Vol 12 No 8, August 2003
1009-1963/2003/12(08)/0831-05

Chinese Physics

© 2003 Chin. Phys. Soc.
and IOP Publishing Ltd

# The calculating formula for radial matrix elements of a relativistic harmonic oscillator

Qiang Wen-Chao(强稳朝)

*Faculty of Science, Xi'an University of Architecture and Technology, Xi'an 710055, China*

(Received 31 December 2002; revised manuscript received 10 April 2003)

A universal practical formula is given for calculating an integral which includes two confluent hypergeometric functions, power and exponential functions; then by means of this formula, the expressions of the radial matrix elements for a relativistic harmonic oscillator are given.

**Keywords:** relativistic harmonic oscillator, radial matrix element, confluent hypergeometric function
**PACC:** 0300, 0330, 0365

## 1. Introduction

In the study and application of the quantum mechanics, it is an important and onerous task to calculate the matrix elements of various operators when the wavefunction was given. To reveal the properties of a quantum system that has a spherical symmetric potential, it is indispensable to calculate radial matrix elements with various powers. Hydrogen atom and isotropic harmonic oscillator, which have a central potential field, are typical examples. Some classical papers have given a few formulae, with some specific condition, for calculating the matrix element of a radial operator of a non-relativistic hydrogen atom.[1,2] Because the wavefunctions of hydrogen atoms and isotropic harmonic oscillators include combination of the confluent hypergeometric function, power and exponential functions, the calculation of radial matrix elements for a central potential field often involves a very complex integral whether in non-relativistic or relativistic quantum mechanics. At present, even though we can use powerful mathematical software, such as Mathematica or Matlab packages, it will take much time or is too difficult to complete the computation for calculating radial matrix elements with higher powers. So it is necessary to look for not only universal but also practical formulae for calculating radial matrix elements either in non-relativistic or relativistic quantum mechanics. In recent years, there have been some studies on calculating radial matrix elements both in non-relativistic and relativistic quantum mechanics.[3−7] Hou[3,4] has given formulae to calculate radial matrix elements of a relativistic hydrogen atom and a non-relativistic isotropic harmonic oscillator. Chen[5−7] obtained recurrence formulae for radial matrix elements of an isotropic harmonic oscillator and a ring-shaped non-spherical harmonic oscillator. On the other hand, we have recently obtained not only the normalized s-wave but also the complete non s-wave wavefunction of a relativistic harmonic oscillator.[8] Obviously, it is now necessary to calculate the radial matrix elements of a relativistic harmonic oscillator. In the present paper, we first give a formula to calculate the integrals that include two confluent hypergeometric functions, exponential and power functions. This formula can always be used to calculate on a computer. In the next two sections, we give the formula to calculate radial matrix elements of a relativistic harmonic oscillator either in Klein–Gordon or Dirac theory.

## 2. A universal formula to calculate the integral including two confluent hypergeometric functions, and exponential and power functions

Landau[1] gave a recurrence formula to calculate

the following integral,

$$J_{\gamma}^{sp}(\alpha,\alpha') = \int_0^{\infty} \mathrm{e}^{\frac{k+k'}{2}z} z^{\gamma-1+s} F(\alpha,\gamma,kz) F(\alpha',\gamma-p,k'z)\mathrm{d}z, \quad (1)$$

where $F$ represents the confluent hypergeometric function $F[(\alpha,\gamma,z],$[1,9] $s$ and $p$ are positive integers. Though Eq.(1) is powerful in calculating a great deal of integrals, that resemble in form

$$\int_0^{\infty} \mathrm{e}^{-\lambda z} z^w F(\alpha,\gamma,kz) F(\alpha',\gamma',k'z)\mathrm{d}z, \quad (2)$$

but when $k=k'$ or $\gamma=\gamma'$ and $w$ are not positive integers, Eq.(1) is not applicable. In Ref.[3] is also given a formula which can calculate integral (2), but the parameters of which are not symmetric about $F(\alpha,\gamma,kz)$ and $F(\alpha',\gamma',k'z)$. Here we give another formula to calculate integral (2). Defining

