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Feynman propagator for an arbitrary half-integral spin*

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Based on the solution to Bargmann–Wigner equation for a particle with arbitrary half-integral spin, a direct derivation of the projection operator and propagator for a particle with arbitrary half-integral spin is worked out. The projection operator constructed by Behrends and Fronsdal is re-deduced and confirmed and simplified, the general commutation rules and Feynman propagator with additional non-covariant terms for a free particle with arbitrary half-integral spin are derived, and explicit expressions for the propagators for spins 3/2, 5/2 and 7/2 are provided.

Keywords: half-integral spin, projection operator, commutation rule, Feynman propagator
PACC: 1100, 1180C, 1480

1. Introduction

In high-energy physics, in order to calculate the relevant Feynman diagrams and perform amplitude analyses for high-energy processes, it is necessary to employ higher spin relativistic wavefunctions, projection operators and Feynman propagators.[1−4] Recently, a systematical method[5−7] has been developed to solve the Bargmann–Wigner[8] equations and derive explicit positive and negative energy wavefunctions for a particle with arbitrary spin, which extended the work by Auvil and Brehm[9] and by Chung[1] who constructed only the positive energy wavefunctions. Based on this work, we have carried out a further investigation on the projection operator, the commutation rules and the Feynman propagator for a free particle with an arbitrary integral spin.[10] In this paper, we extend such an investigation to the case of arbitrary half-integral spin, and thus constitute a complete theoretical system.

The concept of higher spin projection operators was first introduced by Behrends and Fronsdal[11,12] in 1957 when they undertook a calculation of the lifetimes and spectra of Fermi decays of higher spin particles. Based on the properties of these operators derived from the Klein–Gordon and Rarita–Schwinger[13] equations, they constructed an explicit form of the projection operators. The construction was carried out first in the rest system and then generalized to an arbitrary frame. In 1965, Zemach[14] proposed an alternative way to construct the projection operator for an arbitrary spin in the rest frame. However, it is found recently by Chung[1,2] and by Filippini *et al*[4] that the Zemach's formalism is incorrect because it is essentially a nonrelativistic one. Thus the projection operator constructed by Behrends and Fronsda is now the only available formulation. Since the B-F formulation is basically constructed in the rest frame, an independent check of its correctness is necessary. A direct calculation of the projection operator based on the explicit formula of the wavefunctions

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and performed in an arbitrary frame should yield a reliable check. In the case of integral spin, the B-F form has been confirmed previously.[10] The first part of this paper will be devoted to this check for an arbitrary half-integral spin. It is found that the projection operator for half-integral spin should be derived in a different way compared with that for integral spin, because Dirac γ matrices are involved in the present case. Armed with a set of newly found sum relations about γ matrices and spins 1/2 and 1 wavefunctions, we have derived an explicit expression for the projection operator for an arbitrary half-integral spin, which confirms the B-F form. Furthermore, this form is simplified such that it is suitable for the calculation of Feynman propagators.

The second part of this paper is devoted to working out a general theory of the Feynman propagator for a particle with arbitrary half-integral spin. As in the case of integral spin,[10] additional non-covariant terms inevitably appear in the expression of the propagator when the spin is larger than 1/2. It is found that there is a new kind of additional terms originating from the γ factors in the expression of the propagators for half-integral spin, besides that for integral spin. These terms will be derived in a step-by-step way. Especially, explicit expressions for the propagators for spins 3/2, 5/2 and 7/2, which are often useful for experimentalists, are worked out in detail.

2. The projection operator for an arbitrary half-integral spin

For a particle with arbitrary half-integral spin $n+1/2$ (n is an integer), the wavefunction may be expressed as[6]

$$\begin{aligned}&\Psi^{\nu_1\nu_2\cdots\nu_n}(x)\\ =&\frac{1}{\sqrt{V}}\sum_p\sum_{m=-(n+1/2)}^{n+1/2}[a_m(\boldsymbol{p})U_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}\\ &+b_m^+(\boldsymbol{p})V_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px}],\end{aligned}\tag{1}$$

where $U^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})$ and $V^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})$ are, respectively, the positive and negative energy wavefunctions in momentum representations and in the following form

