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A set of Lie symmetrical non-Noether conserved quantity for the relativistic Hamiltonian systems*

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For the relativistic Hamiltonian system, a new type of Lie symmetrical non-Noether conserved quantities are given. On the basis of the theory of invariance of differential equations under infinitesimal transformations, and introducing special infinitesimal transformations for q_s and p_s, we construct the determining equations of Lie symmetrical transformations of the system, which only depend on the canonical variables. A set of non-Noether conserved quantities are directly obtained from the Lie symmetries of the system. An example is given to illustrate the application of the results.

Keywords: relativity, Hamiltonian system, Lie symmetry, non-Noether conserved quantity
PACC: 0320, 0330, 0220

1. Introduction

The conservation laws for a dynamical system play an important role in the fields of mathematics, mechanics and physics. There are three main symmetrical methods used to seek the conserved quantity of the dynamical system: Noether symmetry, Lie symmetry and Mei's form invariance (or Mei symmetry). The Noether symmetry is an invariance of the Hamilton's action under the infinitesimal transformations of groups.[1] The Lie symmetry is an invariance of differential equations under the infinitesimal transformation of groups.[2] The Mei symmetry is one of the equations of motion under the infinitesimal transformations of groups. It is different from the Noether symmetry and the Lie symmetry and can also lead to a conserved quantity under certain conditions.[3–6]

In 1979, Lutzky applied Lie's theory to the differential equations of motion for a mechanical system, and studied Lie symmetries and conserved quantities of a dynamical system.[2] In recent 20 years, researchers developed the method of Lie symmetry, and obtained many important results, e.g. Researchers [7–27]. In general, the Lie symmetry of a dynamical system does not always imply the conserved quantity, it can lead to a conserved quantity only under certain conditions. There are two main methods used to seek the Lie symmetrical conserved quantities. One[2,7–22] is based on the Noether symmetries, and the other[23–27] is directly based on the Lie symmetries of the system.

The dynamics of a Hamiltonian system has been extensively applied to the field of modern physics.[28,29] In 1989, Li[28–31] studied the Noether symmetrical theorems of a Hamiltonian system in phase space and their applications. Later, Mei[12] and Zhang[26,27] gave the Lie symmetrical theorems of the classical Hamiltonian system in phase space. Relativistic analytical mechanics is an important aspect with the modern development in the field of theoretical physics. Since 1987, we have constructed the theory of analytical mechanics for the relativistic system, and given its basic theoretical frame.[32–37] This paper studies the Lie symmetrical non-Noether conserved quantities of the relativistic Hamiltonian system in phase space. A new method for obtaining conserved quantities from Lie symmetries of the system is given. On the basis of the theory of invariance of

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differential equations under infinitesimal transformations of groups, and introducing infinitesimal transformations for q_s and p_s, we construct the determining equations of Lie symmetrical transformations of the system, which only depend on the canonical variables. A set of non-Noether conserved quantities are directly obtained from the Lie symmetries of the relativistic Hamiltonian system. An example is given to illustrate the application of the results.

2. A set of Lie symmetrical determining equations of the relativistic Hamiltonian systems

Here we study a relativistic Hamiltonian system, and its equations of motion can be written as[32]

$$\dot{q}_s = \frac{\partial H^*}{\partial p_s}, \quad \dot{p}_s = -\frac{\partial H^*}{\partial q_s} \quad (s = 1, \cdots, n), \tag{1}$$

where

$$H^*(t, q_s, p_s) = p_s \dot{q}_s - L^*, \quad p_s = \frac{\partial L^*}{\partial \dot{q}_s},$$

$$L^*(t, q_s, \dot{q}_s) = T^* - V, \quad V = V(q_s),$$

$$T^* = \sum_{i=1}^{N} m_{0i} c^2 (1 - \sqrt{1 - \dot{r}_i^2/c^2}),$$

$$m_{0i} = m_i \sqrt{1 - \dot{r}_i^2/c^2} \quad (i = 1, \cdots, N), \tag{2}$$

and (q_s, p_s) are the canonical variables.

