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Exact invariants and adiabatic invariants of the singular Lagrange system*

Chen Xiang-Wei(陈向炜)† and Li Yan-Min(李彦敏)

Department of Physics, Shangqiu Teachers College, Shangqiu 476000, China

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Based on the theory of symmetries and conserved quantities of the singular Lagrange system, the perturbations to the symmetries and adiabatic invariants of the singular Lagrange systems are discussed. Firstly, the concept of higher-order adiabatic invariants of the singular Lagrange system is proposed. Then, the conditions for the existence of the exact invariants and adiabatic invariants are proved, and their forms are given. Finally, an example is presented to illustrate these results.

Keywords: singular Lagrange system, symmetry, exact invariant, adiabatic invariant
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1. Introduction

More and more attention has been paid to the research of symmetries and conserved quantities of mechanical systems and many results have appeared during recent years. These results advanced the knowledge about the essence of mechanical systems.[1−8] In 1917, Burgers first proposed adiabatic invariants which pointed to a special kind of Hamilton system.[9] The adiabatic invariants mean that they are almost not changed when the parameter varies very slowly and they play a very important role in the research on quasi-integrability of a mechanical system. In fact, that the parameter varies very slowly is equal to the action of a small perturbation, so we perform research using the latter model in this paper.

The system described by a singular Lagrangian function is called the singular system for simplicity. The singular system is a physical one. In field theory, many important physical systems are singular systems.[10] In Refs.[10–12] the singular system has been studied.

The study of adiabatic invariants has become a popular subject in mechanics,[13−15] atomic and molecular physics,[16], etc. In this paper, we discuss the adiabatic invariants of the singular Lagrange systems from the viewpoint of symmetries and conserved quantities. Firstly, we construct the equations of the singular Lagrange systems, and propose the concept of higher-order adiabatic invariants. Then, we prove the conditions for existence of the exact invariants and adiabatic invariants, and give their forms. Finally, we present an example to illustrate these results.

2. Equations of motion for singular Lagrange system

Suppose that the position of a system is determined by n generalized coordinates $q_s (s = 1, \cdots, n)$. The Lagrangian function is

$$L = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{1}$$

If

$$\det(h_{sk}) = \det\left(\frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k}\right) = 0, \tag{2}$$

then the system is a singular Lagrange system. Let the nonpotential generalized forces are

$$Q_s = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{3}$$

The equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s \quad (s = 1, \cdots, n), \tag{4}$$

and their expansion are

$$\sum_{k=1}^{n} h_{sk}\ddot{q}_k = \frac{\partial L}{\partial q_s} - \sum_{k=1}^{n}\frac{\partial^2 L}{\partial \dot{q}_s \partial q_k}\dot{q}_k - \frac{\partial^2 L}{\partial \dot{q}_s \partial t} + Q_s. \tag{5}$$

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†E-mail: hnchenxw@sohu.com

From Eqs.(5) we can derive a part of the generalized accelerations

$$\ddot{q}_i = \alpha_i(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \quad (i = 1, \cdots, r; 0 \leq r \leq n), \tag{6}$$

and $(n - r)$ relations

$$\beta_j(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0 \quad (j = 1, \cdots, n - r). \tag{7}$$

3. Infinitesimal transformations and exact invariants

Introducing the infinitesimal transformations of groups

$$t^* = t + \Delta t,$$
$$q_s^*(t^*) = q_s(t) + \Delta q_s, \tag{8}$$

and their expansion

$$t^* = t + \varepsilon\tau^0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}),$$
$$q_s^*(t^*) = q_s(t) + \varepsilon\xi_s^0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{9}$$

where ε is an infinitesimal parameter, τ^0 and ξ_s^0 are infinitesimal generators. Introducing an infinitesimal generator vector

$$X^{(0)} = \tau^0 \frac{\partial}{\partial t} + \sum_{s=1}^{n} \xi_s^0 \frac{\partial}{\partial q_s}, \tag{10}$$

and its first extension

$$X^{(1)} = X^{(0)} + \sum_{s=1}^{n} (\dot{\xi}_s^0 - \dot{q}_s\dot{\tau}^0) \frac{\partial}{\partial \dot{q}_s}. \tag{11}$$