$$\begin{aligned}U_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})=&\sqrt{\frac{n+\frac{1}{2}+m}{2n+1}}\mathrm{e}_{m-\frac{1}{2}}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})u_{\frac{1}{2}}(\boldsymbol{p})\\ &+\sqrt{\frac{n+\frac{1}{2}-m}{2n+1}}\mathrm{e}_{m+\frac{1}{2}}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})u_{-\frac{1}{2}}(\boldsymbol{p}),\end{aligned}\tag{2a}$$

$$\begin{aligned}V_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})=&\sqrt{\frac{n+\frac{1}{2}+m}{2n+1}}\overline{\mathrm{e}}_{m-\frac{1}{2}}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})v_{\frac{1}{2}}(\boldsymbol{p})\\ &+\sqrt{\frac{n+\frac{1}{2}-m}{2n+1}}\overline{\mathrm{e}}_{m+\frac{1}{2}}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})v_{-\frac{1}{2}}(\boldsymbol{p}),\end{aligned}\tag{2b}$$

with $u_r(\boldsymbol{p})$ and $v_r(\boldsymbol{p})(r=\pm1/2)$ being the usual Dirac spinors, $\mathrm{e}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})$ and $\overline{\mathrm{e}}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})$ the positive and negative energy wavefunctions for an integral spin n, namely

$$\begin{aligned}&\mathrm{e}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\\ =&\sum_{\lambda_n=-1}^{1}\langle n-1,m-\lambda_n,1,\lambda_n|n-1,1,n,m\rangle\\ &\times\mathrm{e}_{m-\lambda_n}^{\nu_1\nu_2\cdots\nu_{n-1}}(\boldsymbol{p})\mathrm{e}_{\lambda_n}^{\nu_n}(\boldsymbol{p}),\end{aligned}\tag{3a}$$

$$\begin{aligned}&\overline{\mathrm{e}}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\\ =&\sum_{\lambda_n=-1}^{1}\langle n-1,m-\lambda_n,1,\lambda_n|n-1,1,n,m\rangle\\ &\times\overline{\mathrm{e}}_{m-\lambda_n}^{\nu_1\nu_2\cdots\nu_{n-1}}(\boldsymbol{p})\mathrm{e}_{\lambda_n}^{\nu_n}(\boldsymbol{p}).\end{aligned}\tag{3b}$$

Here $\mathrm{e}_\lambda^\nu(\boldsymbol{p})$ and $\overline{\mathrm{e}}_\lambda^\nu(\boldsymbol{p})$ are the positive and negative energy wavefunctions for spin 1, and are related by

$$\begin{aligned}\overline{\mathrm{e}}_{\lambda_i}^{\nu_i}(\boldsymbol{p})=&g_{\nu_i\mu_i}(\mathrm{e}_{\lambda_i}^{\mu_i}(\boldsymbol{p}))^*=(-1)^{\lambda_i}\mathrm{e}_{-\lambda_i}^{\nu_i}(\boldsymbol{p}),\\ g_{\nu_i\mu_i}=&\mathrm{diag}\{1,1,1,-1\}.\end{aligned}\tag{4}$$

The adjoint field of $\Psi^{\nu_1\nu_2\cdots\nu_n}(x)$ is defined by

$$\begin{aligned}\overline{\Psi}^{\nu_1\nu_2\cdots\nu_n}(x)=&(\Psi^{\nu_1\nu_2\cdots\nu_n}(x))^+\gamma_4\\ =&\frac{1}{\sqrt{V}}\sum_{pm}[a_m^+(\boldsymbol{p})\overline{U}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px}\\ &+b_m(\boldsymbol{p})\overline{V}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}],\end{aligned}\tag{5}$$

where

$$\begin{aligned}\overline{U}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})=&\sqrt{\frac{n+\frac{1}{2}+m}{2n+1}}\overline{\mathrm{e}}_{m-\frac{1}{2}}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\overline{u}_{\frac{1}{2}}(\boldsymbol{p})\\ &+\sqrt{\frac{n+\frac{1}{2}-m}{2n+1}}\overline{\mathrm{e}}_{m+\frac{1}{2}}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\overline{u}_{-\frac{1}{2}}(\boldsymbol{p}),\end{aligned}\tag{6a}$$

