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Reduced-order observer of singular system with unknown inputs*

Wu Jian-Rong(吴健荣)†

Department of Applied Mathematics, University of Science and Technology of Suzhou, Suzhou 215009, *China*

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For the general discrete singular system with unknown inputs, a new reduced-order observer is designed. A numerical example is presented to illustrate the design algorithms.

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1. Introduction

In practice, dynamic systems are often affected by immeasurable disturbances which can be modelled as unknown inputs. To analyse these systems, the design of state estimation or observation is often necessary. Moreover, because of its nice properties the unknown input observer has found wide applicability in the design of a robust observer, decentralized control, and so on.[1] Therefore, the problem of state observation with unknown inputs has received considerable attention (see Refs.[2–7]).

Due to their applications in describing non-causal phenomena, which arise naturally in robotics, economic, electric and chemical systems, the analysis and design of singular systems has received great attention in the last few years.[8−10] It is shown that the solution of a singular system includes differential terms, so that singular systems are very sensitive to slight changes of the inputs, and the presence of immeasurable disturbances or unknown inputs is detrimental to the design of observers. This fact justifies the importance of the observer design for singular systems in the presence of unknown inputs which has led people to begin to study the state estimation for singular systems with unknown inputs recently. However, compared with the normal systems, for singular systems, the achievements have been few. Moreover, most of the existing contexts are about continuous systems. In Refs.[11–15] only square singular systems have been considered under the regularity condition. Additionally, in Refs.[13] and [14], some conditions of existence of observers are not with the original systems, and hence, are inconvenient to justify. To avoid abundant computation, a method for the design of reduced-order observers is presented in Ref.[16]. However, it is not difficult to find that the observer there may not be stable if the related generalized inverses are just normal inverses: that is, if the corresponding matrices are non-singular.

In the present paper, we propose to investigate the problem of state observation of non-square discrete singular systems with unknown inputs. In section 2, a new method for the design of reduced-order observers is presented. The conditions assumed here are slightly different from those in Ref.[16], and are more convenient to justify. In section 3, a numerical example is presented to illustrate the design algorithms.

2. Reduced-order observer design

Consider the singular discrete system

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bu(k) + Dd(k), \\ y(k+1) &= Cx(k+1) + Gu(k+1), \end{aligned} \tag{1}$$

where E, $A \in R^{m\times n}$, $B \in R^{m\times s}$, $D \in R^{m\times q}$, $C \in R^{p\times n}$ and $G \in R^{p\times s}$ are known constant matrices; $x(k)$, $y(k)$, $u(k)$ and $d(k)$ are the state, the

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†Corresponding author. E-mail: jrwu1643@sina.com.cn

output, the known input and the unknown input vectors, respectively.

Without loss of generality, it is assumed that C is of full row rank and D is of full column rank; also it is assumed that rank $E < n \leq m$ and $q < p$.

In the sequel we make the following assumptions:

A1

$$\text{rank}\begin{bmatrix} E & D \\ C & 0 \end{bmatrix} = n + q;$$

A2

$$\text{rank}\begin{bmatrix} sE - A & D \\ C & 0 \end{bmatrix} = n + q, \quad s \in C.$$

These assumptions are slightly different from those in Ref.[16]. When D=0, they generalize the condition of the observability[11] in the sense of Rosenbrock of square (i.e. $m = n$) singular systems. Additionally, in the case that $m = n$ and $E = I$, assumption A1 can be rewritten as rank $\begin{bmatrix} I & D \\ C & 0 \end{bmatrix} = n + q$, or equivalently, rank CD= rank$D = q$, which is the condition generally assumed in the non-singular system with unknown inputs.

To guarantee the uniqueness of solution of (1) the assumption of the regularity of the matrix pencil $sE - A$ is introduced commonly when E and A are square. This condition is of course not fit for non-square singular systems. Fortunately, it is shown in Ref.[17] that the above condition may be left out in the case that rank $\begin{bmatrix} sE - A \\ C \end{bmatrix} = n$, which can be implied by assumption A2.

