Vol 12 No 9, September 2003
1009-1963/2003/12(09)/0946-05

Chinese Physics

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Effect of signal modulating noise in bistable stochastic dynamical systems*

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(Received 1 April 2003; revised manuscript received 7 May 2003)

The effect of signal modulating noise in bistable stochastic dynamical systems is studied. The concept of instantaneous steady state is proposed for bistable dynamical systems. By making a dynamical analysis of bistable stochastic systems, we find that global and local effect of signal modulating noise as well as stochastic resonance can occur in bistable dynamical systems on which both a weak sinusoidal signal and noise are forced. The effect is demonstrated by numerical simulation.

Keywords: bistable systems, effect of signal modulating noise, instantaneous steady state, stochastic resonance

PACC: 0547

1. Introduction

It has been found that under certain nonlinear conditions, increasing the input noise level of a periodically modulated and noise-driven bistable system results in an increase in the output signal-to-noise ratio. That is, a weak signal can be amplified and optimized with the assistance of noise. This nonlinear cooperative phenomenon that is manifest, for instance, in bistable systems, is termed stochastic resonance.[1−3]

The phenomenon of stochastic resonance has attracted considerable interest in the last 20 years. Much attention has been devoted to the study of stochastic resonance from both theoretical and experimental points of view.[1−17] McNamara, Wiesenfeld,[4] Fox[5] and Hu[7] have developed a stochastic resonance equation for bistable systems that are forced by a sinusoidal signal and white noise. Based on this equation, they established the fundamental theories of stochastic resonance, in which the relation between the power spectrum of the stable states transition time series and the input noise intensity was mathematically analysed. These theories are generally in conformity with experimental results.

However, for a continuous bistable system there is more sinusoidal information contained in the output signal than that contained in the stable states transition time series. In this paper we firstly study the response mechanism of bistable systems to the driving signal, and then make an analysis on the output characteristics of bistable systems driven by a sinusoidal signal and white noise. Based on the analysis we show that the effect of signal modulating noise can occur in bistable stochastic dynamical systems when the parameters are properly selected. Finally, this effect is verified by numerical simulation.

2. Dynamical characteristics of bistable systems

2.1. Autonomous bistable systems

Consider the bistable system determined by the differential equation

$$\dot{x} = ax - bx^3 \quad (a > 0, b > 0). \tag{1}$$

The system has two stable steady solutions $x = \pm\sqrt{a/b}$ and an unstable steady solution x=0. Given the initial condition $x_0 > 0$ (or $x_0 < 0$), the system will converge to the steady solution $x = \sqrt{a/b}$ (or $x = -\sqrt{a/b}$). As $t \to \infty$, the system will approach

*Project supported by the National Natural Science Foundation of China (Grant No 10276032), and the Science Foundation of China Academy of Engineering Physics NSAF.

to the respective steady state and never depart from it. Given the initial condition x_0=0, the system will remain at the unstable steady state x=0 for all t.

Now consider the bistable system determined by the differential equation

$$\dot{x} = ax - bx^3 + c \quad (a > 0, b > 0), \tag{2}$$

where c has a critical value $c_c = \sqrt{4a^3/27b}$. When $|c| \geq c_c$, the system becomes monostable, i.e. it has only one stable steady solution. For any initial condition x_0, the system approaches to the stable steady solution. When $|c| < c_c$, the system has two stable steady solutions and an unstable steady solution. In the latter case, if the initial condition x_0 matches to the unstable steady solution, the system will remain at the unstable steady state, otherwise it will approach to one of its two stable steady solutions.