$$\begin{aligned}\overline{V}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})=&\sqrt{\frac{n+\frac{1}{2}+m}{2n+1}}\mathrm{e}_{m-\frac{1}{2}}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\overline{v}_{\frac{1}{2}}(\boldsymbol{p})\\ &+\sqrt{\frac{n+\frac{1}{2}-m}{2n+1}}\mathrm{e}_{m+\frac{1}{2}}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\overline{v}_{-\frac{1}{2}}(\boldsymbol{p}),\end{aligned}\tag{6b}$$

$$\overline{u}_r(\boldsymbol{p}) = u_r^+(\boldsymbol{p})\gamma_4, \quad \overline{v}_r(\boldsymbol{p}) = v_r^+(\boldsymbol{p})\gamma_4. \tag{6c}$$

These wavefunctions are normalized according to

$$\begin{aligned}&\overline{U}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})U_{m'}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\\ &= -(W/E)\delta_{mm'} = \overline{V}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})V_{m'}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p}),\end{aligned} \tag{7a}$$

$$\begin{aligned}&\overline{U}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})V_{m'}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\\ &=\overline{V}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})U_{m'}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p}) = 0,\end{aligned} \tag{7b}$$

where W denotes the rest-mass of the particle. In a similar fashion as in the case of spin $1/2$, the projection operator for spin $n+1/2$ is defined by[15]

$$\begin{aligned}&P_+^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\\ &=\sum_m (U_m^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})\overline{U}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})),\end{aligned} \tag{8a}$$

$$\begin{aligned}&P_-^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\\ &=\sum_m (V_m^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})\overline{V}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})).\end{aligned} \tag{8b}$$

Armed with the explicit expressions for $U_m^{\nu_1\nu_2\cdots\nu_n}$ and $V_m^{\nu_1\nu_2\cdots\nu_n}$, and the projection operators for spin $1/2$ and 1 given by

$$\begin{aligned}\Lambda_+ &= \sum_{r=-1/2}^{1/2} u_r(\boldsymbol{p})\overline{u}_r(\boldsymbol{p}) = \frac{\not{p}+\mathrm{i}W}{2\mathrm{i}E},\\ \Lambda_- &= \sum_{r=-1/2}^{1/2} v_r(\boldsymbol{p})\overline{v}_r(\boldsymbol{p}) = \frac{\not{p}-\mathrm{i}W}{2\mathrm{i}E},\end{aligned} \tag{9a}$$

$$\begin{aligned}P^{\mu\nu}(p) &= \sum_{\lambda=-1}^{1} \overline{\mathrm{e}}_\lambda^\mu(\boldsymbol{p})\mathrm{e}_\lambda^\nu(\boldsymbol{p})\\ &= \sum_{\lambda=-1}^{1} \mathrm{e}_\lambda^\mu(\boldsymbol{p})\overline{\mathrm{e}}_\lambda^\nu(\boldsymbol{p}) = \delta_{\mu\nu} + \frac{p_\mu p_\nu}{W^2},\end{aligned} \tag{9b}$$

together with the following newly found sum relations about γ matrices and spins $1/2$ and 1 wavefunctions

$$\gamma_\nu \mathrm{e}_{+1}^\nu u_{\frac{1}{2}} = 0, \quad \frac{1}{\mathrm{i}\sqrt{2}}\gamma_5\gamma_\nu \mathrm{e}_{+1}^\nu u_{-\frac{1}{2}} = u_{\frac{1}{2}},$$

$$\overline{u}_{\frac{1}{2}}\gamma_\nu \overline{\mathrm{e}}_{+1}^\nu = 0, \quad -\frac{1}{\mathrm{i}\sqrt{2}}\overline{u}_{-\frac{1}{2}}\gamma_\nu \overline{\mathrm{e}}_{+1}^\nu\gamma_5 = \overline{u}_{\frac{1}{2}}, \tag{10a}$$

