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Backlünd transformation and multiple soliton solutions for the (3+1)-dimensional Nizhnik–Novikov–Veselov equation*

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We develop an approach to construct multiple soliton solutions of the (3+1)-dimensional nonlinear evolution equation. We take the (3+1)-dimensional Nizhnik–Novikov–Veselov (NNV) equation as an example. Using the extended homogeneous balance method, one can find a Backlünd transformation to decompose the (3+1)-dimensional NNV into a set of partial differential equations. Starting from these partial differential equations, some multiple soliton solutions for the (3+1)-dimensional NNV equation are obtained by introducing a class of formal solutions.

Keywords: soliton, extended homogeneous balance method, (3+1)-dimensional NNV equation
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1. Introduction

Solving a nonlinear problem plays an important and significant part in nonlinear science. It is well known that there are infinitely many solutions for every nonlinear evolution equation and it is difficult to seek an exact solution of a real nonlinear evolution equation. More and more emphasis is placed on the study of the nonlinear evolution equations by researchers in mathematics, physics, biology and engineering fields. In order to find exact and explicit solitary wave solutions of nonlinear wave equations, scientists have advanced many important and effective methods, such as the inverse scattering approach, Hirota's bilinear method, the Lie symmetry method, Backlünd and Darboux transformation methods, etc. Recently, Wang [1] proposed a method that can construct the soliton solution of nonlinear wave equations, and gave the solitary wave solutions of two types of variant Boussinesq equations. By using this method, the solitary wave solutions of a class of general nonlinear wave equations were obtained in Ref.[2]. Zhang [3] and Bai [4] extended this method to obtain multi-soliton solutions, and the dromion solution.[5] Fan [6,7] improved this method to investigate the Backlünd Transformation, Lax pairs, symmetries and exact solutions for some nonlinear partial differential equations (PDEs). In Refs.[8–10], Fan also extended the homogeneous balance (HB) method to search for similarity reductions of nonlinear PDEs and found the close relations among the HB method, Weiss–Tabor–Carnevale (WTC) method and Clarkson–Kruskal (CK) method.

In this paper, we further extend the HB method and obtain the Backlünd transformation and multi-soliton solutions for the (3+1)-dimensional Nizhnik–Novikov–Veselov (NNV) equation

$$\begin{aligned} &u_{yzt} + au_{yyy}u_{yz} + bu_{xz}u_{xyz} \\ &+ cu_{yy}u_{yyz} + (9c-2b)u_{xx}u_{yzz} \\ &+ (12c-2b-3a)u_{xxz}u_{yz} \\ &+ eu_{yyyyz} + hu_{xxyzz} = 0, \end{aligned} \tag{1}$$

which was originally proposed by Lin by means of the realizations of the generalized centerless Virasoro type symmetry algebra.[11]

2. Extended homogeneous balance (EHB) method

The exact and explicit solutions of the most nonlinear PDEs which are of important applications in

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physics can be obtained by using the HB method introduced by Wang.[1] Given a nonlinear PDE, say, in two variables

$$P(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, \cdots) = 0, \tag{2}$$

where P is in general a polynomial function of its arguments, the subscripts denotes the partial derivatives. A function $\phi(x,t)$ is called a quasi-solution of Eq.(2), if there exists a function $f = f(\phi)$ of one variable only so that a suitable linear combination of the following functions:

$$1, f(\phi), f(\phi)_x, f(\phi)_t, f(\phi)_{xx}, f(\phi)_{tt}, f(\phi)_{xt}, \cdots \tag{3}$$

is actually a solution of Eq.(2). The HB method for finding$f(\phi)$, the quasi-solution $\phi(x,t)$, the suitable linear combination of the functions in Eq.(3) and then obtaining special exact solutions of Eq.(2) consist of four steps (for details, see Ref.[1]).

In order to obtain the new solutions of Eq.(2), we add a function $u_0(x,t)$ which is a solution of Eq.(2) to the right-hand side of the combination obtained in the first step in Ref.[1]. The second step is the same as that in Ref.[1]. In the third step, we obtain a set of equations of $\phi(x,t)$, in which some coefficients may contain $u_0(x,t)$ or/and its derivatives. If the set of equations obtained in the third step is solvable, we solve them and gain a new quasi-solution $\phi(x,t)$. Then proceeding as in the fourth step in Ref.[1], we may obtain new solution(s) of Eq.(2). In the next section, we will use this EHB method to construct some new solutions for Eq.(1).