The underlying idea to the design of the desired observer is to separate the state manifold into three sub-state manifolds by applying similarity transformations. One of the sub-states is not directly influenced by the unknown input; meanwhile, the others can be directly observed by the output $y(k)$. In fact, we have the following theorem.

Theorem 1 Suppose that A1 holds, then there exists a transformation

$$x(k) = J\begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}, \tag{2}$$

so that the system (1) can be rewritten as

$$\begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} & \bar{E}_{13} \\ \bar{E}_{21} & \bar{E}_{22} & 0 \\ \bar{E}_{31} & \bar{E}_{32} & I_{n-p} \end{bmatrix}\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} \\ \bar{A}_{21} & \bar{A}_{22} & \bar{A}_{23} \\ \bar{A}_{31} & \bar{A}_{32} & \bar{A}_{33} \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \bar{B}_3 \end{bmatrix} u(k) + \begin{bmatrix} I_q \\ 0_r \\ 0_{n-p} \end{bmatrix} d(k), \tag{3}$$

$$y(k+1) = \begin{bmatrix} y_1(k+1) \\ y_2(k+1) \end{bmatrix} = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} + Gu(k+1), \tag{4}$$

where

$$\begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} & \bar{E}_{13} \\ \bar{E}_{21} & \bar{E}_{22} & 0 \\ \bar{E}_{31} & \bar{E}_{32} & I_{n-p} \end{bmatrix} = M\ H\ E\ J, \quad \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} \\ \bar{A}_{21} & \bar{A}_{22} & \bar{A}_{23} \\ \bar{A}_{31} & \bar{A}_{32} & \bar{A}_{33} \end{bmatrix} = M\ H\ A\ J, \quad \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \bar{B}_3 \end{bmatrix} = M\ H\ B;$$

M, H, E and J are constant non-singular matrices of appropriate dimensions defined later.

Proof First, noting that D is of full column rank and C is of full row rank, then there exist two non-singular matrices $H \in R^{m\times m}$ and $J \in R^{n\times n}$, so that

$$H\ D = \begin{bmatrix} I_q \\ 0_r \\ 0_{n-p} \end{bmatrix} \quad \text{and} \quad C\ J = \begin{bmatrix} I_p & 0 & 0 \\ 0 & I_{p-q} & 0 \end{bmatrix},$$

where $r = m - n + p - q$.

Now, premultiplying (1) by the matrix H and applying the transformation (2), the system (1) can be rewritten as

$$\begin{bmatrix} * & * & * \\ * & * & E_{23} \\ * & * & E_{33} \end{bmatrix} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = H\ A\ J \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + H\ Bu(k) + \begin{bmatrix} I_q \\ 0_r \\ 0_{n-p} \end{bmatrix} d(k), \tag{5}$$

$$y(k+1) = \begin{bmatrix} y_1(k+1) \\ y_2(k+1) \end{bmatrix} = \begin{bmatrix} I_p & 0 & 0 \\ 0 & I_{p-q} & 0 \end{bmatrix} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} + Gu(k),$$

or

$$\begin{bmatrix} y_1(k+1) \\ y_2(k+1) \end{bmatrix} = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} + Gu(k+1),$$

where

$$\begin{bmatrix} * & * & * \\ * & * & E_{23} \\ * & * & E_{33} \end{bmatrix} = H\ E\ J.$$

Next, notice that

$$\begin{bmatrix} H & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} E & D \\ C & 0 \end{bmatrix} \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} HEJ & HD \\ CJ & 0 \end{bmatrix} = \begin{bmatrix} * & * & * & I_q \\ * & * & E_{23} & 0 \\ * & * & E_{33} & 0 \\ I_q & 0 & 0 & 0 \\ 0 & I_{p-q} & 0 & 0 \end{bmatrix}.$$

From A1, we know that $\operatorname{rank} \begin{bmatrix} E_{23} \\ E_{33} \end{bmatrix} = n-p$. Then there exists a non-singular matrix $L \in R^{(m-q)\times(m-q)}$, so that $L \begin{bmatrix} E_{23} \\ E_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ I_{n-p} \end{bmatrix}$.

