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Chaos in a gravitational field with dipoles

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In this paper we investigate the dynamics of a test particle in the gravitational field with dipoles. At first we study the gravitational potential by numerical simulations, we find that, for appropriate parameters, there are two different cases in the potential curve: one is the one-well case with a stable critical point, and the other is the three-well case with three stable critical points and two unstable critical points. By performing Poincaré sections for different values of the parameters and initial conditions, we find a regular motion and a chaotic motion. From these Poincaré sections, we further confirm that the chaotic motion of the test particle originates mainly from the dipoles.

Keywords: chaos, Poincaré section, dipole
PACC: 0314, 0400, 0545

1. Introduction

Chaotic behaviours have been considered as an interesting phenomenon by many physicists since Lorenz[1] found the deterministic non-periodic flow in 1963. In the past decades, chaos is one of the most important ideas used to explain various nonlinear phenomena in nature. After Poincaré studied the three-body problem, many investigations on chaos in celestial mechanics and astrophysics have been done.[2,3] There are two main lines of the research in general relativity, some deal with the chaos associated with inhomogeneous cosmological models. For example, Oliveira *et al*[4−7] have studied the chaotic behaviour in Bianchi IX model. The others assume a given metric and then look for chaotic behaviours of the geodesic motion in a given background. Although we know many features of the chaos in the Newtonian dynamics, we do not know, so far, much about those in general relativity. Because the gravitational field around a black hole is very strong and nonlinear, we expect to find a new type of the chaotic behaviour in such a strong gravitational field which does not appear in Newtonian dynamics.[8−10] Some authors[11−19] found the chaotic behaviour of a test particle in a relativistic system. Letelier *et al*[20−22] investigated the chaos in black holes with halos. Chen[23] and Wang[24] investigated the dynamics of an extreme charged black hole. Saa[25,16] extended the investigation on the integrability of oblique orbits of the test particle in several gravitational fields. And there are many examples in the literature of chaotic motions involving black holes, such as: the fixed two centres problem,[12,27,28] a black hole surrounded by gravitational waves,[29,30] and several core-shell models with relevance to the description of galaxies.[31] As a Newtonian case, recent works of Chicone *et al*[32] concentrated on the chaotic behaviour of the Hill system. In order to investigate the chaotic behaviour of the dynamic system, many researchers concentrated on the study of the chaotic dynamics in general relativity with the Poincaré–Melnikov method.[33,34] The Melnikov method is an analytical criterion to determine the occurrence of chaos in integrable systems in which homoclinic (or heteroclinic) manifolds biasymptotic to unstable critical points or to periodic orbits (more generally to invariant tori) are subjected to small perturbations.

It is well known that the charge non-spherical-symmetrical distribution is a common phenomenon. So it is interesting to investigate the motion of the test particle in the space–time of gravitational field with dipoles. In this paper, by performing the numeral simulations, we work out the orbital evolution in the phase space and Poincaré sections for different initial conditions, and confirm that there are regular and chaotic motions of the test particle in the grav-

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itational field with dipoles. In the next section we investigate the Hamiltonian of the gravitational field with dipoles. In section 3, we perform numerical simulations to study the potential and the evolution of orbits of the test particle, and at the same time, present Poincaré sections for different parameters. In the final section, a brief conclusion is given.

2. Hamiltonian of the gravitational field with dipoles

Wang *et al*[35−37] gave the metric of the gravitational field with dipoles. Because the space–time of our system is of a static axial-symmetry, using cylindrical coordinates (r, θ, z) to describe the space–time, we obtain the potential to describe the gravitational field with dipoles in Newtonian limits

$$V = -\frac{M}{\sqrt{r^2+z^2}} + \frac{Q^2}{2(r^2+z^2)} + \frac{P^2 z^2}{2(r^2+z^2)^3}, \quad (1)$$

where M, Q and P are mass, charges and dipoles of the gravitational source, respectively, and $r^2 + z^2 = x^2 + y^2 + z^2$, where x, y, z are the usual Cartesian coordinates.