$$J_{n_1 n_2 \mu_1 \mu_2}^{k_1 k_2}(w,\lambda) = \int_0^1 z^w \mathrm{e}^{-\lambda z} F(-n_1,\mu_1,k_1 z) F(-n_2,\mu_2,k_2 z)\mathrm{d}z, \quad (3)$$

where $n_1$ and $n_2$ are non-negative integers, substituting the series expression of the confluent hypergeometric function

$$F(\alpha,\gamma,z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!(\gamma)_n} z^n \quad (4)$$

into Eq.(3), and noticing that when $\alpha$ are non-positive integers, according to the definition of Pochhammer symbol,[9] the terms with $n > -\alpha$ are zero in Eq.(4), we can express Eq.(3) as

$$J_{n_1 n_2 \mu_1 \mu_2}^{k_1 k_2}(w,\lambda) = \sum_{q_1=0}^{n_1} \sum_{q_2=0}^{n_2} \frac{k_1^{q_1} k_2^{q_2} (-n_1)_{q_1} (-n_2)_{q_2}}{q_1! q_2! (\mu_1)_{q_1} (\mu_2)_{q_2}} \times \int_0^1 z^{q_1+q_2+w} \mathrm{e}^{-\lambda z}\mathrm{d}z. \quad (5)$$

Using the integral formula

$$\int_0^{\infty} x^n \mathrm{e}^{-\mu x}\mathrm{d}x = n!\mu^{-(n+1)}, \quad (\mathrm{Re}\ \mu > 0), \quad (6)$$

$J_{n_1 n_2 \mu_1 \mu_2}^{k_1 k_2}(w,\lambda)$ can be expressed as

$$J_{n_1 n_2 \mu_1 \mu_2}^{k_1 k_2}(w,\lambda) = \sum_{q_1=0}^{n_1} \sum_{q_2=0}^{n_2} \frac{k_1^{q_1} k_2^{q_2} (-n_1)_{q_1} (-n_2)_{q_2} (q_1+q_2+w)!}{q_1! q_2! (\mu_1)_{q_1} (\mu_2)_{q_2}} \times \lambda^{-(q_1+q_2+w+1)}, \quad (\mathrm{Re}\ \lambda > 0). \quad (7)$$

Though formula (7) contains two finite summations, it can not only always be executed on a computer, but also the time it takes is far less than that for directly calculating integral (3). Moreover, integral (3) will sometimes be too complicated to complete. It will be seen that if we use formula (7) to calculate the radial matrix elements, $w$ will often be non-negative integers or semi-integers. When $w$ is a non-negative integer, because

$$(q_1+q_2+w)!\lambda^{-(q_1+q_2+w+1)} = (-1)^w \Gamma(q_1+q_2+1) \frac{\mathrm{d}^w}{\mathrm{d}\lambda^w} \lambda^{-(q_1+q_2+1)}, \quad (8)$$

$J_{n_1 n_2 \mu_1 \mu_2}^{k_1 k_2}(w,\lambda)$ can be written as

$$J_{n_1 n_2 \mu_1 \mu_2}^{k_1 k_2}(w,\lambda) = (-1)^w \sum_{q_1=0}^{n_1} \sum_{q_2=0}^{n_2} \frac{k_1^{q_1} k_2^{q_2} (-n_1)_{q_1} (-n_2)_{q_2} \Gamma(q_1+q_2+1)}{q_1! q_2! (\mu_1)_{q_1} (\mu_2)_{q_2}} \times \frac{\mathrm{d}^w}{\mathrm{d}\lambda^w} \lambda^{-(q_1+q_2+1)}. \quad (9)$$

Similarly when $w$ is a semi-integer and larger than or equal to $-1/2$, we have

$$(q_1+q_2+w)!\lambda^{-(q_1+q_2+w+1)} = (-1)^{w+1/2} \Gamma(q_1+q_2+1/2) \frac{\mathrm{d}^{w+1/2}}{\mathrm{d}\lambda^{w+1/2}} \lambda^{-(q_1+q_2+1/2)}. \quad (10)$$