Premultiply (5) by the non-singular matrix M=diag (I_q, L) to obtain

$$\begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} & \bar{E}_{13} \\ \bar{E}_{21} & \bar{E}_{22} & 0 \\ \bar{E}_{31} & \bar{E}_{32} & I_{n-p} \end{bmatrix} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = M\ H\ A\ J \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + M\ H\ B\ u(k) + \begin{bmatrix} I_q \\ 0_r \\ 0_{n-p} \end{bmatrix} d(k),$$

where

$$\begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} & \bar{E}_{13} \\ \bar{E}_{21} & \bar{E}_{22} & 0 \\ \bar{E}_{31} & \bar{E}_{32} & I_{n-p} \end{bmatrix} = M \begin{bmatrix} * & * & * \\ * & * & E_{23} \\ * & * & E_{33} \end{bmatrix}.$$

The proof ends.

The formula (4) tells us that $x_1(k+1)$ and $x_2(k+1)$ can be observed by $y(k+1)$ and $u(k+1)$ directly. So, to observe $x(k+1)$ we only need to design an observer for sub-state $x_3(k+1)$.

Theorem 2 Under the assumptions of A1 and A2, there exists an asymptotic stable observer of the form

$$\bar{x}_3(k+1) = N\bar{x}_3(k) + Ky(k) + Uu(k),$$
$$\hat{x}_3(k+1) = \bar{x}_3(k+1) + K_1 y(k+1) + K_3 u(k+1),$$

which provides an estimate of the sub-state $x_3(k+1)$ defined in (2).

Proof From (3), the following reduced order dynamical system of the sub-state $x_3(k)$ can be written down directly:

$$\begin{aligned} x_3(k+1) &= \bar{A}_{33}x_3(k) + g(k+1), \\ z(k+1) &= \bar{A}_{23}x_3(k), \end{aligned} \tag{6}$$

where

$$\begin{aligned} g(k+1) =& \bar{A}_{31}x_1(k) + \bar{A}_{32}x_2(k) - \bar{E}_{31}x_1(k+1) \\ & - \bar{E}_{32}x_2(k+1) + \bar{B}_3u(k), \\ z(k+1) =& \bar{E}_{21}x_1(k+1) + \bar{E}_{22}x_2(k+1) \\ & - \bar{A}_{21}x_1(k) - \bar{A}_{22}x_2(k) - \bar{B}_2u(k). \end{aligned} \tag{7}$$

By substituting (4) for $x_1(k)$ and $x_2(k)$ in (7), we obtain

$$\begin{aligned} g(k+1) =& \bar{A}_3(y(k) - Gu(k)) - \bar{E}_3(y(k+1) \\ & - Gu(k+1)) + \bar{B}_3u(k), \\ z(k+1) =& \bar{E}_2(y(k+1) - Gu(k+1)) \\ & - \bar{A}_2(y(k) - Gu(k)) - \bar{B}_2u(k), \end{aligned} \tag{8}$$

where $\bar{A}_2=[\bar{A}_{21} \quad \bar{A}_{22}]$, $\bar{A}_3=[\bar{A}_{31} \quad \bar{A}_{32}]$, $\bar{E}_3=[\bar{E}_{31} \quad \bar{E}_{32}]$, $\bar{E}_2=[\bar{E}_{21} \quad \bar{E}_{22}]$.

Obviously, both $g(k)$ and $z(k)$ are measurable. Meanwhile, subsystem (6) is separated from the unknown input $d(k)$.