We know that the angular momentum L in the z direction is conserved and we can also easily reduce the originally three-dimensional (3D) problem to a two-dimensional (2D) one in coordinates (r, z). Now we consider the bounded orbit for the test particle under the potential (1). In Newtonian limits the Hamiltonian is

$$H = \frac{\dot{r}^2 + \dot{z}^2}{2} + \frac{L^2}{2r^2} - \frac{M}{\sqrt{r^2+z^2}} + \frac{Q^2}{2(r^2+z^2)} + \frac{P^2 z^2}{2(r^2+z^2)^3}. \quad (2)$$

Hamiltonian–Jacobi equations corresponding to Hamiltonian (2) can be properly separated in parabolic coordinates,[38,39] leading to the second constant of the motion

$$C = R_z - \alpha \frac{r^2}{2}, \quad (3)$$

where R_z is the z component of the Laplace–Runge–Lenz vector

$$R = \frac{M}{\sqrt{r^2+z^2}}(r\hat{r} + z\hat{z}) + V \times L, \quad (4)$$

where L denotes the total angular momentum. In this case, with two constants of motion H and C, the equations for the trajectories of the test particle can be reduced to quadrature in parabolic coordinates[38,39] and we note that the equations of the motion are invariant under the following rescaling:

$$r \to \lambda r, z \to \lambda z, t \to \lambda t, M \to \lambda M,$$
$$Q \to \lambda Q, P \to \lambda P, H \to \lambda H. \quad (5)$$

3. Numerical simulations

The gravitational potential (1) changes from the one-well case to the three-well case when we change one parameter while fixed the other parameters. Figure 1 shows the potential for different values of P ($P^2 = 7$ (dashed) and $P^2 = 2$(solid)) with other parameters fixed ($M = 1, Q^2 = 0.2, r = 1$). From Fig.1 we can see that the motion in the one-well case is an oscillation, however, in the three-well potential case, the motion is quite different from that in the one-well case. There are three stable critical points (A) and two unstable critical points (B). When the kinetic energy of the test particle is much higher than the gravitational potential, the motion is similar to that in the one-well case; however, when the kinetic energy of the test particle is close to the gravitational potential, the motion is a random oscillation in one of the three wells, particularly near the two unstable critical points. So the motion of the test particle in this case depends sensitively on the initial conditions and the parameters, and becomes chaotic.

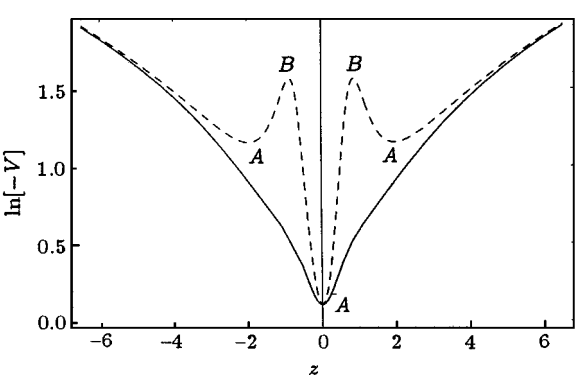


Fig.1. Plots of gravitational potential (1) for $M = 1, Q^2 = 0.2, r = 1$ and $P^2 = 7$ (dashed line), $P^2 = 2$ (solid line). From the plot we can see that when $P^2 = 2$ there is only one well in the potential curve and a stable critical point (A), while for $P^2 = 7$ there are three wells in the potential curve, with three stable critical points (A) and two unstable critical points (B). In the latter case the test particle oscillates randomly in one of the three wells.

