc.[21] It may exhibit the particle or wave nature closely related to the fundamentals of quantum theory.[22]

2. The novel N-solitary wave solutions to the fifth-order KdV equation

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Let us now construct the exact N-solitary wave solutions to Eq.(1). By the dependent variable transformation

$$u = 2(\log f)_{xx}, \tag{2}$$

Equation (1) is rewritten as

$$D_x[D_x(D_t+D_x^5)f\cdot f]f^2 - 5D_x(D_x^4 f\cdot f)\cdot(D_x^2 f\cdot f) = 0, \tag{3}$$

where the operators D_x and D_t are defined as

$$\begin{aligned} &D_x^m D_t^n f\cdot g \\ =&(\partial_x-\partial_{x'})^m(\partial_t-\partial_{t'})^n f(x,t)g(x',t')|_{x'=x,t'=t}. \end{aligned}$$

Equation (3) is not in the bilinear form as it stands. However, if we introduce an independent variable τ and impose a subsidiary condition

$$D_x(D_\tau + D_x^3)f\cdot f = 0, \tag{4}$$

we can reduce Eq.(3) to

$$\left[D_x(D_t+D_x^5) - \frac{5}{3}D_\tau + D_x^3\right]f\cdot f = 0, \tag{5}$$

in which the boundary condition $u\to 0$ at $|\ x\ |\to\infty$ has been used.

In what follows, we only consider the simultaneous bilinear equations. By using the perturbation technique, f can be expanded as a formal series

$$f = 1 + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \varepsilon^3 f^{(3)} + \cdots, \tag{6}$$

and equating coefficients with the same powers of ε yields

$$f_{x\tau}^{(1)} + f_{xxxx}^{(1)} = 0, \tag{7a}$$

$$f_{tx}^{(1)} + f_{xxxxxx}^{(1)} - \frac{5}{3}(f_{\tau\tau}^{(1)} + f_{\tau xxx}^{(1)}) = 0, \tag{7b}$$

$$f_{x\tau}^{(2)} + f_{xxxx}^{(2)} = -\frac{1}{2}D_x(D_\tau + D_x^3)f^{(1)}\cdot f^{(1)}, \tag{7c}$$

$$\begin{aligned} &f_{tx}^{(2)} + f_{xxxxxx}^{(2)} - \frac{5}{3}(f_{\tau\tau}^{(1)} + f_{\tau xxx}^{(1)}) \\ =&-\frac{1}{2}\left[D_x(D_t+D_x^5) - \frac{5}{3}D_\tau(D_\tau+D_x^3)\right]f^{(1)}\cdot f^{(1)}, \end{aligned} \tag{7d}$$

and so on.

The novel multi-solitary wave solutions to the fifth-order KdV equation are found by assuming $f^{(1)}$ to be of the form

$$f^{(1)} = \sum_{j=1}^{n}\eta_j \mathrm{e}^{\xi_j}, \tag{8a}$$

where

$$\xi_j = w_j t + k_j x + p_j\tau + \xi_j^{(0)}, \tag{8b}$$

$$\eta_j = \alpha_j t + \beta_j x + \gamma_j\tau + \delta_j, \tag{8c}$$

and k_j, ω_j, $\xi_j^{(0)}$, p_j, α_j, β_j, γ_j, δ_j are constants, provided that the series (6) is truncated.

In computing, we have used the following formula:

$$\begin{aligned} &D_t^s D_x^r(\eta_1\eta_2\cdots\eta_l \mathrm{e}^{\xi_1}\cdot\eta_{l+1}\eta_{l+2}\cdots\eta_m \mathrm{e}^{\xi_2}) \\ =&\mathrm{e}^{\xi_1+\xi_2}\prod_{p=1}^{l}(\eta_p + \alpha_p\partial_{k_1} + \beta_p\partial_{\omega_1}) \\ &\times\prod_{q=l+1}^{m}(\eta_q + \alpha_q\partial_{k_2} + \beta_q\partial_{\omega_2}) \\ &\times(k_1-k_2)^r(\omega_1-\omega_2)^s, \end{aligned} \tag{9}$$

where $\partial_{k_j} = \dfrac{\partial}{\partial k_j}$, $\partial_{\omega_j} = \dfrac{\partial}{\partial\omega_j}$.