2.2.Nonautonomous bistable systems

Substituting the variable $c(t)$ for the parameter c in Eq.(2), we obtain a nonautonomous bistable system:

$$\dot{x} = ax - bx^3 + c(t) \quad (a > 0, b > 0), \tag{3}$$

where $c(t)$ can be regard as a perturbation to the bistable system. Assume that $c(t)$ is a continuous function for $t \in [0, L]$. Divide the interval $[0, L]$ into n small intervals $\Delta_i = [t_{i-1}, t_i]$ with $0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = L$, $i = 1, 2, \cdots, n$, where the length of Δ_i is $\Delta t_i = t_i - t_{i-1}$. Write $||T|| = \max\limits_{1 \leq i \leq n} \{\Delta t_i\}$. Suppose that $||T||$ is sufficiently small, then $c(t)$ can be regard as a constant on every small interval Δ_i. Let us write $c(t_i) = h_i$, then on the interval Δ_i Eq.(3) can be rewritten as

$$\dot{x} = ax - bx^3 + h_i, \tag{4}$$

where the initial condition is $x_0 = x(t_{i-1})$. The solution of Eq.(4) will converge to the stable steady state which matches to the initial condition $x_0 = x(t_{i-1})$. For convenience, this stable steady state is termed the instantaneous steady state of Eq.(3) at the time $t = t_i$ and written as $x_s(t_i)$. When the time required for the solution of Eq.(4) to converge to a small neighbourhood of the instantaneous steady state $(x_s(t_i) - \varepsilon, x_s(t_i) + \varepsilon)$ (ε is a sufficiently small parameter) is less than Δt_i, that is to say, the solution of Eq.(4) converges to the instantaneous steady state faster than the instantaneous steady state, then the solution of Eq.(3) on the interval Δ_i will be approximate to the instantaneous steady state $x_s(t_i)$, whereas on the interval $[0, L]$ the solution of Eq.(3) will be approximate to the instantaneous steady state on every small interval Δ_i subsection by subsection. When $||T|| \to 0$, the solution of Eq.(3) can be regarded as the instantaneous steady state $x_s(t)$ that changes with $c(t)$, i.e. $x(t) = x_s(t)$.

When the time required for the solution of Eq.(4) to converge to a small neighbourhood of the instantaneous steady state $(x_s(t_i) - \varepsilon, x_s(t_i) + \varepsilon)$ is greater than Δt_i, i.e. the speed of the convergence of the solution of Eq.(4) is less than the speed at which instantaneous steady state varies, the solution cannot follow up the instantaneous steady state, it will approach to the instantaneous steady state to a certain extent on every interval Δ_i, without reaching it. When $||T|| \to 0$, the solution of the system (3) will show a dynamical characteristic that will vary with the instantaneous steady state synchronously, but the amplitude of the variation is less than that of the instantaneous steady state. This characteristic can be considered as a perturbation of the nonautonomous term $c(t)$ to the bistable system. Since the perturbation cannot come up to the variation of the instantaneous steady state, the tendency that the bistable system converges to its stable steady state $\sqrt{a/b}$ (or $-\sqrt{a/b}$) cannot be changed entirely by the perturbation. In this case the solution will gradually approach to the stable steady state $\sqrt{a/b}$ (or $-\sqrt{a/b}$) and have a synchronous perturbation with the instantaneous steady state $x_s(t)$.

From the above analysis, we can conclude that the output characteristics of a bistable system are mainly determined by the competition between the speed of the global change of the instantaneous steady state with time, and the converging speed of the system to local instantaneous steady state on each time interval. The faster the system approaches to its instantaneous steady state, the better the output exhibits the characteristics of $c(t)$, i.e. the more effectively the system responds to $c(t)$. Similarly as the instantaneous steady state relates with $c(t)$, the faster $c(t)$ varies, the less effectively the system responds to $c(t)$. From Eq.(4), we know that the speed the system converges to the instantaneous steady state is mainly determined by the parameters a and b, and a in particular. The greater a, the greater $|\dot{x}(t)|$ is, that is, the faster the system converges to the instantaneous steady state. In addition the varying speed of the instantaneous steady state is related to the amplitude and frequency of $c(t)$.