In this paper, using the numerical method, we work out the evolution of the test particle in the phase space and using the Poincaré section method, we investigate the properties of the dynamics of the test particle in the gravitational field with dipoles. In order to investigate the evolutions of the test particle in the gravitational field with dipoles and its chaotic behaviour, we use variables $(\dot{r}, \dot{z}, r, z)$ and the package POINCARÉ[40] to perform the following numerical experiments and to investigate the chaotic behaviour of a dynamical system. Based on the nonlinear dynamical theory, if the motion is not chaotic, the plotted points form a closed curve in the 2D $(\dot{r}, r)$ plane, because a regular orbit will move on a torus in the phase space and the curve is a cross section of the torus. If the orbit is chaotic, some of those tori will be broken and the Poincaré section does not consist of a set of closed curves, while the points will be distributed randomly in the allowed region to form a chaotic sea. From the distribution of the points in the Poincaré section, we can judge whether the motion is chaotic or not. Figures 2–7 show the typical Poincaré sections across the plane $z = 0$ for different initial conditions. In Fig.2, we present the Poincaré section for $M = 1, L^2 = 0.4, P^2 = 12$, $Q^2 = 2$ and $H = -0.18800554$, and the section consists of a KAM tori structure which characterizes a quasi-integrable Hamilton system with quasi-periodic orbits. With these values of the parameters, there is no chaotic behaviour. We plot the Poincaré sections for $M = 1, L^2 = 0.4, P^2 = 1.2$, $Q^2 = 0.2$ and $H = -0.28340554$ in Fig.3, then we can see that some points are distributed randomly in a finite region to form a chaotic sea, and there are two islands which are surrounded by the chaotic sea. If we go on changing the parameters and the initial conditions to draw Figs.4 and 5 for $M = 1, L^2 = 0.4, P^2 = 12$, $Q^2 = 0.2$, $H = -0.27800554$ and $M = 1, L^2 = 0.2, P^2 = 12$, $Q^2 = 0.2$, $H = -0.28911665$, respectively, a far larger chaotic sea will be obtained in an allowed region and the islands will almost be submerged by the chaotic sea. This means that the quasi-periodic orbits will change into chaos. By comparing the parameters in Figs.2–5, we can find that the chaotic behaviour mainly originates from the dipoles, i.e. the nonspherical-symmetry of the charge distribution.

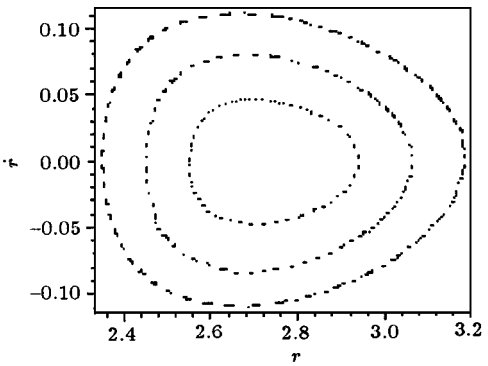


Fig.2. Poincaré section $(\dot{r}, r)$ across the plane $z = 0$, with $\dot{z} > 0$ for $M = 1, L^2 = 0.4, P^2 = 12$, $Q^2 = 2$ and $H = -0.18800554$. For these values of the parameters we have the section of an integrable motion.

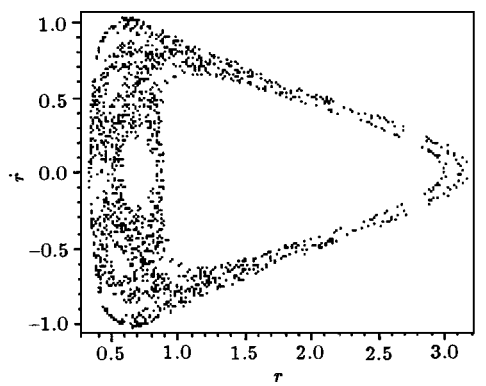


Fig.3. Poincaré section $(\dot{r}, r)$ across the plane $z = 0$, with $\dot{z} > 0$ for $M = 1, L^2 = 0.4, P^2 = 1.2$, $Q^2 = 0.2$ and $H = -0.28340554$. Under this condition there is a large chaotic sea, which means a chaotic motion of the test particle, in the section.

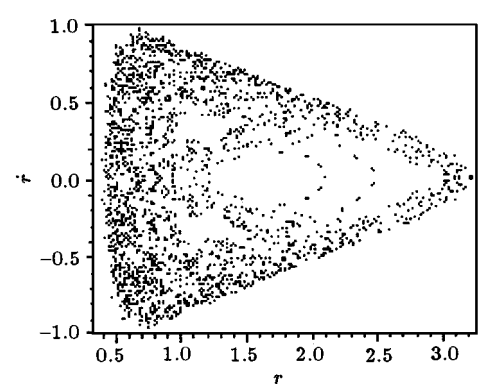


Fig.4. Poincaré section $(\dot{r}, r)$ across the plane $z = 0$, with $\dot{z} > 0$ for $M = 1, L^2 = 0.4, P^2 = 12$, $Q^2 = 0.2$ and $H = -0.27800554$. Under this condition there is a larger chaotic sea than that in Fig.3 in the section, which means a chaotic motion of the test particle.