The equations of motion of the relativistic nonsingular Lagrangian system can be written in the canonical forms under the Legendre transformation. All of them are the relativistic Lagrangian system, the relativistic holonomic conservative system, the relativistic holonomic system in which the generalized forces have a generalized potential and the systems of inverse problem of the relativistic Lagrangian mechanics, so their equations of motion can be always written in the Hamiltonian canonical form of Eqs.(1).

Equations (1) can be expressed as follows

$$\dot{q}_s = g_s(t, q_k, p_k), \quad \dot{p}_s = h_s(t, q_k, p_k),$$

$$(s, k = 1, \cdots, n). \tag{3}$$

We introduce the special infinitesimal transformations for q_s and p_s

$$t^* = t, \quad q_s^*(t^*) = q_s(t) + \Delta q_s,$$

$$p_s^*(t^*) = p_s(t) + \Delta p_s. \tag{4}$$

Their expanding forms are

$$t^* = t, \quad q_s^* = q_s + \varepsilon \xi_s(t, q_k, p_k),$$

$$p_s^* = p_s + \varepsilon \eta_s(t, q_k, p_k),$$

$$(s, k = 1, \cdots, n), \tag{5}$$

where ξ_s and η_s are called generators of q_s and p_s respectively under the infinitesimal transformations (5), and ε is a small parameter. We introduce an infinitesimal transformation vector of the generator

$$X^{(0)} = \xi_s \frac{\partial}{\partial q_s} + \eta_s \frac{\partial}{\partial p_s}, \tag{6}$$

and its first extension reads

$$X^{(1)} = X^{(0)} + \dot{\xi}_s \frac{\partial}{\partial \dot{q}_s} + \dot{\eta}_s \frac{\partial}{\partial \dot{p}_s}. \tag{7}$$

From the theory of invariance of differential equation under infinitesimal transformations, we know that the invariance of Eqs.(3) under the infinitesimal transformations (5) is

$$\dot{\xi}_s = X^{(0)}(g_s), \quad \dot{\eta}_s = X^{(0)}(h_s),$$

$$(s = 1, \cdots, n). \tag{8}$$

Eqs. (8) are called the Lie symmetrical determining equations of the relativistic Hamiltonian system. If the generators ξ_s and η_s of the infinitesimal transformations (5) satisfy the determining Eqs.(8), then the transformations (5) are called Lie symmetrical transformations.

Theorem 1. For the relativistic Hamiltonian system (1), if the generators ξ_s and η_s of the infinitesimal transformations (5) satisfy the determining Eqs.(8), then the corresponding symmetries are Lie symmetries of the system.

3. A set of Lie symmetrical non-Noether conserved quantities of relativistic Hamiltonian systems

We now obtain directly the conserved quantities from the Lie symmetries of the relativistic Hamiltonian system.

Theorem 2. For the relativistic Hamiltonian system (1), if the generators ξ_s and η_s of the infinitesimal transformations (5) satisfy the determining Eqs.(8), and there is a function $\lambda(q_s, p_s)$ satisfying the following condition

$$\frac{1}{\lambda} \frac{\partial}{\partial q_s}(\lambda g_s) + \frac{1}{\lambda} \frac{\partial}{\partial p_s}(\lambda h_s) = 0, \tag{9}$$

then the system has the Lie symmetrical non-Noether conserved quantities as follows:

$$\Phi = \frac{1}{\lambda}\frac{\partial}{\partial q_s}(\lambda\xi_s) + \frac{1}{\lambda}\frac{\partial}{\partial p_s}(\lambda\eta_s) = \text{constant}. \tag{10}$$

In fact,

$$\begin{aligned}\frac{\mathrm{d}\Phi}{\mathrm{d}t} =&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial q_s}\right)\xi_s + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial p_s}\right)\eta_s + \frac{1}{\lambda}\frac{\partial\lambda}{\partial q_s}\dot{\xi}_s\\ &+ \frac{1}{\lambda}\frac{\partial\lambda}{\partial p_s}\dot{\eta}_s + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\xi_s}{\partial q_s} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\eta_s}{\partial p_s}.\end{aligned} \tag{11}$$