$J_{n_1 n_2 \mu_1 \mu_2}^{k_1 k_2}(w,\lambda)$ can also be written as

$$J_{n_1 n_2 \mu_1 \mu_2}^{k_1 k_2}(w,\lambda) = (-1)^{w+1/2} \times \sum_{q_1=0}^{n_1} \sum_{q_2=0}^{n_2} \frac{k_1^{q_1} k_2^{q_2} (-n_1)_{q_1} (-n_2)_{q_2} \Gamma(q_1+q_2+1/2)}{q_1! q_2! (\mu_1)_{q_1} (\mu_2)_{q_2}} \times \frac{\mathrm{d}^{w+1/2}}{\mathrm{d}\lambda^{w+1/2}} \lambda^{-(q_1+q_2+1/2)}. \quad (11)$$

Formulae (9) and (11) can be combined as one formula

$$J_{n_1 n_2 \mu_1 \mu_2}^{k_1 k_2}(w,\lambda) = (-1)^{w+\delta} \left[ \frac{\mathrm{d}^{w+\delta}}{\mathrm{d}p^{w+\delta}} J_{n_1 n_2 \mu_1 \mu_2}^{k_1 k_2}(-\delta,p) \right] \Bigg|_{p=\lambda}. \quad (12)$$

In the above formula, $\delta$=0 or $\delta = 1/2$ when $w$ is a non-negative integer or a semi-integer and larger than or equal to $-1/2$ respectively.

In quantum mechanics, one usually needs calculating the integral

$$I_{n_1 n_2 \mu_1 \mu_2}^{k_1 k_2}(w) = \int_0^{\infty} \mathrm{e}^{\frac{k+k'}{2}r^2} r^w F(-n_1,\mu_1,k_1 r^2) F(-n_2,\mu_2,k_2 r^2)\mathrm{d}r. \quad (13)$$

This integral can easily be expressed as

$$I^{k_1k_2}_{n_1n_2\mu_1\mu_2}(w)=\frac{1}{2}J^{k_1k_2}_{n_1n_2\mu_1\mu_2}\left(\frac{w-1}{2},\frac{k_1+k_2}{2}\right). \quad (14)$$

## 3. The radial matrix elements of a relativistic isotropic harmonic oscillator in Klein–Gordon theory

In Ref.[8] we have obtained the radial wavefunction of a relativistic isotropic harmonic oscillator with equal scalar potential and vector potential ($\hbar=c=1$)

$$R_{n,l}(r)=\frac{\sqrt{2}\beta^{2\alpha}_{n,l}}{2^{\alpha_l}\Gamma(4\alpha_l)}\mathrm{e}^{-\frac{\beta_{n,l}}{2\sqrt{2}}r^2}r^l\sqrt{\frac{\Gamma(n+4\alpha_l)}{n!}} \times F\left[-n,4\alpha_l,\frac{\beta_{n,l}}{\sqrt{2}}r^2\right], \quad (15)$$

where $F[\alpha,\gamma,z]$ is the confluent hypergeometric function,[9]

$$\alpha_l=(3+2l)/8, \quad \beta_{n,l}=\sqrt{k(E_{n,l}+\mu)}, \quad (16)$$

where $k$ is the elastic coefficient, $E_{n,l}$ and $\mu$ are the energy and rest mass of a harmonic oscillator respectively. The radial matrix elements are

$$\begin{aligned}&\langle n_1,l_1|r^w|n_2,l_2\rangle\\ =&\int_0^\infty r^{w+2}R^*_{n_1,l_1}R_{n_2,l_2}\mathrm{d}r\\ =&\frac{\beta^{2\alpha_{l_1}}_{n_1,l_1}\beta^{2\alpha_{l_2}}_{n_2,l_2}}{2^{\alpha_{l_1}+\alpha_{l_2}-1}\Gamma(4\alpha_{l_1})\Gamma(4\alpha_{l_2})}\\ &\times\sqrt{\frac{\Gamma(n_1+4\alpha_{l_1})\Gamma(n_2+4\alpha_{l_2})}{n_1!n_2!}}\\ &\times\int_0^\infty \mathrm{e}^{-\frac{1}{2\sqrt{2}}(\beta_{n_1,l_1}+\beta_{n_2,l_2})r^2}r^{l_1+l_2+w+2}\\ &\times F\left(-n_1,4\alpha_{l_1},\frac{\beta_{n_1,l_1}}{\sqrt{2}}r^2\right)\\ &\times F\left(-n_2,4\alpha_{l_2},\frac{\beta_{n_2,l_2}}{\sqrt{2}}r^2\right)\mathrm{d}r. \end{aligned} \quad (17)$$