3. Bistable systems driven by a sinusoidal signal and white noise

3.1. Bistable systems driven by a sinusoidal signal

Assume that $c(t) = A\sin\omega t$, then the Eq.(3) can be rewritten as

$$\dot{x} = ax - bx^3 + A\sin\omega t. \tag{5}$$

As $A\sin\omega t$ is a periodic signal, the instantaneous steady state $x_s(t)$ will also exhibit a periodical behaviour, where the period of $x_s(t)$ is the same as that of $A\sin\omega t$, $T = 2\pi/\omega$. The amplitude of $x_s(t)$ is not only connected with $A\sin\omega t$, but also connected with the parameters a and b. For the properly selected parameters a and b, the amplitude of the instantaneous steady state $x_s(t)$ can be greater than A. When $A\sin\omega t$ is a slowly varying periodic signal, i.e. the period T is comparatively long, the instantaneous steady state $x_s(t)$ can be considered as the approximate solution of Eq.(5). So for the properly selected parameters a and b, we can obtain an amplified output signal compared with the input signal $A\sin\omega t$. From the preceding analysis, we see that the bistability of the system will be destroyed when $|A\sin\omega t| \geq c_c$, therefore A has a critical value $A_c = \sqrt{4a^3/27b}$. For $A < A_c$, the system will periodically oscillate in the neighbourhood of the stable steady state $\sqrt{a/b}$ (or $-\sqrt{a/b}$). For $A \geq A_c$, the system will periodically oscillate, enveloping the two stable steady states $\pm\sqrt{a/b}$. When $A\sin\omega t$ is a fast varying periodic signal, i.e. the period T is comparatively short, the variation of the solution of the Eq.(5) cannot follow up the variation of the instantaneous steady state, therefore the response amplitude of the system to the fast sinusoidal signal is less than that to the slow sinusoidal signal for the same given parameters a and b. In this case, the solution of Eq.(5) will periodically oscillate with $A\sin\omega t$ in the neighbourhood of the stable steady state $x = \sqrt{a/b}$ (or $x = -\sqrt{a/b}$) with a transient process that approaches to the stable steady state from the initial point, while the transient process is also perturbed by $A\sin\omega t$. That is, the solution will be a quasi-periodic signal.

3.2. Bistable systems driven by white noise

Assume that $c(t) = w(t)$, then the Eq.(3) can be rewritten as

$$\dot{x} = ax - bx^3 + w(t), \tag{6}$$

where $w(t)$ is a white noise with $\langle w(t)\rangle$=0 and $\langle w(t)w(t')\rangle = 2D\delta(t - t')$. Since the instantaneous steady state will rapidly and randomly vary with $w(t)$, being restricted by the speed at which the solution converges to the instantaneous steady state, the variation amplitude and randomicity of the solution of Eq.(6) will decrease. This indicates that bistable systems have the function of suppressing noise. When we substitute the band-limited white noise for the white noise, for the higher frequency and higher intensity noise, the suppression of noise in the bistable systems is more effective. As decreasing the parameter a of Eq.(6) will result in decreasing the efficiency of the system's response to the variation of the instantaneous steady state, bistable systems with smaller value of the parameter a will be more effective in suppressing noise.

3.3. Bistable systems driven by both a sinusoidal signal and white noise

Assume that $c(t) = A\sin\omega t + w(t)$, then the Eq.(3) can be rewritten as

$$\dot{x} = ax - bx^3 + A\sin\omega t + w(t), \tag{7}$$

where $a > 0$, $b > 0$, $w(t)$ is a white noise. Equation (7) is just the dynamical model of stochastic resonance.[4,5] From the preceding analyses, we know that, for an input sinusoidal signal the output signal of the bistable system still exhibits the periodic characteristics and its amplitude may be little reduced or even magnified, whereas an input noise will be suppressed by the system and its amplitude can be largely reduced. When a bistable system is driven by both a sinusoidal signal and a white noise, as the correlation time of noise is far shorter than that of the sinusoidal signal, the variation of noise is far quicker than that of the sinusoidal signal. Being restricted by the speed at which the system converges to the instantaneous steady state, the noise in the output of the system will be largely reduced, while the signal may be little reduced or even amplified, so the signal contained in the output of the system will show a decrease of noise. Therefore, we can imagine that by properly selecting parameters of a bistable system driven by both a weak sinusoidal signal and white noise, one of the following cases may be made to take place in the output:

(1) The system exhibits the local effect of signal modulating noise in the neighbourhood of the stable steady state.

