Vol 12 No 7, July 2003
1009-1963/2003/12(07)/0753-06

Chinese Physics

© 2003 Chin. Phys. Soc.
and IOP Publishing Ltd

Influence of second sideband excitation on the dynamics of trapped ions in a cavity*

Liu Xiang(刘 翔)[a)b)] and Fang Mao-Fa(方卯发)[a)†]

[a)] *Department of Physics, Hunan Normal University, Changsha* 410081, *China*
[b)] *Anhui Institute of Optics and Fine Mechanics, Chinese Academy of Sciences, Hefei* 230031, *China*

(Received 26 December 2002; revised manuscript received 5 March 2003)

We study the dynamics of a trapped ion placed at an antinode of the standing wave inside a high finesse cavity with consideration of the second sideband excitation between the ionic internal levels and the light field. We investigate the entanglement of the three subsystems embodying the ionic internal levels, the vibrational mode of the ion and the cavity field.

Keywords: second sideband excitation, linear entropy, trapped ion
PACC: 4250V, 4250, 3290, 0365

1. Introduction

In recent years, much attention has been attracted by trapped ions manipulated by laser beams, due to not only the fundamental interest of physics involved but also some potential applications, such as the preparation of nonclassical states of the vibrational motion of ions,[1] precision spectroscopy,[2] and quantum computation.[3] The laser fields interacting with the trapped ions are usually treated as classical. However, the quantization of the laser field brings about more possibilities. The system of trapped ions placed inside a high finesse cavity makes the problem more interesting and more complicated because it involves three quantum degrees of freedom, namely, the ionic internal levels, the vibrational mode of the ion and the light field of the cavity. A number relevant papers has been noted in the literature, for example, the investigation of the nonclassical effects of the trapped ions,[4] the schemes for generating motional Schrödinger cat states,[5] the transfer of coherence between the motional states and light,[6] and even the proposition of quantum logic gate.[7] Even though, further work on the dynamics of ionic inversion of the system is still needed because experimental observation of these collapses and revivals provides a novel means of measuring the statistics of the quantum motion of the ion, and thus of detecting nonclassical states of motion.[8] In Ref.[4], the authors studied mainly the full quantum system in first-order Lamb–Dicke approximation. We found, however, that it is inadequate to consider only the first-order approximation in some cases. In this paper, we explore the dynamics of the system of the trapped ion placed at an antinode of the standing wave within a cavity, in which the second-order Lamb–Dicke approximation must be considered. We also investigate the entanglement of the ionic internal levels, the ionic vibrational mode, and the cavity field.

2. Dynamics of the system

Consider a system with a single two-level trapped ion placed inside a cavity that supports a single-mode quantized radiation field with frequency ω_a. For a standing wave travelling in the x direction, the Hamiltonian of the system is given by ($\hbar = 1$)[4,9]

$$H = \omega_a a^+ a + \nu b^+ b + \frac{1}{2}\omega_0 \sigma_z + g(a^+ + a)(\sigma_- + \sigma_+)\sin[\eta(b^+ + b) + \phi], \quad (1)$$

where η is the Lamb–Dicke parameter, b^+ and b are the creation and destruction operators of the centre

*Project supported by the National Natural Science Foundation of China (Grant No 19874020) and the Natural Science Foundation of Hunan Province (Grant No 01JJY3030).

†Corresponding author

of mass of vibrational motion mode of the ion. ϕ accounts for the relative position of the centre of mass of the ion to the standing wave. ν denotes the vibrational frequency and ω_0 is the transition frequency between the internal electronic states $|e\rangle$ and $|g\rangle$. a^+ and a are the creation and the destruction operators of the single mode field. The coupling parameter g is directly proportional to the ion-radiation interaction strength. σ_+, σ_- and σ_z are the familiar pseudo-spin operators for the ion.