Using formulae (13) and (14) and simplifying, we obtain

$$\begin{aligned}&\langle n_1,l_1|r^w|n_2,l_2\rangle\\ =&\frac{\beta^{2\alpha_{l_1}}_{n_1,l_1}\beta^{2\alpha_{l_2}}_{n_2,l_2}}{2^{\alpha_{l_1}+\alpha_{l_2}-1}\Gamma(4\alpha_{l_1})\Gamma(4\alpha_{l_2})}\\ &\times\sqrt{\frac{\Gamma(n_1+4\alpha_{l_1})\Gamma(n_2+4\alpha_{l_2})}{n_1!n_2!}}\\ &\times I^{\frac{\beta_{n_1,l_1}}{\sqrt{2}},\frac{\beta_{n_2,l_2}}{\sqrt{2}}}_{n_1,n_2,4\alpha_{l_1},4\alpha_{l_2}}(l_1+l_2+w+2)\\ =&\frac{\beta^{2\alpha_{l_1}}_{n_1,l_1}\beta^{2\alpha_{l_2}}_{n_2,l_2}}{2^{\alpha_{l_1}+\alpha_{l_2}}\Gamma(4\alpha_{l_1})\Gamma(4\alpha_{l_2})}\\ &\times\sqrt{\frac{\Gamma(n_1+4\alpha_{l_1})\Gamma(n_2+4\alpha_{l_2})}{n_1!n_2!}}\\ &\times J^{\frac{\beta_{n_1,l_1}}{\sqrt{2}},\frac{\beta_{n_2,l_2}}{\sqrt{2}}}_{n_1,n_2,4\alpha_{l_1},4\alpha_{l_2}}\left(\frac{l_1+l_2+w+1}{2},\frac{\beta_{n_1,l_1}+\beta_{n_2,l_2}}{2\sqrt{2}}\right). \end{aligned} \quad (18)$$

Because $l_1,l_2$, can only take integers, when $w$ is a non-negative integer or a semi-integer and larger than or equal to $-1/2$, $\langle n_1,l_1|r^w|n_2,l_2\rangle$ can also be written as

$$\begin{aligned}&\langle n_1,l_1|r^w|n_2,l_2\rangle\\ =&\frac{\beta^{2\alpha_{l_1}}_{n_1,l_1}\beta^{2\alpha_{l_1}}_{n_1,l_1}}{2^{\alpha_{l_1}+\alpha_{l_2}}\Gamma(4\alpha_{l_1})\Gamma(4\alpha_{l_2})}\\ &\times\sqrt{\frac{\Gamma(n_1+4\alpha_{l_1})\Gamma(n_2+4\alpha_{l_2})}{n_1!n_2!}}(-1)^{s+\delta}\\ &\times\frac{\mathrm{d}^{s+\delta}}{\mathrm{d}p^{s+\delta}}J^{\frac{\beta_{n_1,l_1}}{\sqrt{2}},\frac{\beta_{n_2,l_2}}{\sqrt{2}}}_{n_1,n_2,4\alpha_{l_1},4\alpha_{l_2}}(-\delta,p)\Big|_{p=\frac{\beta_{n_1,l_1}+\beta_{n_2,l_2}}{2\sqrt{2}}}. \end{aligned} \quad (19)$$

In Eq.(19), for brevity, $s=(l_1+l_2+w+1)/2$.

## 4. The radial matrix elements of a relativistic isotropic harmonic oscillator in Dirac theory

In Dirac theory, the wavefunction of a relativistic isotropic harmonic oscillator, with equal scalar potential and vector potential, is ($\hbar=c=1$)[8]

$$\psi=\frac{1}{r}\begin{bmatrix}F(r)\phi^A_{jm_j}\\ \mathrm{i}G(r)\phi^B_{jm_j}\end{bmatrix}, \quad K=j+1/2, \quad (20)$$

$$\psi=\frac{1}{r}\begin{bmatrix}F(r)\phi^B_{jm_j}\\ \mathrm{i}G(r)\phi^A_{jm_j}\end{bmatrix}, \quad K=-(j+1/2), \quad (21)$$