The invariance of Eqs.(6) and (7) under the infinitesimal transformations (9) leads to the following determining equations and restriction equations respectively

$$\ddot{\xi}_i^0 - \dot{q}_i\ddot{\tau}^0 - 2\dot{\tau}^0\alpha_i = X^{(1)}(\alpha_i) \quad (i = 1, \cdots, r), \tag{12}$$

$$X^{(1)}(\beta_j(t, \boldsymbol{q}, \dot{\boldsymbol{q}})) = 0 \quad (j = 1, \cdots, n - r). \tag{13}$$

Definition 1[12] If the generators τ^0 and ξ_s^0 satisfy the determining Eqs. (12) and (13), then the corresponding transformations are called the Lie symmetrical transformations, and the corresponding symmetries are called the Lie symmetries.

The Lie symmetries do not always imply exact invariants. The following theorem gives the conditions of the existence of exact invariants and their forms.

Theorem 1 For the infinitesimal generators τ^0 and ξ_s^0 satisfying the determining Eqs. (12) and (13), if there exists a gauge function $G^0 = G^0(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the following structure equation

$$L\dot{\tau}^0 + X^{(1)}(L) + \sum_{s=1}^{n} Q_s(\xi_s^0 - \dot{q}_s\tau^0) + \dot{G}^0 = 0, \tag{14}$$

then the singular Lagrange system possesses the following exact invariants

$$I_0 = L\tau^0 + \sum_{s=1}^{n} \frac{\partial L}{\partial \dot{q}_s}(\xi_s^0 - \dot{q}_s\tau^0) + G^0 = \text{constant}. \tag{15}$$

Proof

Differentiating I_0 with respect to time t, and considering Eqs.(14) and (4), we have

$$\begin{aligned}\frac{\mathrm{d}I_0}{\mathrm{d}t} =& \dot{L}\tau^0 + L\dot{\tau}^0 + \sum_{s=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}(\xi_s^0 - \dot{q}_s\tau^0) \\ &+ \sum_{s=1}^{n} \frac{\partial L}{\partial \dot{q}_s}(\dot{\xi}_s^0 - \ddot{q}_s\tau^0 - \dot{q}_s\dot{\tau}^0) \\ &- L\dot{\tau}^0 - \frac{\partial L}{\partial t}\tau^0 - \sum_{s=1}^{n} \frac{\partial L}{\partial q_s}\xi_s^0 \\ &- \sum_{s=1}^{n} \frac{\partial L}{\partial \dot{q}_s}(\dot{\xi}_s^0 - \dot{q}_s\dot{\tau}^0) - \sum_{s=1}^{n} Q_s(\xi_s^0 - \dot{q}_s\tau^0) \\ =& \sum_{s=1}^{n} \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} - Q_s\right)(\xi_s^0 - \dot{q}_s\tau^0) = 0.\end{aligned}$$

4. Perturbation of the symmetries and adiabatic invariants

Firstly, we give the concept of higher-order adiabatic invariants.

Definition 2 If $I_z(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \varepsilon)$ is a physical quantity including ε in which the highest power is z in a mechanical system, and its derivative with respect to time t is in direct proportion to ε^{z+1}, then I_z is called a zth-order adiabatic invariant of the mechanical system.

Suppose the singular Lagrange system corresponding to Eq.(4) is perturbed by small quantities εW_s, then the equations of motion of the system become

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + \varepsilon W_s \quad (s = 1, \cdots, n). \tag{16}$$

Due to the action of εW_s, the primary symmetries and invariants of the system may vary. Suppose the variation is a small perturbation based on the symmetrical transformations of the system without perturbation, and $\tau(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ and $\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ express the generators of time and space respectively after being perturbed, then

$$\tau = \tau^0 + \varepsilon\tau^1 + \varepsilon^2\tau^2 + \cdots,$$

$$\xi_s = \xi_s^0 + \varepsilon\xi_s^1 + \varepsilon^2\xi_s^2 + \cdots, \tag{17}$$

and they satisfy

$$L\dot{\tau} + X^{(1)'}(L) + \sum_{s=1}^{n} Q_s(\xi_s - \dot{q}_s\tau) + \varepsilon\sum_{s=1}^{n} W_s(\xi_s - \dot{q}_s\tau) + \dot{G} = 0, \tag{18}$$