If we take $\phi = \pi/2$ (i.e., the ion is centred at an antinode of the standing wave), the interaction Hamiltonian will become

$$H_i = g(a^+ + a)(\sigma_- + \sigma_+)\cos[\eta(b^+ + b)]. \quad (2)$$

In the Lamb–Dicke limit($\eta \ll 1$), it is not enough to expand the trigonometric function term up to first order in η. An expansion up to second order in η becomes necessary, and we expand the cosine in Eq.(2) and obtain

$$\cos[\eta(b^+ + b)] \approx 1 - \frac{1}{2}\eta^2(b^+ + b)^2. \quad (3)$$

If the light field is tuned to the second red vibrational sideband, i.e., $\omega_a = \omega_o - 2\nu$, after applying the rotating wave approximation and discarding the rapidly oscillating terms, we obtain the interaction Hamiltonian of Jaynes–Cumming-like type in interaction picture:

$$H_i^{\rm I} = \frac{1}{2}\eta^2 g(b^{+2}a^+\sigma_- + b^2 a\sigma_+). \quad (4)$$

Similarly, if the light field is tuned to the second blue vibrational sideband ($\omega_a = \omega_o + 2\nu$), we have

$$H_i^{\rm I} = \frac{1}{2}\eta^2 g(b^{+2}a^+\sigma_+ + b^2 a\sigma_-), \quad (5)$$

an "anti-Jaynes–Cummings-like" type. In our work, we only consider the first case.

The evolution operator associated with the Hamiltonian (4) is given by

$$\begin{aligned} U(t) =& C_{m+2,n+1}|e\rangle\langle e| + C_{m,n}|g\rangle\langle g| \\ & - {\rm i}D_{m+2,n+1}b^2 a|e\rangle\langle g| \\ & - {\rm i}b^{+2}a^+ D_{m+2,n+1}|g\rangle\langle e|, \end{aligned} \quad (6)$$

where

$$\begin{aligned} & C_{m+2,n+1} \\ =& \cos\left[\frac{1}{2}\eta^2\sqrt{(b^+b+2)(b^+b+1)(a^+a+1)}gt\right], \end{aligned} \quad (7)$$

$$C_{m,n} = \cos\left[\frac{1}{2}\eta^2\sqrt{b^+b(b^+b-1)a^+a}gt\right], \quad (8)$$

and

$$\begin{aligned} & D_{m+2,n+1} \\ =& \frac{\sin\left[\frac{1}{2}\eta^2\sqrt{(b^+b+2)(b^+b+1)(a^+a+1)}gt\right]}{\sqrt{(b^+b+2)(b^+b+1)(a^+a+1)}}. \end{aligned} \quad (9)$$

We assume now the initial state of the ion-field system to be described by the state vector

$$\begin{aligned} |\Psi(0)\rangle &= |\Psi(0)\rangle_a \otimes |\Psi(0)\rangle_b \otimes |\Psi(0)\rangle_i \\ &= \sum_n a_n|n\rangle_a \otimes \sum_m b_m|m\rangle_b \otimes |e\rangle_i, \end{aligned} \quad (10)$$

i.e., the cavity field is initially prepared in a superposition of number(Fock) state $\sum_n a_n|n\rangle_n$, the ionic vibrational centre-of-mass motion prepared in a state $\sum_m b_m|m\rangle_b$, and the ionic internal levels prepared in the excited state $|e\rangle_i$. The state vector of the system at any time t is

$$\begin{aligned} |\Psi(t)\rangle =& U(t)|\Psi(0)\rangle \\ =& \sum_{m,n} b_m a_n[\cos(\Omega_{m,n}t)|m\rangle_b|n\rangle_a|e\rangle_i \\ & - {\rm i}\sin(\Omega_{m,n}t)|m+2\rangle_b|n+1\rangle_a|g\rangle_i], \end{aligned} \quad (11)$$

where the generalized Rabi frequency $\Omega_{m,n}$ is given by the relation

$$\Omega_{m,n} = \frac{1}{2}\eta^2 g\sqrt{(m+2)(m+1)(n+1)}. \quad (12)$$

From Eq.(11), we obtain the ionic population inversion of the system including the three subsystems

$$\begin{aligned} W_i(t) &= \langle\Psi(t)|\sigma_z|\Psi(t)\rangle \\ &= \frac{1}{2}\sum_{m,n}|a_n|^2|b_m|^2\cos(2\Omega_{m,n}t). \end{aligned} \quad (13)$$