For example, a one-solitary wave solution may be taken as

$$f^{(1)} = \eta_1 \mathrm{e}^{\xi_1}, \tag{10}$$

and by solving Eq.(7), we obtain

$$f^{(2)} = -\frac{\beta_1^2}{4k_1^2}\mathrm{e}^{2\xi_1}, \tag{11}$$

where

$$\begin{aligned} &\omega_1 = -k_1^5, \quad p_1 = -k_1^3, \\ &\alpha_1 = -5k_1^5\beta_1, \quad \gamma_1 = -3\beta_1 k_1^2, \end{aligned} \tag{12}$$

$$f^{(n)} = 0 \qquad (n\geq 3). \tag{13}$$

With this result, taking $\varepsilon = 1$, a one-solitary wave solution to Eq.(1) is given by

$$u = 2\left[\ln\left(1+\eta_1\mathrm{e}^{\xi_1} - \frac{-\beta_1^2}{4k_1^2}\mathrm{e}^{2\xi_1}\right)\right]_{xx}. \tag{14}$$

Similarly, a two-solitary wave solution can be deduced. By taking

$$f^{(1)} = \eta_1\mathrm{e}^{\xi_1} + \eta_2\mathrm{e}^{\xi_2}, \tag{15a}$$

it follows from Eq.(7) that

$$\begin{aligned} f^{(2)} =& -\frac{\beta_1^2}{4k_1^2}\mathrm{e}^{2\xi_1} - \frac{\beta_2^2}{4k_2^2}\mathrm{e}^{2\xi_2} \\ &+\left[\left(\frac{(k_1-k_2)^2}{(k_1^2+k_2^2)^2}\eta_1\eta_2 - \frac{4k_1(k_1-k_2)\beta_2}{(k_1+k_2)^3}\eta_1\right.\right. \\ &+\frac{4k_2(k_1-k_2)\beta_1}{(k_1+k_2)^3}\eta_2 \\ &\left.\left.+\frac{4\beta_1\beta_2(k_1^2-4k_1k_2+k_2^2)}{(k_1+k_2)^4}\right)\right]\mathrm{e}^{\xi_1+\xi_2}, \end{aligned} \tag{15b}$$

$$f^{(3)} = -\left[\frac{\beta_2^2(k_1-k_2)^4}{4k_2^2(k_1+k_2)^4}\eta_1 - 2\beta_1\beta_2^2\frac{(k_1-k_2)^3}{k_2(k_1+k_2)^5}\right]e^{\xi_1+2\xi_2} - \left[\frac{\beta_1^2(k_2-k_1)^4}{4k_1^2(k_1+k_2)^4}\eta_2 + 2\beta_2\beta_1^2\frac{(k_1-k_2)^3}{k_1(k_1+k_2)^5}\right]e^{\xi_2+2\xi_1}, \quad (15c)$$

$$f^{(4)} = \frac{\beta_1^2\beta_2^2(k_1-k_2)^8}{16k_1^2k_2^2(k_1+k_2)^8}e^{2\xi_1+2\xi_2}, \quad (15d)$$

$$f^{(n)} = 0 \qquad n \geq 5. \quad (15e)$$

Generally, the novel N-solitary wave solutions have the structure

$$f = \sum_{\mu=0,1,2}\left\{\prod_{j=1}^{n}\left(\frac{\beta_j}{2ik_j}\right)^{\mu_j(\mu_j-1)}(\beta_j\partial_{k_j}+\delta_j)^{\mu_j(2-\mu_j)} \times \exp\left(\sum_{j=1}^{n}\mu_j\xi_j + \sum_{1\leq j<l}^{n}\mu_j\mu_l A_{jl}\right)\right\}, \quad (16a)$$

where

$$e^{A_{jl}} = \left(\frac{k_j-k_l}{k_j+k_l}\right)^2, \quad (16b)$$

$$\omega_j = -k_j^5, \quad (16c)$$

and the first sum is taken over all possible combinations of $\mu = 0, 1, 2$.