Next, we will prove that subsystem (6) is also observable under the condition of A2. Indeed, we have

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sE - A & D \\ C & 0 \end{bmatrix} \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} s\bar{E}_{11} - \bar{A}_{11} & s\bar{E}_{12} - \bar{A}_{12} & s\bar{E}_{13} - \bar{A}_{13} & I_q \\ s\bar{E}_{21} - \bar{A}_{21} & s\bar{E}_{22} - \bar{A}_{22} & -\bar{A}_{23} & 0 \\ s\bar{E}_{31} - \bar{A}_{31} & s\bar{E}_{32} - \bar{A}_{32} & sI - \bar{A}_{33} & 0 \\ I_q & 0 & 0 & 0 \\ 0 & I_{p-q} & 0 & 0 \end{bmatrix},$$

which together with the assumption A2 implies that $\text{rank} \begin{bmatrix} sI - \bar{A}_{33} \\ \bar{A}_{23} \end{bmatrix} = n - p$, $s \in C$. That is, the subsystem (6) is observable.

Therefore, we can construct the following an asymptotic stable observer of order $n-p$ for the sub-state $x_3(k)$:

$$\begin{aligned} \hat{x}_3(k+1) =& \bar{A}_{33}\hat{x}_3(k) + \bar{K}(z(k+1) \\ & - \bar{A}_{23}\hat{x}_3(k)) + g(k+1), \end{aligned} \tag{9}$$

where $\bar{K} \in R^{(n-p)\times p}$ is selected by stabilizing the above observer.

From (8), we know

$$\begin{aligned} \hat{x}_3(k+1) =& N\hat{x}_3(k) + K_1y(k+1) - K_2y(k) \\ & + K_3u(k+1) + K_4u(k), \end{aligned} \tag{10}$$

where $K_1 = \bar{K}\bar{E}_2 - \bar{E}_3$, $K_2 = \bar{K}\bar{A}_2 - \bar{A}_3$, $K_3=(\bar{E}_3 - \bar{K}\bar{E}_2)G$, $K_4 = \bar{K}(\bar{A}_2G - \bar{B}_2) - \bar{A}_3G + \bar{B}_3$, $N = \bar{A}_{33} - \bar{K}\bar{A}_{23}$. Thus, to obtain $\hat{x}_3(k+1)$ we have to use the output and input values at time $k+1$. To circumvent this problem, we employ the following procedure.

Let

$$\bar{x}_3(k+1) = \hat{x}_3(k+1) - K_1y(k+1) - K_3u(k+1). \tag{11}$$

Then, from (10) we have

$$\bar{x}_3(k+1) = N\bar{x}_3(k) + Ky(k) + Uu(k), \tag{12}$$

where $K = NK_1 - K_2$, $U = NK_3 + K_4$.

Obviously, $\bar{x}_3(k+1)$ can be obtained without using the output and input values at time $k+1$. The proof ends.

By applying the observer (12) and back-tracing the different variable transformations, we can obtain the following theorem.

Theorem 3 Under the assumptions A1 and A2, the estimate of the state $x(k)$ of (1) is given by

$$\hat{x}(k) = J \begin{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \\ \hat{x}_3(k) \end{bmatrix} = J \begin{bmatrix} y(k) - Gu(k) \\ \hat{x}_3(k) \end{bmatrix} = J \left(\begin{bmatrix} 0 \\ I \end{bmatrix} \bar{x}_3(k) + \begin{bmatrix} I \\ K_1 \end{bmatrix} y(k) + \begin{bmatrix} -G \\ K_3 \end{bmatrix} u(k) \right), \tag{13}$$

where $\bar{x}_3(k)$ is computed by (12) recursively.