$$\mathrm{i}\gamma_5\gamma_\nu \mathrm{e}_0^\nu u_{\frac{1}{2}} = u_{\frac{1}{2}}, \quad -\mathrm{i}\gamma_5\gamma_\nu \mathrm{e}_0^\nu u_{-\frac{1}{2}} = u_{-\frac{1}{2}},$$

$$-\mathrm{i}\overline{u}_{\frac{1}{2}}\gamma_\nu \mathrm{e}_0^\nu\gamma_5 = \overline{u}_{\frac{1}{2}}, \quad \mathrm{i}\overline{u}_{-\frac{1}{2}}\gamma_\nu \mathrm{e}_0^\nu\gamma_5 = \overline{u}_{-\frac{1}{2}}, \tag{10b}$$

$$-\frac{1}{\mathrm{i}\sqrt{2}}\gamma_5\gamma_\nu \mathrm{e}_{-1}^\nu u_{\frac{1}{2}} = u_{-\frac{1}{2}}, \quad \gamma_\nu \mathrm{e}_{-1}^\nu u_{-\frac{1}{2}} = 0,$$

$$\frac{1}{\mathrm{i}\sqrt{2}}\overline{u}_{\frac{1}{2}}\gamma_\nu \overline{\mathrm{e}}_{-1}^\nu\gamma_5 = \overline{u}_{-\frac{1}{2}}, \quad \overline{u}_{-\frac{1}{2}}\gamma_\nu \overline{\mathrm{e}}_{-1}^\nu = 0, \tag{10c}$$

and

$$\gamma_5\Lambda_+\gamma_5 = -\Lambda_-, \quad \gamma_\sigma P^{\sigma\mu}\Lambda_- = -\Lambda_+\gamma_\sigma P^{\sigma\mu},$$

$$\gamma_\sigma P^{\sigma\mu}\gamma_5\Lambda_+\gamma_5 = \Lambda_+\gamma_\sigma P^{\sigma\mu}, \tag{11}$$

one can perform a direct calculation of the projection operator for an arbitrary half-integral spin. For example, in the case of spin $3/2$, by utilizing the explicit form of the wavefunctions listed below

$$U_{\frac{3}{2}}^\mu(\boldsymbol{p}) = \mathrm{e}_{+1}^\mu(\boldsymbol{p})u_{\frac{1}{2}}(\boldsymbol{p}),$$

$$U_{\frac{1}{2}}^\mu(\boldsymbol{p}) = \sqrt{\frac{1}{3}}\mathrm{e}_{+1}^\mu(\boldsymbol{p})u_{-\frac{1}{2}}(\boldsymbol{p}) + \sqrt{\frac{2}{3}}\mathrm{e}_0^\mu(\boldsymbol{p})u_{\frac{1}{2}}(\boldsymbol{p}),$$

$$U_{-\frac{1}{2}}^\mu(\boldsymbol{p}) = \sqrt{\frac{2}{3}}\mathrm{e}_0^\mu(\boldsymbol{p})u_{-\frac{1}{2}}(\boldsymbol{p}) + \sqrt{\frac{1}{3}}\mathrm{e}_{-1}^\mu(\boldsymbol{p})u_{\frac{1}{2}}(\boldsymbol{p}),$$

$$U_{-\frac{3}{2}}^\mu(\boldsymbol{p}) = \mathrm{e}_{-1}^\mu(\boldsymbol{p})u_{-\frac{1}{2}}(\boldsymbol{p}),$$

a straightforward calculation leads to

$$\begin{aligned}P_+^{\mu\nu}\left(\frac{3}{2},p\right) &= \sum_{m=-3/2}^{3/2} U_m^\mu(\boldsymbol{p})\overline{U}_m^\nu(\boldsymbol{p})\\ &=\Lambda_+\left[P^{\mu\nu}(p) - \frac{1}{3}\gamma_\sigma\gamma_\rho P^{\sigma\mu}(p)P^{\rho\nu}(p)\right].\end{aligned} \tag{12}$$