where

$$\begin{aligned} X^{(1)'} =& X^{(0)'} + \sum_{s=1}^{n}(\dot{\xi}_s - \dot{q}_s\dot{\tau})\frac{\partial}{\partial\dot{q}_s} \\ =& \tau\frac{\partial}{\partial t} + \sum_{s=1}^{n}\xi_s\frac{\partial}{\partial q_s} + \sum_{s=1}^{n}(\dot{\xi}_s - \dot{q}_s\dot{\tau})\frac{\partial}{\partial\dot{q}_s}, \end{aligned}$$

with G in Eq.(18) being a gauge function. Let

$$G = G^0 + \varepsilon G^1 + \varepsilon^2 G^2 + \cdots, \tag{19}$$

$$X^{(1)k} = \tau^k\frac{\partial}{\partial t} + \sum_{s=1}^{n}\xi_s^k\frac{\partial}{\partial q_s} + \sum_{s=1}^{n}(\dot{\xi}_s^k - \dot{q}_s\dot{\tau}^k)\frac{\partial}{\partial\dot{q}_s}. \tag{20}$$

Substituting Eqs.(17) and (19) into Eq.(18), we obtain

$$L\dot{\tau}^k + X^{(1)k}(L) + \sum_{s=1}^{n} Q_s(\xi_s^k - \dot{q}_s\tau^k) + \sum_{s=1}^{n} W_s(\xi_s^{k-1} - \dot{q}_s\tau^{k-1}) + \dot{G}^k = 0 \quad (k = 0, 1, 2, \cdots), \tag{21}$$

when k=0, the condition W_s=0 holds.

Theorem 2 For a singular Lagrange system perturbed by small quantities εW_s, if the generators $\tau^k(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ and $\xi_s^k(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ under infinitesimal transformations satisfy the structure Eq. (21), and $G^k(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is the gauge function, then

$$I_z = \sum_{k=0}^{z}\varepsilon^k\left[L\tau^k + \sum_{s=1}^{n}\frac{\partial L}{\partial\dot{q}_s}(\xi_s^k - \dot{q}_s\tau^k) + G^k\right]$$

is a zth-order adiabatic invariant of the mechanical system.

Proof

$$\begin{aligned} \frac{\mathrm{d}I_z}{\mathrm{d}t} =& \sum_{k=0}^{z}\varepsilon^k\Bigg[\dot{L}\tau^k + L\dot{\tau}^k + \sum_{s=1}^{n}\frac{\partial L}{\partial\dot{q}_s}(\dot{\xi}_s^k - \ddot{q}_s\tau^k - \dot{q}_s\dot{\tau}^k) \\ &+ \sum_{s=1}^{n}(\xi_s^k - \dot{q}_s\tau^k)\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s} - L\dot{\tau}^k - X^{(1)k}(L) \\ &- \sum_{s=1}^{n} Q_s(\xi_s^k - \dot{q}_s\tau^k) - \sum_{s=1}^{n} W_s(\xi_s^{k-1} - \dot{q}_s\dot{\tau}^{k-1})\Bigg] \\ =& \sum_{k=0}^{z}\varepsilon^k\Bigg[\sum_{s=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s} - \frac{\partial L}{\partial q_s} - Q_s\right)(\xi_s^k - \dot{q}_s\tau^k) \\ &- \sum_{s=1}^{n} W_s(\xi_s^{k-1} - \dot{q}_s\dot{\tau}^{k-1})\Bigg]. \end{aligned}$$

Using Eq.(16), we have

$$\frac{\mathrm{d}I_z}{\mathrm{d}t} = \sum_{k=0}^{z}\varepsilon^k\Bigg[\sum_{s=1}^{n}\varepsilon W_s(\xi_s^k - \dot{q}_s\tau^k) - \sum_{s=1}^{n} W_s(\xi_s^{k-1} - \dot{q}_s\dot{\tau}^{k-1})\Bigg].$$

Expanding the above formula and making summation, we have

$$\frac{\mathrm{d}I_z}{\mathrm{d}t} = \varepsilon^{z+1}\sum_{s=1}^{n} W_s(\xi_s^z - \dot{q}_s\tau^z).$$

It shows that $\dfrac{\mathrm{d}I_z}{\mathrm{d}t}$ is in direct proportion to ε^{z+1}, so I_z is a zth-order adiabatic invariant of the mechanical system.