From Eqs.(3) and (6), the determining Eqs.(8) can be written in the form

$$\begin{aligned}\dot{\xi}_s =&\xi_k\frac{\partial g_s}{\partial q_k} + \eta_k\frac{\partial g_s}{\partial p_k},\\ \dot{\eta}_s =&\xi_k\frac{\partial h_s}{\partial q_k} + \eta_k\frac{\partial h_s}{\partial p_k} \quad (k,s=1,\cdots,n),\end{aligned} \tag{12}$$

so we have

$$\begin{aligned}\frac{\partial\dot{\xi}_s}{\partial q_s} &= \frac{\partial\xi_k}{\partial q_s}\frac{\partial g_s}{\partial q_k} + \frac{\partial\eta_k}{\partial q_s}\frac{\partial g_s}{\partial p_k} + \xi_k\frac{\partial^2 g_s}{\partial q_k\partial q_s} + \eta_k\frac{\partial^2 g_s}{\partial p_k\partial q_s},\\ \frac{\partial\dot{\eta}_s}{\partial p_s} &= \frac{\partial\xi_k}{\partial p_s}\frac{\partial h_s}{\partial q_k} + \frac{\partial\eta_k}{\partial p_s}\frac{\partial h_s}{\partial p_k} + \xi_k\frac{\partial^2 h_s}{\partial q_k\partial p_s} + \eta_k\frac{\partial^2 h_s}{\partial p_k\partial p_s}.\end{aligned} \tag{13}$$

Noticing

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\xi_s}{\partial q_s} &= \frac{\partial^2\xi_s}{\partial q_s\partial q_k}g_k + \frac{\partial^2\xi_s}{\partial q_s\partial p_k}h_k + \frac{\partial^2\xi_s}{\partial q_s\partial t},\\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\eta_s}{\partial p_s} &= \frac{\partial^2\eta_s}{\partial p_s\partial q_k}g_k + \frac{\partial^2\eta_s}{\partial p_s\partial p_k}h_k + \frac{\partial^2\eta_s}{\partial p_s\partial t},\\ \dot{\xi}_s &= \frac{\partial\xi_s}{\partial q_k}g_k + \frac{\partial\xi_s}{\partial p_k}h_k + \frac{\partial\xi_s}{\partial t},\\ \dot{\eta}_s &= \frac{\partial\eta_s}{\partial q_k}g_k + \frac{\partial\eta_s}{\partial p_k}h_k + \frac{\partial\eta_s}{\partial t},\end{aligned} \tag{14}$$

we have

$$\begin{aligned}\frac{\partial\dot{\xi}_s}{\partial q_s} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\xi_s}{\partial q_s} + \frac{\partial\xi_s}{\partial q_k}\frac{\partial g_k}{\partial q_s} + \frac{\partial\xi_s}{\partial p_k}\frac{\partial h_k}{\partial q_s},\\ \frac{\partial\dot{\eta}_s}{\partial p_s} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\eta_s}{\partial p_s} + \frac{\partial\eta_s}{\partial q_k}\frac{\partial g_k}{\partial p_s} + \frac{\partial\eta_s}{\partial p_k}\frac{\partial h_k}{\partial p_s}.\end{aligned} \tag{15}$$

Substituting Eqs.(13) into Eqs.(15), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\xi_s}{\partial q_s} &= \frac{\partial\eta_k}{\partial q_s}\frac{\partial g_s}{\partial p_k} - \frac{\partial\xi_s}{\partial p_k}\frac{\partial h_k}{\partial q_s} + \xi_k\frac{\partial^2 g_s}{\partial q_k\partial q_s} + \eta_k\frac{\partial^2 g_s}{\partial p_k\partial q_s},\\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\eta_s}{\partial p_s} &= \frac{\partial\xi_k}{\partial p_s}\frac{\partial h_s}{\partial q_k} - \frac{\partial\eta_s}{\partial q_k}\frac{\partial g_k}{\partial p_s} + \xi_k\frac{\partial^2 h_s}{\partial q_k\partial p_s} + \eta_k\frac{\partial^2 h_s}{\partial p_k\partial p_s}.\end{aligned} \tag{16}$$