3. Numerical example

In this section, a numerical example is presented to illustrate the above design method. We consider the system (1) described by

$$E=\begin{bmatrix}1&1&0\\0&1&1\\0&2&1\\1&4&2\end{bmatrix},\ A=\begin{bmatrix}0&1&1\\0&0&1\\1&1&0\\1&1&1\end{bmatrix},\ B=\begin{bmatrix}1&1\\0&1\\2&0\\1&0\end{bmatrix},\ D=\begin{bmatrix}1\\1\\0\\0\end{bmatrix},\ C=\begin{bmatrix}1&0&1\\1&0&0\end{bmatrix},\ G=\begin{bmatrix}1&0\\0&1\end{bmatrix}.$$

The equivalent forms (3) and (4) are

$$\begin{bmatrix}0&1&1\\1&-2&0\\0&1&0\\\frac{1}{2}&-\frac{1}{2}&1\end{bmatrix}\begin{bmatrix}x_1(k+1)\\x_2(k+1)\\x_3(k+1)\end{bmatrix}=\begin{bmatrix}1&-1&1\\0&0&-1\\1&-2&-1\\0&\frac{1}{2}&\frac{1}{2}\end{bmatrix}\begin{bmatrix}x_1(k)\\x_2(k)\\x_3(k)\end{bmatrix}+\begin{bmatrix}1&1\\-1&0\\-3&0\\1&0\end{bmatrix}u(k)+\begin{bmatrix}1\\0\\0\\0\end{bmatrix}d(k),$$

$$y(k+1)=\begin{bmatrix}y_1(k+1)\\y_2(k+1)\end{bmatrix}=\begin{bmatrix}x_1(k+1)\\x_2(k+1)\end{bmatrix}+u(k+1).$$

Thus, the dynamical system of the sub-state $x_3(k)$ is

$$x_3(k+1)=\frac{1}{2}x_3(k)+g(k+1),$$

$$z(k+1)=\begin{bmatrix}-1\\-1\end{bmatrix}x_3(k),$$

where

$$g(k+1)=\begin{bmatrix}-\frac{1}{2}&\frac{1}{2}\end{bmatrix}(y(k+1)-u(k+1))+\begin{bmatrix}0&\frac{1}{2}\end{bmatrix}(y(k)-Gu(k))+\begin{bmatrix}1&0\end{bmatrix}u(k),$$

$$z(k+1)=\begin{bmatrix}1&-2\\0&1\end{bmatrix}(y(k+1)-u(k+1))-\begin{bmatrix}0&0\\1&-2\end{bmatrix}(y(k)-Gu(k))-\begin{bmatrix}-1&0\\-3&0\end{bmatrix}u(k).$$

Choose $\bar{K}=\begin{bmatrix}-1&0\end{bmatrix}$, then the asymptotic stable observer for the sub-state $x_3(k)$ is given by

$$\bar{x}_3(k+1)=-\frac{1}{2}\bar{x}_3(k)+\begin{bmatrix}\frac{3}{4}&-\frac{3}{4}\end{bmatrix}y(k)+\begin{bmatrix}-\frac{3}{4}&\frac{3}{4}\end{bmatrix}u(k),$$

$$\hat{x}_3(k+1)=\bar{x}_3(k+1)+\begin{bmatrix}-\frac{3}{2}&\frac{5}{2}\end{bmatrix}y(k+1)-\begin{bmatrix}-\frac{3}{2}&\frac{5}{2}\end{bmatrix}u(k+1).$$

The estimate of the state $x(k)$ of (1) can be written as

$$\hat{x}(k)=J\begin{bmatrix}\begin{bmatrix}x_1(k)\\x_2(k)\end{bmatrix}\\\hat{x}_3(k)\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\1&-1&0\end{bmatrix}\begin{bmatrix}y(k)-u(k)\\\hat{x}_3(k)\end{bmatrix}=\begin{bmatrix}0&1\\0&0\\1&-1\end{bmatrix}(y(k)-u(k))+\begin{bmatrix}0\\1\\0\end{bmatrix}\hat{x}_3(k).$$

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