5. An illustrative example

Let the Lagrangian function be

$$L = \frac{1}{2}\dot{q}_1^2 + \dot{q}_1 q_2 + \frac{1}{2}(q_1 - q_2)^2, \tag{22}$$

and the nonpotential generalized forces be

$$Q_1 = -\dot{q}_2, \quad Q_2 = \dot{q}_1. \tag{23}$$

We study the exact invariants and adiabatic invariants of the system. Equation (4) give

$$\ddot{q}_1 = q_1 - q_2 - 2\dot{q}_2, \quad -2\dot{q}_1 + q_1 - q_2 = 0. \tag{24}$$

The determining Eq. (12) gives

$$\begin{aligned} &\ddot{\xi}_1^0 - \dot{q}_1\ddot{\tau}^0 - 2\dot{\tau}^0(q_1 - q_2 - 2\dot{q}_2) \\ =& \xi_1^0 - \xi_2^0 - 2(\dot{\xi}_2^0 - \dot{q}_2\dot{\tau}^0). \end{aligned} \tag{25}$$

The restriction Eq. (13) gives

$$-2(\dot{\xi}_1^0 - \dot{q}_1\dot{\tau}^0) + \xi_1^0 - \xi_2^0 = 0. \tag{26}$$

Eqs. (25) and (26) have the following solutions

$$\tau^0 = 1, \quad \xi_1^0 = \xi_2^0 = 0.$$

The structure Eq. (14) gives

$$\begin{aligned} &L\dot{\tau}^0 + \xi_1^0(q_1 - q_2) + \xi_2^0(\dot{q}_1 - q_1 + q_2) \\ &+ (\dot{q}_1 + q_2)(\dot{\xi}_1^0 - \dot{q}_1\dot{\tau}^0) - \dot{q}_2(\xi_1^0 - \dot{q}_1\tau^0) \\ &+ \dot{q}_1(\xi_2^0 - \dot{q}_2\tau^0) + \dot{G}^0 = 0, \end{aligned} \tag{27}$$

it has a solution

$$G^0 = 0. \tag{28}$$

So we can obtain the following exact invariant by Theorem 1

$$I_0 = -\frac{1}{2}\dot{q}_1^2 + \frac{1}{2}(q_1 - q_2)^2 = \text{constant}. \qquad (29)$$

Lastly, we study the adiabatic invariants of the system. Suppose the system is perturbed by the small quantities as follows

$$\varepsilon W_1 = \varepsilon q_1, \quad \varepsilon W_2 = \varepsilon q_2. \qquad (30)$$

The structure Eq. (19) gives

$$\begin{aligned} &L\dot{\tau}^1 + \xi_1^1(q_1 - q_2) + \xi_2^1(\dot{q}_1 - q_1 + q_2) \\ &+ (\dot{q}_1 + q_2)(\dot{\xi}_1^1 - \dot{q}_1\dot{\tau}^1) - \dot{q}_2(\xi_1^1 - \dot{q}_1\tau^1) + \dot{q}_1(\xi_2^1 - \dot{q}_2\tau^1) \\ &+ W_1(\xi_1^0 - \dot{q}_1\tau^0) + W_2(\xi_2^0 - \dot{q}_2\tau^0) + \dot{G}^1 = 0, \end{aligned} \qquad (31)$$

it has solutions as

$$\tau^1 = 1, \quad \xi_1^1 = \xi_2^1 = 0, \quad G^1 = \frac{1}{2}(q_1^2 + q_2^2), \qquad (32)$$

so we can obtain the following first-order adiabatic invariants by Theorem 2

$$\begin{aligned} I_1 = &-\frac{1}{2}\dot{q}_1^2 + \frac{1}{2}(q_1 - q_2)^2 \\ &+ \varepsilon\left[-\frac{1}{2}\dot{q}_1^2 + \frac{1}{2}(q_1 - q_2)^2 + \frac{1}{2}(q_1^2 + q_2^2)\right]. \end{aligned}$$

Further we can obtain more higher-order adiabatic invariants.

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