3. Other multi-solitary wave solutions to the fifth-order KdV equation

An advantage of the bilinear Bäcklund transformation is that the transformed equations are linear with respect to each dependent variable. In this section, we shall show how a reconstructed bilinear Bäcklund transformation for the Eq.(1) may be used to deduce multi-solitary wave solutions. It is also significant to note that the form of multi-solitary wave solution is similar to the one obtained by Hirota's method. As to the Bäcklund transformation in bilinear form to the fifth-order KdV equation, Hirota has given three equations of Bäcklund transformation.[2] In Ref.[23], a simple form has been presented as

$$(D_x^2-\lambda)f\cdot g = 0, \quad (17a)$$

$$(D_t+15\lambda^2D_x+D_x^5)f\cdot g = 0, \quad (17b)$$

which are related to the pair to solutions of the fifth-order KdV equation.

In order to give some new multi-solitary wave solution to Eq.(1), we reconstruct the Bäcklund transformation mentioned above, our aim being to derive novel exact solutions. We make the following transformation:

$$f \to e^{\xi}f, \quad (18a)$$

$$g \to e^{\eta}g, \quad (18b)$$

where $\xi = \omega_1 t + k_1 x + \xi^{(0)}$, $\eta = \sigma_1 t + h_1 x + \eta^{(0)}$, $\xi^{(0)}$, $\eta^{(0)}$ are arbitrary constants. Expanding f and g as power series in a parameter ε as above

$$f = 1 + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \varepsilon^3 f^{(3)} + \cdots, \quad (19a)$$

$$g = 1 + \varepsilon g^{(1)} + \varepsilon^2 g^{(2)} + \varepsilon^3 g^{(3)} + \cdots, \quad (19b)$$

and using the following bilinear operator identity:

$$D_t^m D_x^n e^{\xi}f\cdot e^{\eta}g = e^{\xi+\eta}(D_t+\omega_1-\sigma_1)^m(D_x+k_1-h_1)^n f\cdot g,$$

Eqs.(17a) and (17b) are transformed into

$$\{[D_x+(h_1-k_1)]^2-\lambda\}f\cdot g = 0, \quad (20a)$$

$$\{D_t+(\sigma_1-\omega_1)+15\lambda_1-k_1+(h_1-k_1)^5\}f\cdot g = 0. \quad (20b)$$

respectively. Equating the constant terms, we have

$$(h_1-k_1)^2 = \lambda, \quad (21a)$$

$$\sigma_1-\omega_1+15\lambda^2(h_1-k_1)+(h_1-k_1)^5 = 0. \quad (21b)$$

Thus we arrive at the reconstructed Bäcklund transformation of the fifth-order KdV equation as follows:[24]

$$(D_x^2+2kD_x)f\cdot g = 0, \quad (22a)$$

$$(D_t+D_x^5+5kD_x^4+10k^2D_x^3)f\cdot g = 0, \quad (22b)$$

respectively, where k is a parameter.

From Eqs.(22a) and (22b), we can recover various exact novel solutions to the fifth-order KdV equation. Substituting Eq.(19) into Eq.(22) and equating the terms with the same powers in ε, we have in the order of ε

$$f_{xx}^{(1)}+2kf_x^{(1)}+g_{xx}^{(1)}-2kg_{xx}^{(1)} = 0, \quad (23a)$$

$$f_t^{(1)}+f_{xxxxx}^{(1)}+5kf_{xxxx}^{(1)}+10k^2f_{xxx}^{(1)} + 10k^3f_{xx}^{(1)}+20k^4f_x^{(1)}-g_t^{(1)}-g_{xxxxx}^{(1)}+5kg_{xxxx}^{(1)} - 10k^2g_{xxx}^{(1)}+10k^3g_{xx}^{(1)}-20k^4g_x^{(1)} = 0, \quad (23b)$$

and in the order of ε^2,

$$f_{xx}^{(2)}+2kf_x^{(2)}+g_{xx}^{(2)}-2kg_x^{(2)} = -(D_x^2+2kD_x)f^{(1)}\cdot g^{(1)}, \quad (24a)$$