Similarly,

$$\begin{aligned}P_-^{\mu\nu}\left(\frac{3}{2},p\right) &= \sum_{m=-3/2}^{3/2} V_m^\mu(\boldsymbol{p})\overline{V}_m^\nu(\boldsymbol{p})\\ &=\Lambda_-\left[P^{\mu\nu}(p) - \frac{1}{3}\gamma_\sigma\gamma_\rho P^{\sigma\mu}(p)P^{\rho\nu}(p)\right].\end{aligned} \tag{13}$$

Combining Eqs.(12) and (13), we obtain

$$P_\pm^{\mu\nu}\left(\frac{3}{2},p\right) = \Lambda_\pm Q^{\mu\nu}\left(\frac{3}{2},p\right),$$

$$Q^{\mu\nu}\left(\frac{3}{2},p\right) =P^{\mu\nu}(p) - \frac{1}{3}\gamma_\sigma\gamma_\rho P^{\sigma\mu}(p)P^{\rho\nu}(p). \tag{14a}$$

By using the following relations

$$P^{\mu\nu} = P^{\nu\mu}, \quad \gamma_\sigma\gamma_\rho P^{\sigma\rho} = 3,$$

$$\gamma_\sigma\gamma_\rho P^{\sigma\nu}P^{\rho\mu} = 2P^{\mu\nu} - \gamma_\sigma\gamma_\rho P^{\sigma\mu}P^{\rho\nu}, \tag{14b}$$

$Q^{\mu\nu}\left(\frac{3}{2},p\right)$ can be rewritten as

$$Q^{\mu\nu}\left(\frac{3}{2},p\right) = \frac{2}{5}\gamma_\sigma\gamma_\rho P^{\sigma\mu\rho\nu}(2,p), \tag{14c}$$

where $P^{\sigma\mu\rho\nu}(2,p)$ is the projection operator for spin 2 (see Ref.[10]). Similarly, in the case of spin 5/2, a direct calculation yields

$$P_{\pm}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2},p\right)=\Lambda_{\pm}Q^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2},p\right),$$

$$Q^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2},p\right)=\frac{3}{7}\gamma_\sigma\gamma_\rho P^{\sigma\mu_1\mu_2\rho\nu_1\nu_2}(3,p), \quad (15)$$

where $P^{\sigma\mu_1\mu_2\rho\nu_1\nu_2}(3,p)$ is the projection operator for spin 3. From these results it is possible to write down a general expression for the spin $n+1/2$ projection operators such as

$$P_{\pm}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2},p\right)$$
$$=\Lambda_{\pm}Q^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2},p\right), \quad (16a)$$

$$Q^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2},p\right)$$
$$=\frac{n+1}{2n+3}\gamma_\mu\gamma_\nu P^{\mu\mu_1\cdots\mu_n\nu\nu_1\cdots\nu_n}(n+1,p), \quad (16b)$$

where $P^{\mu\mu_1\cdots\mu_n\nu\nu_1\cdots\nu_n}(n+1,p)$ is the spin $n+1$ projection operator defined by

$$P^{\mu\mu_1\cdots\mu_n\nu\nu_1\cdots\nu_n}(n+1,p)$$
$$=\sum_{m=-(n+1)}^{n+1}\mathrm{e}_m^{\mu\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})\overline{\mathrm{e}}_m^{\nu\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p}). \quad (17)$$

Expressions given by Eqs.(16a) and (16b) are consistent with that constructed by Behrends and Fronsdal.[11] In what follows, we give a general proof for these expressions. Considering the positive energy projection operator, it can be written as (using Eqs.(2a) and (6a))

$$\begin{aligned}
&P_{+}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2},p\right)\\
=&\sum_{m=-(n+1/2)}^{n+1/2}U_m^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})\overline{U}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\\
=&\frac{1}{2n+1}\sum_{m=-(n+1)}^{n+1}[(n+1+m)\mathrm{e}_m^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})u_{\frac{1}{2}}(\boldsymbol{p})\overline{u}_{\frac{1}{2}}(\boldsymbol{p})\overline{\mathrm{e}}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\\
&+\sqrt{(n+1+m)(n-m)}\mathrm{e}_m^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})u_{\frac{1}{2}}(\boldsymbol{p})\overline{u}_{-\frac{1}{2}}(\boldsymbol{p})\overline{\mathrm{e}}_{m-1}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\\
&+\sqrt{(n+1+m)(n-m)}\mathrm{e}_{m+1}^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})u_{-\frac{1}{2}}(\boldsymbol{p})\overline{u}_{\frac{1}{2}}(\boldsymbol{p})\overline{\mathrm{e}}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\\
&+(n-m)\mathrm{e}_{m+1}^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})u_{-\frac{1}{2}}(\boldsymbol{p})\overline{u}_{-\frac{1}{2}}(\boldsymbol{p})\overline{\mathrm{e}}_{m+1}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})], \quad (18)
\end{aligned}$$