3. The entanglement parameter

In order to distinguish between pure states and statistical mixtures of the system including the three subsystems (i.e., the trapped ion, the vibrational mode and the cavity field), we investigate the time evolution of the entanglement parameter. For a particular subsystem described by the reduced density matrix $\rho_x = {\rm Tr}_{\{y\neq x\}}\rho$ (ρ is the density matrix of the whole system, x, y denotes the three respective subsystems), the entanglement parameter associated with linear entropy[10] is

$$S_x^{\rm pur}(t) = 1 - {\rm Tr}_x\{\rho_x^2(t)\}. \quad (14)$$

The parameter, also called "purity parameter",[11] measures the deviation of an actual state from a pure

state. The deviation of its value from zero expresses a loss of the state purity, i.e., for a pure state $S_x^{\rm pur}=0$, and is positive otherwise. Note that the entanglement parameter is a lower bound for von Neumann entropy $S_x=-{\rm Tr}_x\{\rho_x\ln\rho_x\}$, i.e. $S_x^{\rm pur}\leq S_x$. Here, we use the parameter $S_x^{\rm pur}$ as an appropriate measure of the entanglement between the three quantum- mechanical subsystems involved in the dynamics.

The density of the whole system described by Eq.(11) is

$$\rho(t)=|\Psi(t)\rangle\langle\Psi(t)|. \tag{15}$$

For simplicity, we only give here the reduced density matrix ρ_i of the trapped ion:

$$\rho_i(t)={\rm Tr}_{\{a,b\}}\{\rho\}=\left[\begin{array}{cc}\rho_{11}(t) & \rho_{12}(t)\\ \rho_{21}(t) & \rho_{22}(t)\end{array}\right], \tag{16}$$

where

$$\begin{aligned}\rho_{11}(t)=&\sum_{m,n}|a_n|^2|b_m|^2\cos^2(\Omega_{m,n}t),\\ \rho_{22}(t)=&1-\rho_{11}(t),\\ \rho_{12}(t)=&\sum_{m,n}a_{n+1}^*a_nb_{m+2}b_m\\ &\times\cos(\Omega_{m+2,n+1}t)\sin(\Omega_{m,n}t),\\ \rho_{21}(t)=&\rho_{12}^*(t)\end{aligned} \tag{17}$$

with $\Omega_{m,n}$ defined by Eq.(12). Thus, the entanglement parameter $S_i^{\rm pur}$ is

$$S_i^{\rm pur}(t)=1-\lambda_i^{+2}-\lambda_i^{-2}, \tag{18}$$

$\lambda_i^{\pm}$ being the eigenvalues of the reduced density matrices $\rho_i(t)$.

4. Numerical calculation and discussion

Case 1 We first assume the cavity field initially in the coherent state $|\alpha\rangle_a$, the vibrational mode prepared in the Fock state $|M\rangle_b$, i.e., in Eq.(11), $a_n=\exp(-|\alpha|^2/2)\alpha^n/\sqrt{n!}$, $b_m=\delta_{M,m}$. The expression for the state vector $|\psi(t)\rangle$ can be written as

$$\begin{aligned}|\Psi(t)\rangle=&={\rm e}^{-\frac{|\alpha|^2}{2}}\sum_n\frac{\alpha^n}{\sqrt{n!}}\\ &\times[\cos(\Omega_M\sqrt{n+1}t)|n\rangle_a|M\rangle_b|e\rangle_i\\ &-{\rm i}\sin(\Omega_M\sqrt{n+1}t)|n+1\rangle_a|M+2\rangle_b|g\rangle_i],\end{aligned} \tag{19}$$

where $\Omega_M=\frac{1}{2}\eta^2g\sqrt{(M+2)(M+1)}$.