$$
\begin{aligned}
&f_t^{(2)} + f_{xxxxx}^{(2)} + 5kf_{xxxx}^{(2)} + 10k^2 f_{xxx}^{(2)} \\
&+ 10k^3 f_{xx}^{(2)} + 20k^4 f_x^{(2)} - g_t^{(2)} - g_{xxxxx}^{(2)} + 5kg_{xxxx}^{(2)} \\
&- 10k^2 g_{xxx}^{(2)} + 10k^3 g_{xx}^{(2)} - 20k^4 g_x^{(2)} \\
=&-(D_t + D_x^5 + 5kD_x^4 + 10k^2 D_x^3 \\
&+ 10k^3 D_x^2 + 20k^4 D_x) f^{(1)} \cdot g^{(1)}, \qquad (24b)
\end{aligned}
$$

and so on.

Equations in the higher order of ε are linear nonhomogeneous equations; we can find solutions solving these equations successively.

In what follows, we distinguish parameter k from k_1 and k_2, so as to acquire various exact solutions.

1. $k = -\frac{1}{2}k_1$

We start out, by choosing the simplest nontrivial solution as follows:

$$g^{(1)} = \mathrm{e}^{\xi_1}, \qquad (25)$$

where $\xi_1 = k_1 x + \omega_1 t + \xi_1^{(0)}$, k_1 and $\xi_1^{(0)}$ are arbitrary constants.

Substituting Eq.(25) into Eq.(23) gives

$$f^{(1)} = \eta_1 \mathrm{e}^{\xi_1}, \qquad (26a)$$

where

$$\eta_1 = 10k_1^5 t - 2k_1 x + \gamma_1, \qquad (26b)$$

γ_1 is an arbitrary constant,

$$\omega_1 = -k_1^5. \qquad (26c)$$

Substituting Eqs.(25) and (26) into Eq.(24) yields

$$f^{(2)} = -\mathrm{e}^{2\xi_1}. \qquad (27)$$

As all higher order terms in Eq.(22) can be taken to be zero, we have a solution to Eq.(19a),

$$f = 1 + \eta_1 \mathrm{e}^{\xi_1} - 2\mathrm{e}^{2\xi_1}. \qquad (28)$$

Similarly, if we choose $g^{(1)} = \eta_1 \mathrm{e}^{\xi_1}, g^{(2)} = -\mathrm{e}^{\xi_1}$ and put them into Eq.(26), we arrive at

$$f^{(1)} = \left(\frac{1}{2}\eta_1^2 - \eta_1 - 40k_1^5 t\right)\mathrm{e}^{\xi_1}, \qquad (29)$$

$$f^{(2)} = \left(-\frac{1}{2}\eta_1^2 - \eta_1 - 40k_1^5 t - 2\right)\mathrm{e}^{2\xi_1}, \qquad (30)$$

$$f^{(3)} = -\mathrm{e}^{3\xi_1}, \qquad (31)$$

$$f^{(i)} = 0, \quad i \geq 4. \qquad (32)$$

Therefore series (19a) is truncated and becomes

$$
\begin{aligned}
f =& 1 + \left(\frac{1}{2}\eta_1^2 - \eta_1 - 40k_1^5 t\right)\mathrm{e}^{\xi_1} \\
&- \left(\frac{1}{2}\eta_1^2 + \eta_1 + 40k_1^5 t + 2\right)\mathrm{e}^{2\xi_1} - \mathrm{e}^{3\xi_1}. \qquad (33)
\end{aligned}
$$

Substituting Eq.(33) into Eq.(2), we obtain the explicit solitary wave solutions.

2. $k = -\frac{1}{2}k_2$

Assuming that $g^{(1)} = \mathrm{e}^{\xi_1}$, we have

$$f = 1 - \frac{k_1 + k_2}{k_1 - k_2}\mathrm{e}^{\xi_1} + \mathrm{e}^{\xi_2} - \frac{k_1 - k_2}{k_1 + k_2}\mathrm{e}^{\xi_1 + \xi_2}, \qquad (34)$$

where $\xi_2 = k_2 x + \omega_2 t + \xi_2^{(0)}$ and $\omega_2 = -k_2^5$, k_2 and $\xi_2^{(0)}$ are arbitrary constants.