(2) The system exhibits the global effect of signal modulating noise enveloping the two stable steady states.

(3) When the signal cannot induce the system to globally move between the two stable steady states, if the average waiting time between two noise-induced continuous transitions between the two stable steady states is comparable with half the period of the sinusoidal signal, the nonlinear cooperative effect of the system between the signal and noise makes the system to transit with the modulation frequency of the signal between the two stable steady states, then the sinusoidal signal is amplified and the signal-to-noise ratio in the output increases. This case is just the stochastic resonance in the bistable system.[11]

Now let us qualitatively consider how can we select the proper parameters so that the above cases be realized. When the input sinusoidal signal is a slow signal, if for certain values of the parameters a and b, the effect of signal modulating noise do not occur in the output, for the sake of the occurrence of this effect in the system we can select smaller values for a that the noise will be suppressed more. From $A_c = \sqrt{4a^3/27b}$, we know that for the same parameter b, as a decreases, A_c decreases more quickly, thus the system will transit more frequently between the two stable steady states. For the occurrence of the local effect of signal modulating noise in the neighbourhood of a certain stable steady state, we ought to let $A_c > A$. Since the role of a is far greater than that of b in A_c, the minification of b ought to be greater than that of a. As for the occurrence of the global effect of signal modulating noise enveloping the two stable steady states, in the case of decreasing a to make $A \geq A_c$, we can let the minification of b equal that of a and even can increase b. As for the occurrence of stochastic resonance, under the condition $A < A_c = \sqrt{4a^3/27b}$, we can select a greater value for a, and also select a greater value for b so as to make A_c reduced. Apparently the selection of a and b in this case will be more difficult than that in the above cases.

When the input sinusoidal signal is a fast signal, being restricted by the speed at which the system converges to the instantaneous steady state, the input noise will be suppressed greatly. If the parameter a is not too great, the suppression of the sinusoidal signal will be far less than that of noise. Therefore, the effect of signal modulating noise can be easily seen in the output of the system.

4. Numerical analysis

Assume that the input signal is a composite signal $c(t) = 0.05\sin(0.002 \times 2\pi t) + 0.5w(t)$, $w(t)$ is a band-limited white noise with the root-mean-square $d = 1$. Here the weak sinusoidal signal is almost completely submerged in the noise. The system equation can be written as

$$\dot{x} = ax - bx^3 + 0.05\sin(0.002 \times 2\pi t) + 0.5w(t). \quad (8)$$

We take $a = 0.1$, b=1, then $A_c = \sqrt{4 \times 0.1^3/27} < 0.05$, the movement of the system is a global oscillation induced by the sinusoidal signal. We numerically solve the Eq.(8) using the fourth-order Runge–Kutta method and taking h=0.05 as the iteration step, $x_0 = 0$ as the starting value. Figure 1(a) shows the output signal of the system, where we can see the global effect of signal modulating noise. Again we consider the local effect of signal modulating noise in bistable systems. We take a=0.1, b=0.01, then $A_c > 0.05$. Figure 1(b) shows the output signal of the system, where we can see the local effect of signal modulating noise.