where

$$\phi^A_{jm_j}=\frac{1}{\sqrt{2l+1}}\begin{bmatrix}\sqrt{l+m+1} & Y_{lm}\\ \sqrt{l-m} & Y_{l,m+1}\end{bmatrix},$$

$$\phi^B_{jm_j}=\frac{1}{\sqrt{2l+3}}\begin{bmatrix}-\sqrt{l+1-m} & Y_{l+1,m}\\ \sqrt{l+1+m+1} & Y_{l+1,m+1}\end{bmatrix}, \tag{22}$$

$$F(r)=C_{n,K}\mathrm{e}^{-\frac{\beta_{n,K}}{2\sqrt{2}}r^2}r^K F\left(-n,4\alpha_K,\frac{\beta_{n,K}}{\sqrt{2}}r^2\right), \tag{23}$$

$$\begin{aligned}G(r)=&-C_{n,K}\frac{\beta_{n,K}}{2\sqrt{2}\alpha'(\mu+E_{n,K})}\mathrm{e}^{-\frac{\beta_{n,K}}{2\sqrt{2}}r^2}r^{K+1}\\&\times\left[nF\left(1-n,4\alpha_K+1,\frac{\beta_{n,K}}{\sqrt{2}}r^2\right)\right.\\&\left.+2\alpha_K F\left(-n,4\alpha_K,\frac{\beta_{n,K}}{\sqrt{2}}r^2\right)\right],\end{aligned} \tag{24}$$

$$C_{n,K}=\frac{2^{1-\alpha_K}\beta_{n,K}^{2\alpha_K}}{\Gamma(4\alpha_K)}\sqrt{\frac{\Gamma(n+4\alpha_K)(E_{n,K}+\mu)}{n!(3E_{n,K}+\mu)}}, \tag{25}$$

$$\alpha_K=(2K+1)/8,\quad \beta_{n,K}=\sqrt{k(E_{n,K}+\mu)}. \tag{26}$$

In the above formulae, $n,K,l,m$ are quantum numbers, $E_{n,K}$ and $\mu$ are energy and rest mass of a harmonic oscillator respectively. The radial matrix elements are

$$\begin{aligned}&\langle n_1,K_1,l_1,m_1|r^w|n_2,K_2,l_2,m_2\rangle\\=&\int r^{w+2}\,\Psi^+_{n_1K_1l_1m_1}\,\Psi_{n_2K_2l_2m_2}\sin\theta\mathrm{d}\phi\mathrm{d}\theta\mathrm{d}r.\end{aligned} \tag{27}$$

It is easy to prove

$$\begin{aligned}&\int(\phi^A_{j_1m_{j_1}})^+\phi^A_{j_2m_{j_2}}\sin\theta\mathrm{d}\theta\mathrm{d}\phi\\=&\int(\phi^B_{j_1m_{j_1}})^+\phi^B_{j_2m_{j_2}}\sin\theta\mathrm{d}\theta\mathrm{d}\phi=\delta_{l_1l_2}\delta_{m_1m_2},\end{aligned} \tag{28}$$

but

$$\begin{aligned}&\int(\phi^A_{j_1m_{j_1}})^+\phi^B_{j_2m_{j_2}}\sin\theta\mathrm{d}\theta\mathrm{d}\phi\\=&\int(\phi^B_{j_1m_{j_1}})^+\phi^A_{j_2m_{j_2}}\sin\theta\mathrm{d}\theta\mathrm{d}\phi=0.\end{aligned} \tag{29}$$

Substituting formulae (20)–(25) into Eq.(27) and simplifying give the expression of radial matrix elements

$$\begin{aligned}&\langle n_1,K_1,l_1,m_1|r^w|n_2,K_2,l_2,m_2\rangle\\=&\delta_{K_1K_2}\delta_{l_1l_2}\delta_{m_1m_2}(I_1+I_2+I_3+I_4+I_5),\end{aligned} \tag{30}$$