In Fig.1 we plot the time evolution of dynamical properties of the system with $M=8,|\alpha|^2=64,\eta=0.05$. Fig.1(a) shows the time evolution of $W_i(t)$. Obviously, the constancy of the initial parameter M of the vibrational mode makes the ionic inversion $W_i(t)$(oscillating at Rabi frequencies $\Omega_M\sqrt{n+1}$) to exhibit the characteristic pattern of JCM.[12] The corresponding revival time is $T_R^{(1)}\approx 4\pi\sqrt{\bar n}/\eta^2g\sqrt{(M+2)(M+1)}$, with $\bar n=|\alpha|^2$ being the mean excitation number of the cavity field.

Figure 1(b) shows the time evolution of the entanglement parameter $S_a^{\rm pur}$. The value of $S_a^{\rm pur}$ is almost equal to zero at one-half of the revival time $T_R^{(1)}$, which indicates that the cavity field is in a pure superposition state composed of two coherent-like states (the so-called Schrödinger cat states[13]). Noting that under this condition the entanglement parameter $S_i^{\rm pur}$ and $S_b^{\rm pur}$ evolve identically (see Fig.1(c)), they are all equal to the maximum value 1/2 at $\frac{1}{2}T_R^{(1)}$. It shows at this time the generation of the Bell-like state of the ionic internal states and the vibrational states, i.e., $|\psi_{i,b}\rangle=(1/\sqrt{2})(|M\rangle_b|e\rangle_i+|M+2\rangle_b|g\rangle_i)$. During the revival time the entanglement parameters $S_a^{\rm pur}$ and $S_b^{\rm pur}$ display an additional local minimum which never falls to zero. This corresponds to the partial reconstruction of the initial vibrational mode and cavity field.

Case 2 If we take the cavity field initially in a Fock state $|N\rangle_a$, the vibrational mode prepared in the coherent state $|\beta\rangle_b$, i.e., in Eq.(11), $a_n=\delta_{N,n}$, $b_m=\exp(-|\beta|^2/2)\beta^m/\sqrt{m!}$, the expression for the state vector $|\psi(t)\rangle$ can be written as

$$\begin{aligned}|\Psi(t)\rangle=&{\rm e}^{-\frac{|\beta|^2}{2}}\sum_m\frac{\beta^m}{\sqrt{m!}}\\ &\times[\cos(\Omega_N\sqrt{(m+2)(m+1)}t)\\ &\times|m\rangle_b|N\rangle_a|e\rangle_i\\ &-{\rm i}\sin(\Omega_N\sqrt{(m+2)(m+1)}t)\\ &\times|m+2\rangle_b|N+1\rangle_a|g\rangle_i]\end{aligned} \tag{20}$$

with $\Omega_N=\frac{1}{2}\eta^2g\sqrt{(N+1)}$.