3. Backlünd transformations of the (3+1)-dimensional NNV equation

In order to obtain a special solution of model (1), we are able to rewrite Eq.(1) in the following potential form by using $v = u_x$,$w = u_y$, $v_y = w_x$:

$$\begin{aligned} &w_{zt} + aw_{yy}w_z + bv_zv_{yz} \\ &+ cw_yw_{yz} + (9c-2b)v_xw_{zz} \\ &+ (12c-2b-3a)v_{xz}w_z + ew_{yyyz} \\ &+ \frac{3e(7c-b-a)}{a+c}v_{xyzz} = 0. \end{aligned} \tag{4}$$

According to the EHB method, we let

$$\begin{aligned} v(x,y,z,t) &= f_x(\phi) + \nu(x,y,z,t), \\ w(x,y,z,t) &= f_y(\phi) + \omega(x,y,z,t), \end{aligned} \tag{5}$$

where $\nu(x,y,z,t)$ and $\omega(x,y,z,t)$ are a solution of Eq.(4). Substituting Eq.(5) into Eq.(4), we obtain

$$\begin{aligned} &w_{zt} + aw_{yy}w_z + bv_zv_{yz} + cw_yw_{yz} \\ &+ (9c-2b)v_xw_{zz} + (12c-2b-3a)v_{xz}w_z \\ &+ ew_{yyyz} + \frac{3e(7c-b-a)}{a+c}v_{xyzz} \\ =&(ef^{(5)} + af^{(3)}f^{(2)} + cf^{(3)}f^{(2)})\phi_z\phi_y^4 \\ &+ \text{lower power terms of the derivatives of} \\ &\phi(x,y,z,t) \text{ with respect to } x,y,z \text{ and } t. \end{aligned} \tag{6}$$

Setting the coefficients of $\phi_z\phi_y^4$ in Eq.(6) to zero yields an ordinary differential system for $f(\phi)$:

$$ef^{(5)} + af^{(3)}f^{(2)} + cf^{(3)}f^{(2)} = 0. \tag{7}$$

The following special solutions exist for Eq.(7):

$$f(\phi) = \frac{12e}{a+c}\ln(\phi). \tag{8}$$

Thereby $f'^2 = \frac{-12e}{a+c}f''$, $f'f'' = \frac{-6e}{a+c}f^{(3)}$, $f'f^{(3)} = \frac{-4e}{a+c}f^{(4)}$, $f''^2 = \frac{-2e}{a+c}f^{(4)}$. Using these results and letting $\nu(x,y,z,t) = \omega(x,y,z,t) = 0$ for simplicity, expression (6) can be simplified to

$$\begin{aligned} &w_{zt} + aw_{yy}w_z + bv_zv_{yz} + cw_yw_{yz} + (9c-2b)v_xw_{zz} + (12c-2b-3a)v_{xz}w_z + ew_{yyyz} + \frac{3e(7c-b-a)}{a+c}v_{xyzz} \\ =&(3(a+3b-13c)\phi_{xx}\phi_y\phi_z^2 + 2(18c-5b)\phi_x\phi_y\phi_z\phi_{xz} + (b-3a+3c)\phi_x^2\phi_y\phi_{zz} \\ &+ 2(21c-3a-4b)(\phi_{xy}\phi_x\phi_z^2 - \phi_x^2\phi_{yz}\phi_z))f^{(4)} + (6e(b+4a-10c)\phi_x\phi_{xz}\phi_{yz} \\ &+ 3e(a-c)\phi_z\phi_{yy}^2 + 6ae(\phi_y^2\phi_{yyz} - \phi_{yy}\phi_y\phi_{yz}) + 6e(7c-a-b)\phi_x\phi_y\phi_{xzz} + 6e(b+2a-5c)\phi_y\phi_z\phi_{xxz} \\ &+ (a+c)\phi_y\phi_z\phi_t - 2e(a-2c)\phi_y\phi_z\phi_{yyy} + 6e(7c-2b-a)\phi_y\phi_{xz}^2 + 3e(3b-11c-a)\phi_y\phi_{xx}\phi_{zz} \\ &+ 6e(2a+5b-23c)\phi_z\phi_{yz}\phi_{xx} + 3e(7c-a-b)\phi_z^2\phi_{xxy} + 6e(14c-2a-3b)\phi_z\phi_x\phi_{xyz} \\ &+ 6e(7c-a-b)\phi_x\phi_{zz}\phi_{xy} + 6e(14c-2a-3b)\phi_z\phi_{xz}\phi_{xy} + 3e(3b-11c-a)\phi_x^2\phi_{xz})f^{(3)} \\ &+ (12e(7c-2b-a)\phi_{xz}\phi_{xyz} + 3e(7c-a-b)\phi_y\phi_{xxzz} + 4e(a+c)\phi_y\phi_{yyyz} + 6e(a-c)\phi_{yy}\phi_{yyz} \\ &+ 3e(7b-29c-a)\phi_{xx}\phi_{xz} + 4e(c-a)\phi_{yyy}\phi_{yz} + (a+c)\phi_t\phi_{yz} + 6e(5a-17c+3b)\phi_{xxz}\phi_{yz} \end{aligned}$$

$$+6e(7c-a-b)\phi_z\phi_{xxyz}+e(a+c)\phi_{yyyy}\phi_z+(a+c)\phi_{yt}\phi_z+3e(7c-a-b)\phi_{xxy}\phi_{zz}$$
$$+6e(7c-a-b)\phi_{xy}\phi_{xzz}+6e(7c-a-b)\phi_x\phi_{xyzz}+(a+c)\phi_y\phi_{tz})f''$$
$$+((a+c)(\phi_{yzt}+e\phi_{yyyyz})+3e(7c-a-b)\phi_{xxyzz})f'. \quad (9)$$