(note that $\mathrm{e}_m^{\mu_1\mu_2\cdots\mu_n}=0$ when $m>n$ or $m<-n$). On the other hand, we have from Eqs.(3)

$$\begin{aligned}
&\frac{n+1}{2n+3}\Lambda_+\gamma_\mu\gamma_\nu P^{\mu\mu_1\cdots\mu_n\nu\nu_1\cdots\nu_n}(n+1,p)\\
=&\frac{n+1}{2n+3}\Lambda_+\gamma_\mu\sum_{m=-(n+1)}^{n+1}\left[\sum_{\lambda=-1}^{1}\langle n,m-\lambda,1,\lambda|n,1,n+1,m\rangle\mathrm{e}_{m-\lambda}^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})\mathrm{e}_\lambda^\mu(\boldsymbol{p})\right]\\
&\times\left[\sum_{\lambda=-1}^{1}\langle n,m-\lambda,1,\lambda|n,1,n+1,m\rangle\overline{\mathrm{e}}_{m-\lambda}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\overline{\mathrm{e}}_\lambda^\nu(\boldsymbol{p})\right]\gamma_\nu\\
=&\frac{\Lambda_+\gamma_\mu}{(2n+3)(2n+1)}\sum_{m=-(n+1)}^{n+1}\left\{\frac{(n+m+1)(n+m)}{2}\mathrm{e}_{m-1}^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})\mathrm{e}_1^\mu(\boldsymbol{p})\overline{\mathrm{e}}_{m-1}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\overline{\mathrm{e}}_1^\nu(\boldsymbol{p})\right.\\
&+(n+m+1)\sqrt{\frac{(n+m)(n-m+1)}{2}}\mathrm{e}_{m-1}^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})\mathrm{e}_1^\mu(\boldsymbol{p})\overline{\mathrm{e}}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\overline{\mathrm{e}}_0^\nu(\boldsymbol{p})\\
&+\frac{1}{2}\sqrt{(n^2-m^2)[(n+1)^2-m^2]}\mathrm{e}_{m-1}^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})\mathrm{e}_1^\mu(\boldsymbol{p})\overline{\mathrm{e}}_{m+1}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\overline{\mathrm{e}}_{-1}^\nu(\boldsymbol{p})\\
&+(n+m+1)\sqrt{\frac{(n+m)(n-m+1)}{2}}\mathrm{e}_m^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})\mathrm{e}_0^\mu(\boldsymbol{p})\overline{\mathrm{e}}_{m-1}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\overline{\mathrm{e}}_1^\nu(\boldsymbol{p})\\
&\left.+[(n+1)^2-m^2]\mathrm{e}_m^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})\mathrm{e}_0^\mu(\boldsymbol{p})\overline{\mathrm{e}}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\overline{\mathrm{e}}_0^\nu(\boldsymbol{p})\right.
\end{aligned}$$