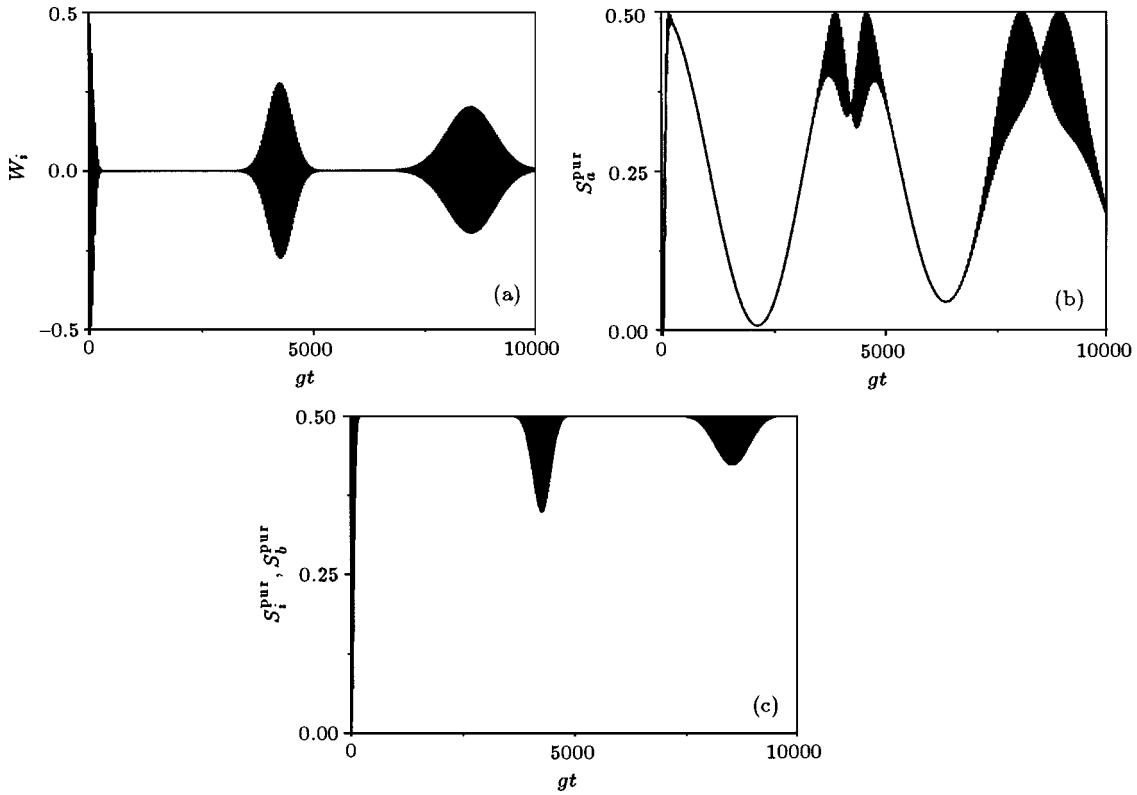


Fig.1. The time evolution of the dynamical properties with $M = 8, |\alpha|^2 = 64, \eta = 0.05$. (a) The ionic inversion $W_i(t)$; (b) The entanglement parameter $S_a(t)$; (c) The entanglement parameters $S_i(t), S_b(t)$.

Figure 2 shows the time evolution of the dynamical properties under this condition with $|\beta|^2 = 8, N = 64, \eta = 0.05$. Due to no fluctuations of the photon number N in the cavity field, the collapses-revival effect of the inversion $W_i(t)$ (beating at Rabi frequencies $\Omega_N\sqrt{(m+2)(m+1)}$) are similar to that in two-photon JCM.[12,14] This is illustrated in Fig.2(a) with the corresponding revival time $T_R^{(2)} \approx 2\pi/\eta^2 g\sqrt{(N+1)}$. The time evolution of the entanglement parameter $S_b^{\rm pur}$ is shown in Fig.2(b). It shows that the $S_b^{\rm pur}$ is close to zero at $\frac{1}{4}T_R^{(2)}$ and $\frac{3}{4}T_R^{(2)}$, i.e., Schrödinger-cat-like states in the vibrational mode are established. At the revival time $T_R^{(2)}$ the value of $S_b^{\rm pur}$ tends to zero, which means that the vibrational mode is disentangled to the initial pure state.

On the other hand, the entropies $S_i^{\rm pur}$ and $S_a^{\rm pur}$ in this case are equal to 1/2 in between two revival times(see Fig.2(c)). This indicates that the degree of mutual entanglement between the cavity mode and the ionic internal levels is maximal. At the revival times, however, the values of $S_i^{\rm pur}$ and $S_a^{\rm pur}$ are almost equal to zero. One finds that at these times the three quantum subsystems are disentangled to pure states. It means the reconstruction of the initial vibrational mode and the cavity field.