Setting the coefficients of $f^{(4)}, f^{(3)}, f'', f'$ in Eq.(9) to zero and simplifying then yields a set of PDEs for $\phi(x,y,z,t)$:

$$3(a+3b-13c)\phi_{xx}\phi_y\phi_z^2+2(18c-5b)\phi_x\phi_y\phi_z\phi_{xz}+2(21c-3a-4b)(\phi_{xy}\phi_x\phi_z^2-\phi_x^2\phi_{yz}\phi_z)$$
$$+(b-3a+3c)\phi_x^2\phi_y\phi_{zz}=0, \quad (10)$$
$$(a-c)(\phi_z\phi_{yy}^2-\phi_y\phi_z\phi_{yyy})+2(14c-2a-3b)(\phi_z\phi_x\phi_{xyz}+\phi_z\phi_{xz}\phi_{xy})+2(b-10c+4a)\phi_{xz}\phi_x\phi_{yz}$$
$$+2(2a+5b-23c)\phi_z\phi_{xx}\phi_{yz}+2a(\phi_y^2\phi_{yyz}-\phi_{yy}\phi_y\phi_{yz})+(7c-b-a)(2\phi_{xy}\phi_x\phi_{zz}+2\phi_y\phi_x\phi_{xzz}+\phi_z^2\phi_{xxy})$$
$$+(3b-a-11c)(\phi_y\phi_{zz}\phi_{xx}+\phi_{xz}\phi_x^2)+2(7c-a-2b)\phi_y\phi_{xz}^2+(5a-17c+3b)\phi_{xxz}\phi_y\phi_z=0, \quad (11)$$
$$(7c-b-a)(2\phi_{xyzz}\phi_x+2\phi_{xzz}\phi_{xy}+\phi_{xxy}\phi_{zz}+\phi_z\phi_{xxyz})+(a+c)\phi_{yyyz}\phi_y+(2a-2c)\phi_{yyz}\phi_{yy}$$
$$(7b-a-29c)\phi_{xz}\phi_{xx}+(c-3a)\phi_{yz}\phi_{yyy}+4(7c-a-2b)\phi_{xz}\phi_{xyz}+(7b-41c+11a)\phi_{yz}\phi_{xxz}=0, \quad (12)$$
$$(a+c)(\phi_t+e\phi_{yyy})+3e(7c-a-b)\phi_{xxz}=0. \quad (13)$$

Thereby, from Eqs.(5) and (8), we can have a class of Backlünd transformation for Eq.(4) as follows:

$$v=\frac{12e}{a+c}\frac{\partial}{\partial x}\ln\phi, \quad w=\frac{12e}{a+c}\frac{\partial}{\partial y}\ln\phi, \quad (14)$$

where ϕ satisfies Eqs.(10)–(13).

4. Multiple soliton solutions of the (3+1)-dimensional NNV equation

Now we can construct the multiple soliton-like solutions of Eq.(4) by means of Eqs.(10)–(13). From Eqs.(10)–(13), three special solutions are obtained as

$$\phi_1=1+\sum_{i=1}^{n}\exp(k_ix+ly+hz+\omega_it), \quad (15)$$

$$\phi_2=1+\sum_{i=1}^{n}\exp(kx+l_iy+hz+m_it), \quad (16)$$

$$\phi_3=1+\sum_{i=1}^{n}\exp(kx+ly+h_iz+n_it), \quad (17)$$

where

$$\omega_i=\frac{3ek_i^2h(a+b-7c)-el^3(a+c)}{a+c},$$
$$m_i=\frac{3ek^2h(a+b-7c)-el_i^3(a+c)}{a+c},$$
$$n_i=\frac{3ek^2h_i(a+b-7c)-el^3(a+c)}{a+c}. \quad (18)$$

Inserting Eqs.(15)–(17) into Eq.(14), we can construct three types of multiple soliton-like solutions of the (3+1)-dimensional NNV equation:

$$v_1=\frac{(12e)\left(\sum_{i=1}^{n}k_i\exp(k_ix+ly+hz+\omega_it)\right)}{(a+c)\phi_1},$$
$$w_1=\frac{12el(\phi_1-1)}{(a+c)\phi_1}, \quad (19)$$
$$v_2=\frac{12ek(\phi_2-1)}{(a+c)\phi_2},$$
$$w_2=\frac{(12e)\left(\sum_{i=1}^{n}l_i\exp(kx+l_iy+hz+m_it)\right)}{(a+c)\phi_2}, \quad (20)$$
$$v_3=\frac{12ek(\phi_3-1)}{(a+c)\phi_3},$$
$$w_3=\frac{12el(\phi_3-1)}{(a+c)\phi_3}, \quad (21)$$

where ϕ_1, ϕ_2, ϕ_3 are determined by Eqs.(15)–(17).

5. Summary

In summary, by using the EHB method for the (3+1)-dimensional NNV equation, some types of single and multi-soliton solutions are obtained. The method used here is standard and direct, so we believe that these solutions may exist for other classes of (3+1)-dimensional nonlinear evolution equation. Further work on these aspects is worthy of study.

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