In order to obtain a new two-solitary wave solution, we start with the following Hirota standard-form solution of Eq.(24):

$$g^{(1)} = \mathrm{e}^{\xi_1} + \mathrm{e}^{\xi_2}, \qquad (35)$$

$$g^{(2)} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \mathrm{e}^{\xi_1 + \xi_2}, \qquad (36)$$

where the meanings of ξ_1 and ξ_2 have been mentioned above.

Substituting Eqs.(17) and (35) into Eqs.(23) and (24), and solving the resulting equation, we have

$$f^{(1)} = -\frac{k_1 + k_2}{k_1 - k_2}\mathrm{e}^{\xi_1} + \eta_2 \mathrm{e}^{\xi_2}, \qquad (37a)$$

where

$$\eta_2 = 10k_2^5 t - 2k_2 x + \gamma_2, \qquad (37b)$$

γ_2 is an arbitrary constant,

$$\omega_2 = -k_2^5, \quad \omega_1 = -k_1^5, \qquad (27c)$$

$$f^{(2)} = \left[\frac{k_2 - k_1}{k_1 + k_2}\eta_2 - \frac{8k_1 k_2}{(k_1 + k_2)^2}\right]\mathrm{e}^{\xi_1 + \xi_2} - \mathrm{e}^{2\xi_2}, \qquad (38)$$

and

$$f^{(3)} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^3 \mathrm{e}^{\xi_1 + 2\xi_2}. \qquad (39)$$

$$f^{(i)} = 0, \quad i \geq 4. \qquad (40)$$

If we replace $f^{(1)} \to g^{(1)}$, $f^{(2)} \to g^{(2)}$, $f^{(3)} \to g^{(3)}$, similarly, we can find

$$f^{(1)} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \mathrm{e}^{\xi_1} + \left(\frac{1}{2}\eta_2^2 - \eta_2 - 40k_2^5 t\right)\mathrm{e}^{\xi_2}, \qquad (41)$$

$$
\begin{aligned}
f^{(2)} =& -\left(\frac{1}{2}\eta_2^2 + \eta_2 + 40k_2^5 t + 2\right)\mathrm{e}^{2\xi_2} \\
&+ \left[\frac{1}{2}\eta_2^2 + \frac{-k_1^2 + 8k_1 k_2 + k_2^2}{k_1^2 - k_2^2}\eta_2 - 40k_2^5 t\right. \\
&\left. - 8k_1 k_2 \frac{k_1^2 - 4k_1 k_2 + k_2^2}{(k_1^2 - k_2^2)^2 (k_1^2 + k_2^2)^2}\right]\mathrm{e}^{\xi_1 + \xi_2}, \qquad (42)
\end{aligned}
$$

$$f^{(3)} = -\mathrm{e}^{3\xi_2} - \left[\frac{(k_1-k_2)^2}{2(k_1+k_2)^2}\eta_2^2 + \frac{k_1^3+7k_1^2k_2-9k_1k_2^2+k_2^3}{(k_1+k_2)^3}\eta_2 + k_2^5\frac{40(k_1-k_2)^2}{(k_1+k_2)^2}t + \frac{2k_1^4+8k_2k_1^3+28k_1^2k_2^2+8k_1k_2^3+2k_2^4}{(k_1+k_2)^4}\right]\mathrm{e}^{\xi_1+2\xi_2}, \tag{43}$$

$$f^{(4)} = -\left(\frac{k_1-k_2}{k_1+k_2}\right)^4 \mathrm{e}^{\xi_1+3\xi_2}, \tag{44}$$

$$f^{(i)} = 0, \quad i \geq 5. \tag{45}$$

Substituting Eqs.(41–45) into Eq.(19a) and then inserting the resulting equation into Eq.(2), we can obtain an exact multi-solitary wave solution.

On the basis of the above results, it is possible to derive more novel solitary wave solutions in a similar manner. However, the calculations are so tedious that we are unable to show any solution associated with the N-solitary wave solutions here.

In summary, by using the extended Hirota bilinear method and bilinear Bäcklund transformation method to the fifth-order KdV equation, we have shown various novel solitary wave solutions. The common singularity characteristic of all the novel solitary wave solutions may reflect a certain physical meaning.

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