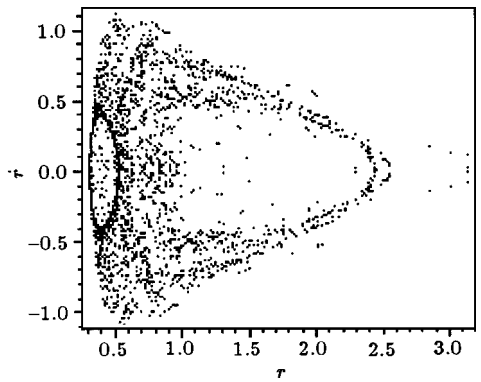


Fig.5. Poincaré section $(\dot{r}, r)$ across the plane $z = 0$, with $\dot{z} > 0$ for $M = 1, L^2 = 0.2, P^2 = 12$, $Q^2 = 0.2$ and $H = -0.28911665$. Under this condition there is a much larger chaotic sea than those in Figs.3 and 4 in the section, which means a chaotic motion of the test particle.

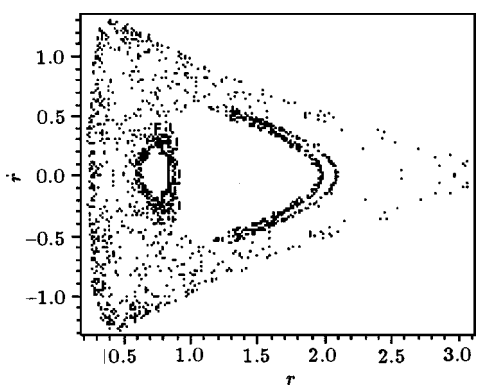


Fig.6. Poincaré section $(\dot{r}, r)$ across the plane $z = 0$, with $\dot{z} > 0$ for $M = 1, L^2 = 0.4, P^2 = 1.2$, $Q^2 = 0$ and $H = -0.29340554$. Under this extreme condition there is a large chaotic sea, which means a chaotic motion of the test particle, in the section.

By performing further investigations, we can work out the next two extreme cases, in Fig.6 we present the Poincaré section for $M = 1, L^2 = 0.4, P^2 = 1.2$, $Q^2 = 0$ and $H = -0.29340554$. In this case due to the non-spherical symmetry of the charge distribution, there are dipoles P in the gravitational source. From Fig.5, we can find that there are two islands surrounded by a large chaotic sea; that is to say, with above parameters our system is a chaotic one. However, if the charge distribution is spherical-symmetric, i.e. there is no dipole in the gravitational source, we work out the Poincaré section for $M = 1, L^2 = 0.4, P^2 = 0$, $Q^2 = 0.2$ and $H = -0.28400554$ in Fig.7. We can see that there is a KAM tori structure in this section. This case corresponds to an integrable motion.

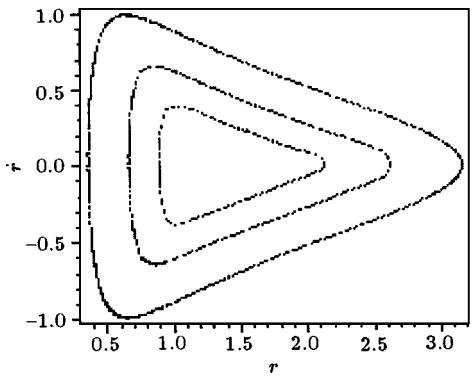


Fig.7. Poincaré section $(\dot{r}, r)$across the plane $z = 0$, with $\dot{z} > 0$ for $M = 1, L^2 = 0.4, P^2 = 0$, $Q^2 = 0.2$ and $H = -0.28400554$. For these extreme values of the parameters we have the section of an integrable motion just as in Fig.2.

4. Conclusions and discussion

In this paper we have investigated the gravitational potential (1) by numerical simulations. We find that, for appropriate parameters of the system, there are two cases in the potential curves: one is the one-well case with a stable critical point, and the other is the three-well case with three stable and two unstable critical points. In order to verify the chaotic motion, we have presented Poincaré sections for different values of the parameters and initial conditions. From these Poincaré sections we have found the regular motion and the chaotic motion. By comparing these Poincaré sections, we confirm that the chaotic motion of the test particle originates mainly from dipoles.

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