Case 3 Now consider both the cavity light field and the vibrational mode are initially prepared in coherent states. Given the cavity mode in coherent state $|\alpha\rangle$, $\alpha = \exp(-|\alpha|^2/2)\alpha^n/\sqrt{n!}$; the vibrational mode in $|\beta\rangle$, $\beta = \exp(-|\beta|^2/2)\beta^m/\sqrt{m!}$, then at time t, the state vector of the whole system Eq.(11) can be rewritten as

$$
\begin{aligned}
|\Psi(t)\rangle =& {\rm e}^{-(\frac{|\alpha|^2}{2}+\frac{|\beta|^2}{2})}\sum_{m,n}\frac{\alpha^n\beta^m}{\sqrt{m!n!}} \\
&\times[\cos(\Omega_{m,n}t)|m\rangle_b|n\rangle_a|e\rangle_i \\
&-{\rm i}\sin(\Omega_{m,n}t)|m+2\rangle_b|n+1\rangle_a|g\rangle_i], \quad (21)
\end{aligned}
$$

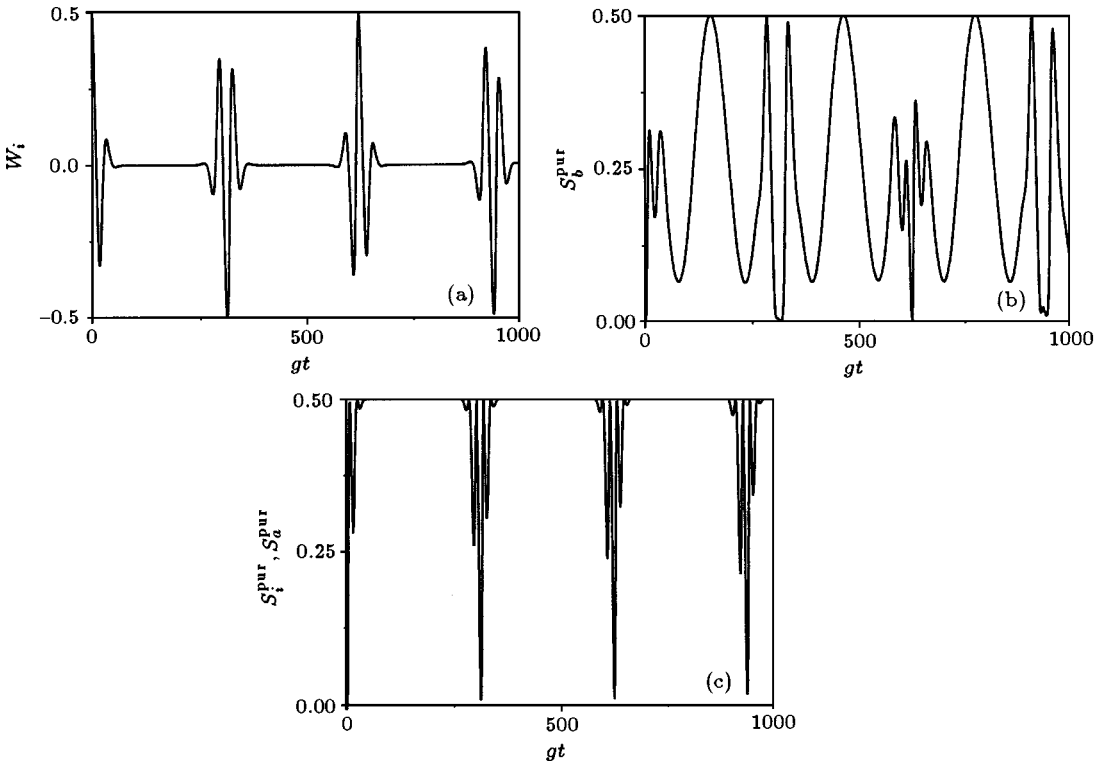


Fig.2. The time evolution of the dynamical properties with $|\beta|^2 = 8, N = 64, \eta = 0.05$. (a) The ionic inversion $W_i(t)$; (b) The entanglement parameter $S_b(t)$; (c) The entanglement parameters $S_i(t), S_a(t)$.