where

$$\begin{aligned}I_1=&C_{n_1K_1}C_{n_2K_2}\int_0^\infty\mathrm{e}^{-\frac{\beta_{n_1K_1}+\beta_{n_2K_2}}{2\sqrt{2}}r^2}r^{K_1+K_2+w}\\&\times F\left(-n,4\alpha_{K_1},\frac{\beta_{n_1K_1}}{\sqrt{2}}r^2\right)\\&\times F\left(-n,4\alpha_{K_2},\frac{\beta_{n_2K_2}}{\sqrt{2}}r^2\right)\mathrm{d}r\\=&\frac{1}{2}C_{n_1K_1}C_{n_2K_2}J^{\frac{\beta_{n_1K_1}}{\sqrt{2}},\frac{\beta_{n_2K_2}}{\sqrt{2}}}_{n_1,\ 4\alpha_{K_1},\ n_2,\ 4\alpha_{K_2}}\\&\times\left(\frac{K_1+K_2+w-1)}{2},\frac{\beta_{n_1K_1}+\beta_{n_2K_2}}{2\sqrt{2}}\right),\end{aligned} \tag{31}$$

$$\begin{aligned}I_2=&\frac{n_1n_2C_{n_1K_1}C_{n_2K_2}\beta_{n_1K_1}\beta_{n_2K_2}}{8\alpha_{K_1}\alpha_{K_2}(\mu+E_{n_1,K_1})(\mu+E_{n_2,K_2})}\\&\times\int_0^\infty\mathrm{e}^{-\frac{\beta_{n_1K_1}+\beta_{n_2K_2}}{2\sqrt{2}}r^2}r^{K_1+K_2+w+2}\\&\times F\left(1-n_1,4\alpha_{K_1}+1,\frac{\beta_{n_1K_1}}{\sqrt{2}}r^2\right)\\&\times F\left(1-n_2,4\alpha_{K_2}+1,\frac{\beta_{n_2K_2}}{\sqrt{2}}r^2\right)\mathrm{d}r\\=&\frac{n_1n_2C_{n_1K_1}C_{n_2K_2}\beta_{n_1K_1}\beta_{n_2K_2}}{16\alpha_{K_1}\alpha_{K_2}(\mu+E_{n_1,K_1})(\mu+E_{n_2,K_2})}\\&\times J^{\frac{\beta_{n_1K_1}}{\sqrt{2}},\frac{\beta_{n_2K_2}}{\sqrt{2}}}_{n_1-1,\ 4\alpha_{K_1}+1,\ n_2-1,\ 4\alpha_{K_2}+1}\\&\times\left(\frac{K_1+K_2+w+1)}{2},\frac{\beta_{n_1K_1}+\beta_{n_2K_2}}{2\sqrt{2}}\right),\end{aligned} \tag{32}$$

$$\begin{aligned}I_3=&\frac{n_1C_{n_1K_1}C_{n_2K_2}\beta_{n_1K_1}\beta_{n_2K_2}}{4\alpha_{K_1}}\\&\times\int_0^\infty\mathrm{e}^{-\frac{\beta_{n_1K_1}+\beta_{n_2K_2}}{2\sqrt{2}}r^2}r^{K_1+K_2+w+2}\\&\times F\left(1-n_1,4\alpha_{K_1}+1,\frac{\beta_{n_1K_1}}{\sqrt{2}}r^2\right)\\&\times F\left(-n_2,4\alpha_{K_2},\frac{\beta_{n_2K_2}}{\sqrt{2}}r^2\right)\mathrm{d}r\\=&\frac{n_1C_{n_1K_1}C_{n_2K_2}\beta_{n_1K_1}\beta_{n_2K_2}}{8\alpha_{K_1}}\\&\times J^{\frac{\beta_{n_1K_1}}{\sqrt{2}},\frac{\beta_{n_2K_2}}{\sqrt{2}}}_{n_1-1,\ 4\alpha_{K_1}+1,\ n_2,\ 4\alpha_{K_2}}\\&\times\left(\frac{K_1+K_2+w+1)}{2},\frac{\beta_{n_1K_1}+\beta_{n_2K_2}}{2\sqrt{2}}\right),\end{aligned} \tag{33}$$