Substituting Eqs.(12) and (16) into Eq.(11), we can obtain

$$\begin{aligned}\frac{\mathrm{d}\Phi}{\mathrm{d}t} =&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial q_s}\right)\xi_s + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial p_s}\right)\eta_s\\ &+ \frac{1}{\lambda}\frac{\partial\lambda}{\partial q_s}\left(\xi_k\frac{\partial g_s}{\partial q_k} + \eta_k\frac{\partial g_s}{\partial p_k}\right)\\ &+ \frac{1}{\lambda}\frac{\partial\lambda}{\partial p_s}\left(\xi_s\frac{\partial h_s}{\partial q_k} + \eta_k\frac{\partial h_s}{\partial p_k}\right)\\ &+ \frac{\partial^2 g_s}{\partial q_k\partial q_s}\xi_k + \frac{\partial^2 h_s}{\partial q_k\partial p_s}\xi_k\\ &+ \frac{\partial^2 g_s}{\partial p_k\partial q_s}\eta_k + \frac{\partial^2 h_s}{\partial p_k\partial p_s}\eta_k.\end{aligned} \tag{17}$$

From the condition (9), we have

$$\begin{aligned}&\frac{\partial^2 g_s}{\partial q_s\partial q_k} + \frac{\partial^2 h_s}{\partial p_s\partial q_k}\\ =&-\frac{\partial}{\partial q_k}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial q_s}\right)g_s - \frac{1}{\lambda}\frac{\partial\lambda}{\partial q_s}\frac{\partial g_s}{\partial q_k}\\ &-\frac{\partial}{\partial q_k}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial p_s}\right)h_s - \frac{1}{\lambda}\frac{\partial\lambda}{\partial p_s}\frac{\partial h_s}{\partial q_k},\end{aligned} \tag{18}$$

$$\begin{aligned}&\frac{\partial^2 g_s}{\partial q_s\partial p_k} + \frac{\partial^2 h_s}{\partial p_s\partial p_k}\\ =&-\frac{\partial}{\partial p_k}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial q_s}\right)g_s - \frac{1}{\lambda}\frac{\partial\lambda}{\partial q_s}\frac{\partial g_s}{\partial q_k}\\ &-\frac{\partial}{\partial q_k}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial p_s}\right)h_s - \frac{1}{\lambda}\frac{\partial\lambda}{\partial p_s}\frac{\partial h_s}{\partial p_k}.\end{aligned} \tag{19}$$

Noticing

$$\begin{aligned}\frac{\partial}{\partial q_k}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial q_s}\right)g_k\xi_s &= \frac{\partial}{\partial q_k}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial q_s}\right)g_s\xi_k,\\ \frac{\partial}{\partial q_k}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial p_s}\right)g_k\eta_s &= \frac{\partial}{\partial p_k}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial q_s}\right)g_s\eta_k,\\ \frac{\partial}{\partial p_k}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial q_s}\right)h_k\xi_s &= \frac{\partial}{\partial q_k}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial p_s}\right)h_s\xi_k,\\ \frac{\partial}{\partial p_k}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial p_s}\right)h_k\eta_s &= \frac{\partial}{\partial p_k}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial p_s}\right)h_s\eta_k,\end{aligned} \tag{20}$$

and substituting Eqs.(18),(19)and (20) into Eq.(17), we obtain

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = 0, \quad \text{i.e.,} \quad \Phi = \text{constant}. \tag{21}$$

From theorem 2, we can obtain an important deduction.