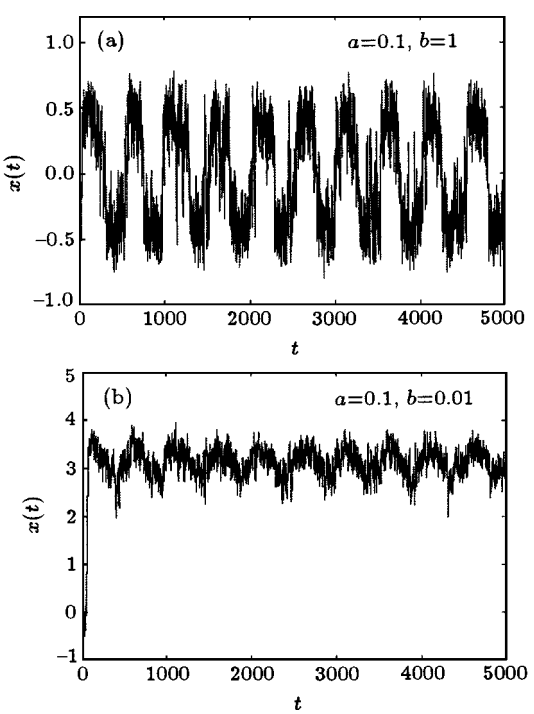


Fig.1. When the system input signal is $c(t) = 0.05\sin(0.002 \times 2\pi t) + 0.5w(t)$, the output signals for different system parameters.

Now assume that the input signal is $0.05\sin 2\pi t + 0.5w(t)$, the equation of the system then can be written as

$$\dot{x} = ax - bx^3 + 0.05\sin 2\pi t + 0.5w(t). \qquad (9)$$

We numerically solve the Eq.(9) using the Runge–Kutta method and taking h=0.0001 as the iteration step, x_0=0 as the starting value. Figures 2 (a) and (b) show the output signal of the system for a=0.1, b=1, and a=0.5, b=10, respectively. The effect of signal modulating noise with a transient process that approaches to the stable steady state is shown in the figure.

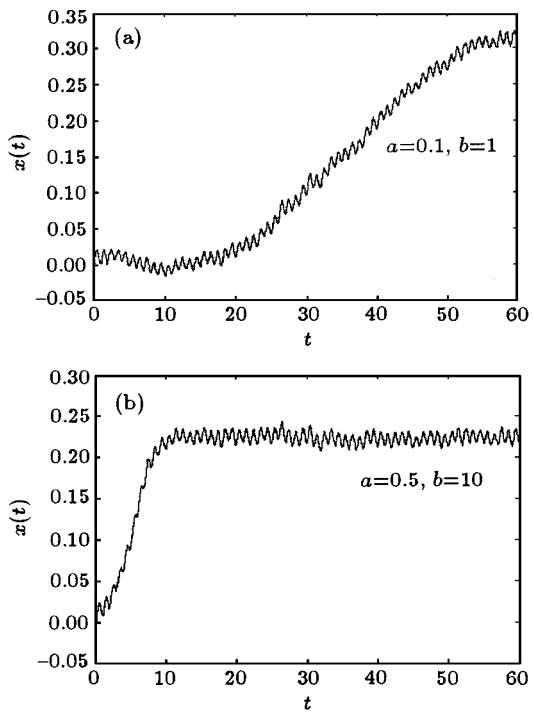


Fig.2. When the system input signal is $c(t) = 0.05\sin 2\pi t + 0.5w(t)$, the output signals for different system parameters.

In the above analysis, we numerically show that bistable systems have the function of suppressing noise, and the effect of signal modulating noise can manifest in a bistable dynamical system that is driven by both a weak sinusoidal signal and white noise when the parameters are properly selected. The results correspond with the preceding analysis.

5. Conclusion

In this paper we introduced the instantaneous steady state for bistable dynamical systems. The response mechanism of bistable systems to the driving signal has been analysed with the help of this term. In particular, we have analysed the output characteristics of bistable systems driven by a sinusoidal signal and white noise. The analyses show that bistable systems have the function of suppressing noise, however, a sinusoidal signal still has the periodical features and its amplitude may be little reduced or even can be magnified in the output. We indicate that the local and global effect of signal modulating noise can manifest in bistable systems driven by both a weak sinusoidal signal and white noise when the parameters are properly selected. Numerical simulation shows the effect. The investigation of this paper shows that not only stochastic resonance but also the effect of signal modulating noise can occur in bistable systems on which both a sinusoidal signal and white noise are forced. This effect can be used in the application of detecting weak signals.

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