the ionic inversion is

$$W_i(t) = \mathrm{e}^{-(|\alpha|^2+|\beta|^2)} \sum_{m,n} \frac{|\alpha|^{2n}|\beta|^{2m}}{m!n!} \cos(2\Omega_{m,n}t). \tag{22}$$

Due to the double coherent summation of terms oscillating at generalized Rabi frequencies $\Omega_{m,n} = \frac{1}{2}\eta^2 g\sqrt{(m+2)(m+1)(n+1)}$, we expect the double structures in the inversion. That is, the dynamics of the population inversion predicted by Eq.(22) may be interpreted in terms of two families of revival times. The revival time associated with the field is

$$T_R^a \approx \frac{4\pi\sqrt{\bar{n}}}{\eta^2 g(\bar{m}+3/2)}, \tag{23}$$

which depends on $\bar{m} = |\alpha|^2$ and $\bar{n} = |\beta|^2$, the mean excitation number of the centre-of-mass motion and the cavity field, respectively. On the other hand, after performing the summation over m in above equation, we obtain another expression of Eq.(22)

$$W_i(t) = \mathrm{e}^{-|\alpha|^2} \sum_n \frac{\alpha^n}{n!} w_n(t), \tag{24}$$

with

$$w_n(t) = \exp\left[-2\bar{m}\sin^2\left(\frac{1}{2}\eta^2 g\sqrt{n+1}t\right)\right] \times \cos\left[\bar{m}\sin(\eta^2 g\sqrt{n+1}t) + \frac{3}{2}\eta^2 g\sqrt{n+1}t\right]. \tag{25}$$

Thus the revival time associated with the vibrational mode is

$$T_R^b \approx \frac{2\pi}{\eta^2 g\sqrt{(\bar{n}+1)}}, \tag{26}$$

depending only on $\bar{n}$. Considering $\bar{n} \gg \bar{m}$, then $T_R^a \gg T_R^b$, the "rapid revivals" corresponding to T_R^b in the ionic inversion $W_i(t)$ are modulated by the "long" revival time T_R^a. We show this modulation of rapid revivals in the time evolution of $W_i(t)$ in Fig.3(a), with the initial parameters given as $|\alpha|^2 = 64, |\beta|^2 = 8$, and $\eta = 0.05$. In Fig.3(b) we plot the time evolution of the entanglement parameter S_i^{pur}. The parameter S_i^{pur}, describing the entanglement between the ionic internal state and the two coherent modes (the vibrational mode and the cavity field), evolves to its minimum at times $(2k+1)T_R^b/4, k = 0, 1, 2\cdots$. When $\bar{m}, \bar{n} \to \infty$, the first minimal value $\left(\text{at } \frac{1}{4}T_R^b\right)$ is almost equal to

zero, which means that at this time a pure superposition of the ionic internal levels is generated. Due to the complexity of the system, we cannot obtain the respective time evolution of the entanglement parameter $S_a^{\rm pur}$ and $S_b^{\rm pur}$. However, we can obtain the information of the coupled system of the vibrational mode and the cavity field from the entanglement parameter $S_i^{\rm pur}$; for instance, at $\frac{1}{4}T_R^b$ the coupled system is in a macroscopic superposition — the two mode Schrödinger-cat-like states.

In conclusion, we investigated the dynamics of a trapped ion placed at an antinode of the standing wave inside a high finesse cavity with consideration of the second sideband excitation between the ionic internal levels and the light field. In the Lamb–Dicke approximation (here up to the second order) it shows that the ionic population inversion as a function of time exhibits different structures of beats depending on the initial preparation of the cavity field and the vibrational motion of the trapped ion. We also analysed the entanglement between the trapped ion, the vibrational mode and the cavity field by using the entanglement parameter (associated with linear entropy). It shows that under particular conditions we can reconstruct the initial vibrational mode and the cavity field, or prepare the two mode Schrödinger-cat-like states between the vibrational mode and the cavity field in this system.