Deduction For the relativistic Hamiltonian system (1), if the generators ξ_s and η_s of the infinitesimal transformations (5) satisfy the determining Eqs.(8) and the following condition

$$\frac{\partial g_s}{\partial q_s} + \frac{\partial h_s}{\partial p_s} = 0, \tag{22}$$

then the system has the Lie symmetrical non-Noether conserved quantities as follows

$$\Phi' = \frac{\partial \xi_s}{\partial q_s} + \frac{\partial \eta_s}{\partial p_s} = \text{constant}. \tag{23}$$

4. Example

As an example, we study the Lie symmetries and conserved quantities of a relativistic vibrator, of which the Lagrangian is

$$L^* = m_0 c^2 (1 - \sqrt{1 - \dot{q}^2/c^2}) - \frac{1}{2} k q^2, \tag{24}$$

where k=constant. We have

$$p = \frac{\partial L^*}{\partial \dot{q}} = m_0 (1 - \dot{q}^2/c^2)^{-1/2} \dot{q}. \tag{25}$$

Noticing

$$(mc^2)^2 = (m_0 c^2)^2 + (cp)^2, \tag{26}$$

$$\frac{p^2}{m_0^2 c^2} = \frac{\dot{q}^2/c^2}{1 - \dot{q}^2/c^2}, \tag{27}$$

we have

$$H^* = m_0 c^2 \left(\sqrt{1 + \frac{p^2}{m_0^2 c^2}} - 1 \right) + \frac{1}{2} k q^2, \tag{28}$$

$$g = \frac{\partial H^*}{\partial p} = \frac{2p}{m_0} \left(1 + \frac{p^2}{m_0^2 c^2} \right)^{-1/2},$$

$$h = -\frac{\partial H^*}{\partial q} = -kq. \tag{29}$$

If we introduce the infinitesimal transformations (5), then the determining Eqs.(8) give

$$\dot{\xi} = \frac{2}{m_0} \left(1 + \frac{p^2}{m_0^2 c^2} \right)^{-3/2} \eta, \quad \dot{\eta} = -k\xi, \tag{30}$$

and the above equations have the following solutions

$$\xi = \mathrm{i}\varphi \mathrm{e}^{-\mathrm{i}k\varphi t}, \quad \eta = \mathrm{e}^{-\mathrm{i}k\varphi t}, \tag{31}$$

where

$$\varphi = \sqrt{\frac{2}{k m_0}} \left(1 + \frac{p^2}{m_0^2 c^2} \right)^{-3/4}. \tag{32}$$

From theorem 1, we know that the generators (31) correspond to the Lie symmetry of the system under the infinitesimal transformations (5). Substituting Eqs.(29) into the condition (9), we can obtain

$$\lambda = \frac{1}{2} a k q^2 + 2 a m_0 c^2 \left(1 + \frac{p^2}{m_0^2 c^2} \right)^{1/2} + b,$$

$$(a, b = \text{constant}). \tag{33}$$

From theorem 2, we obtain the following conserved quantity

$$\Phi = (g - \mathrm{i}h\varphi) \frac{1}{\varphi} \mathrm{e}^{-\mathrm{i}k\varphi t} = \text{constant}. \tag{34}$$

Obviously, Φ is the non-Noether conserved quantity of the system. Noticing $\lambda \neq$ constant, the system does not have the conserved quantity as Φ'.

Under the classical approximation $\dot{q}_s \ll c$, we have $m \approx m_0, p = m\dot{q}$ and $p^2/m_0^2 c^2 \approx 0$. For the L^*, choosing the former two terms of the power-series expression of $\sqrt{1 - \dot{q}^2/c^2}$ with respect to $\dot{q}/c$ we can obtain

$$L^* \approx L = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} k q^2, \tag{35}$$

$$H^* \approx H = \frac{1}{2} m \dot{q}^2 + \frac{1}{2} k q^2. \tag{36}$$

Therefore, in this example we obtain the Lie symmetries and the conserved quantities of the classical vibrator.

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