$$\begin{aligned}I_4=&\frac{n_2C_{n_1K_1}C_{n_2K_2}\beta_{n_1K_1}\beta_{n_2K_2}}{4\alpha_{K_2}(\mu+E_{n_1,K_1})(\mu+E_{n_2,K_2})}\\&\times\int_0^\infty\mathrm{e}^{-\frac{\beta_{n_1K_1}+\beta_{n_2K_2}}{2\sqrt{2}}r^2}r^{K_1+K_2+w+2}\\&\times F\left(-n_1,4\alpha_{K_1},\frac{\beta_{n_1K_1}}{\sqrt{2}}r^2\right)\\&\times F\left(1-n_2,4\alpha_{K_2}+1,\frac{\beta_{n_2K_2}}{\sqrt{2}}r^2\right)\mathrm{d}r\\=&\frac{n_2C_{n_1K_1}C_{n_2K_2}\beta_{n_1K_1}\beta_{n_2K_2}}{8\alpha_{K_2}(\mu+E_{n_1,K_1})(\mu+E_{n_2,K_2})}\\&\times J^{\frac{\beta_{n_1K_1}}{\sqrt{2}},\frac{\beta_{n_2K_2}}{\sqrt{2}}}_{n_1,\ 4\alpha_{K_1},\ n_2-1,\ 4\alpha_{K_2}+1}\\&\times\left(\frac{K_1+K_2+w+1)}{2},\frac{\beta_{n_1K_1}+\beta_{n_2K_2}}{2\sqrt{2}}\right),\end{aligned} \tag{34}$$

$$\begin{aligned}I_5=&\frac{C_{n_1K_1}C_{n_2K_2}\beta_{n_1K_1},\beta_{n_2K_2}}{2(\mu+E_{n_1K_1})(\mu+E_{n_2K_2})}\\&\times\int_0^\infty\mathrm{e}^{-\frac{\beta_{n_1K_1}+\beta_{n_2K_2}}{2\sqrt{2}}r^2}r^{K_1+K_2+w+2}\end{aligned}$$

$$\times F\left(-n_1, 4\alpha_{K_1}, \frac{\beta_{n_1K_1}}{\sqrt{2}}r^2\right) \times F\left(-n_2, 4\alpha_{K_2}, \frac{\beta_{n_2K_2}}{\sqrt{2}}r^2\right)\mathrm{d}r$$
$$= \frac{C_{n_1K_1}C_{n_2K_2}\beta_{n_1K_1}\beta_{n_2K_2}}{4(\mu+E_{n_1,K_1})(\mu+E_{n_2,K_2})} \times J^{\frac{\beta_{n_1K_1}}{\sqrt{2}},\frac{\beta_{n_2K_2}}{\sqrt{2}}}_{n_1,\ 4\alpha_{K_1},\ n_2,\ 4\alpha_{K_2}} \times \left(\frac{K_1+K_2+w+1)}{2}, \frac{\beta_{n_1K_1}+\beta_{n_2K_2}}{2\sqrt{2}}\right). \quad (35)$$

## 5. Conclusion

In the above sections we have given the complete formulae of radial matrix elements of a relativistic isotropic harmonic oscillator. Formulae given here have well executed in the Mathematica 4.0 software package. Additionally, formulae (7), (12) and (14) can also be used to calculate the radial matrix elements of relativistic or non- relativistic hydrogen atoms as well as other quantum system of central potential field.

## References


[1] Landau L D and Lifshitz E M 1958 *Quantum Mechanics* (London: Pergamon) p124, 503

[2] Wu T U 1986 *Quantum Mechanics* (Singapore: World Scientific) p189

[3] Hou C F, Jiang Y Y, Sun X D and Sun W J 1999 *Acta Phys. Sin.* **48** 47 (in Chinese)

[4] Hou C F, Sun X D, Jzhou Z X and Li Y 1999 *Acta Phys. Sin.* **48** 385 (in Chinese)

[5] Chen C Y 2000 *Chin. Phys.* **9** 737

[6] Chen C Y 2000 *Acta Phys. Sin.* **49** 607 (in Chinese)

[7] Chen C Y and Sun D S 2002 *Acta Phys. Sin.* **51** 468 (in Chinese)

[8] Qiang W C 2002 *Chin. Phys.* **12** 757

[9] Wang Z X and Guo D R 2000 *Introduction To Special Function* (Peking: Peking University Press) (in Chinese)