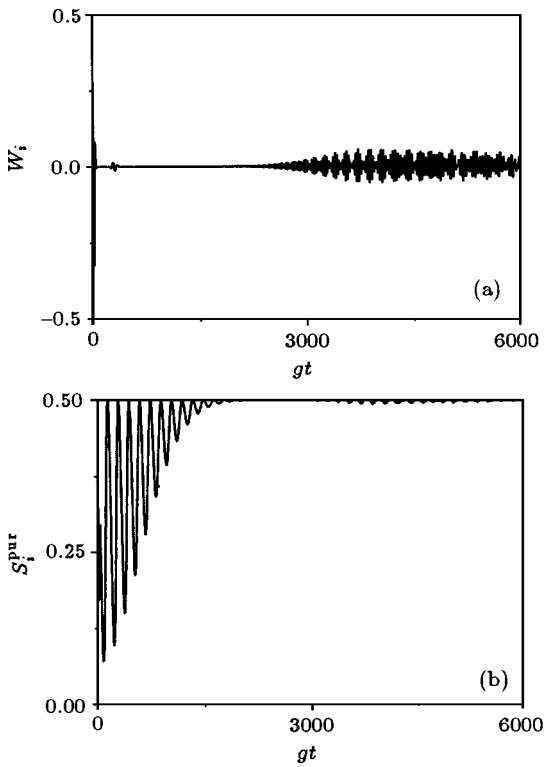


Fig.3. The time evolution of the dynamical properties with $|\alpha|^2 = 64, |\beta|^2 = 8, \eta = 0.05$. (a) The ionic inversion $W_i(t)$; (b) The entanglement parameter $S_i(t)$.

References

[1] Heinzen D J and Wineland D J 1997 *Phys. Rev.* A **42** 2977
Cirac J I, Blatt R, Parkins A S and Zoller P 1993 *Phys. Rev. Lett.* **70** 762 and references therein

[2] Wineland D J, Bollinger J J, Itano W M, Moore F L and Heinzen D J 1992 *Phys. Rev.* A **46** R6797
Wu Y and Yang X X 1997 *Phys. Rev. Lett.* **78** 3086

[3] Cirac J I and Zoller P 1995 *Phys. Rev. Lett.* **74** 4091

[4] Buzek V, Drobny G, Kim M S, Adam G and Kinght P L 1997 *Phys. Rev.* A **56** 2352
Luo X L *et al* 1999 *Acta Phys. Sin.* **48** 0218 (in Chinese)

[5] Liu X and Fang M F 2002 *Chin. Phys.* **11** 0635

[6] Parkins A S and Kimble H J 1999 *J. Opt. B: Quantum Semiclass. Opt.* **1** 496

[7] Jane E, Plenio M B and Jonathan D 2002 *Phys. Rev.* A **65** 050302
Zou X B, Pahlke K and Mathis W 2002 *Phys. Rev.* A **65** 064303
Semiâo F L, Vidiella-Barranco A and Roversi J A 2002 *Phys. Lett.* A **299** 423

[8] Cirac J I, Blatt R, Parkins A S and Zoller P 1994 *Phys. Rev.* A **49** 1202

[9] Luo X L, Zhu X W, Wu Y, Feng M and Gao K L 1998 *Phys. Lett.* A **237** 354

[10] Wehrl A 1978 *Rev. Mod. Phys.* **50** 221

[11] Buzek V and Drobny 1993 *Phys. Rev.* A **47** 1237

[12] Shore B and Knight P L 1993 *J. Mod. Opt.* **40** 1195

[13] Gea-Banancloche J 1990 *Phys. Rev. Lett.* **65** 3385
Gea-Banancloche J 1991 *Phys. Rev.* A 44 5913.

[14] Fang M F and Zhou G H 1994 *Phys. Lett.* A **39** 397