$$
\begin{aligned}
&+(n-m+1)\sqrt{\frac{(n-m)(n+m+1)}{2}}\mathrm{e}_m^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})\mathrm{e}_0^{\mu}(\boldsymbol{p})\overline{\mathrm{e}}_{m+1}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\overline{\mathrm{e}}_{-1}^{\nu}(\boldsymbol{p})\\
&+\frac{1}{2}\sqrt{(n^2-m^2)[(n+1)^2-m^2]}\mathrm{e}_{m+1}^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})\mathrm{e}_{-1}^{\mu}(\boldsymbol{p})\overline{\mathrm{e}}_{m-1}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\overline{\mathrm{e}}_{1}^{\nu}(\boldsymbol{p})\\
&+(n-m+1)\sqrt{\frac{(n-m)(n+m+1)}{2}}\mathrm{e}_{m+1}^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})\mathrm{e}_{-1}^{\mu}(\boldsymbol{p})\overline{\mathrm{e}}_{m}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\overline{\mathrm{e}}_{0}^{\nu}(\boldsymbol{p})\\
&+\frac{(n-m+1)(n-m)}{2}\mathrm{e}_{m+1}^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})\mathrm{e}_{-1}^{\mu}(\boldsymbol{p})\overline{\mathrm{e}}_{m+1}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\overline{\mathrm{e}}_{-1}^{\nu}(\boldsymbol{p})\Bigg\}\gamma_\nu. \qquad (19)
\end{aligned}
$$

Utilizing

$$
\begin{aligned}
\Lambda_+\gamma_\mu\mathrm{e}_\lambda^\mu(\boldsymbol{p}) =&\gamma_\mu\mathrm{e}_\lambda^\mu(\boldsymbol{p})\gamma_5\Lambda_+\gamma_5\\
=&\gamma_\mu\mathrm{e}_\lambda^\mu(\boldsymbol{p})\gamma_5(u_{\frac{1}{2}}\overline{u}_{\frac{1}{2}}+u_{-\frac{1}{2}}\overline{u}_{-\frac{1}{2}})\gamma_5,
\end{aligned}
$$

and Eq.(10), it is not difficult to find that the right-hand side of Eq.(19) is exactly the same as the right-hand side of Eq.(18). Hence

$$
\begin{aligned}
&P_+^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2},p\right)\\
=&\frac{n+1}{2n+3}\Lambda_+\gamma_\mu P^{\mu\mu_1\cdots\mu_n\nu\nu_1\cdots\nu_n}(n+1,p)\gamma_\nu.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&P_-^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2},p\right)\\
=&\frac{n+1}{2n+3}\Lambda_-\gamma_\mu P^{\mu\mu_1\cdots\mu_n\nu\nu_1\cdots\nu_n}(n+1,p)\gamma_\nu.
\end{aligned}
$$

Therefore, the B-F form (Eq.(16)) of the projection operator is proved. In view of the calculation of the propagator, however, the B-F form is not the simplest one because it contains many non-independent terms. Fortunately, this problem can be solved by the following tricks. On one hand, by using the explicit expression for the spin $n+1$ projection operator (see Ref.[10]) and Eq.(14b), one can rewrite $Q^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2},p\right)$ in a form that contains only the independent terms, namely, $3P^{\mu_1\nu_1}P^{\mu_2\nu_2}\cdots P^{\mu_n\nu_n}$, $\gamma_\sigma\gamma_\rho P^{\sigma\mu_1}P^{\rho\nu_1}P^{\mu_2\nu_2}\cdots P^{\mu_n\nu_n}$ and their permutations among the tensor indices $(\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots,\nu_n)$. On the other hand, by using

$$
P^{\mu\nu}=\delta_{\mu\nu}+\frac{p_\mu p_\nu}{W^2},
$$

$$
p^2=-W^2,\quad \Lambda_\pm\not{p}=\pm\mathrm{i}W\Lambda_\pm, \qquad (20)
$$

one can further re-express terms like $\Lambda_\pm\gamma_\sigma\gamma_\rho\ P^{\sigma\mu_1}P^{\rho\nu_1}\ P^{\mu_2\nu_2}\cdots P^{\mu_n\nu_n}$ as

$$
\begin{aligned}
&\Lambda_\pm\gamma_\sigma\gamma_\rho P^{\sigma\mu_1}P^{\rho\nu_1}P^{\mu_2\nu_2}\cdots P^{\mu_n\nu_n}\\
=&\Lambda_\pm\left[\gamma_{\mu_1}\gamma_{\nu_1}\mp\frac{\mathrm{i}}{W}(\gamma_{\mu_1}p_{\nu_1}-\gamma_{\nu_1}p_{\mu_1})\right.\\
&\left.+\frac{1}{W^2}p_{\mu_1}p_{\nu_1}\right]P^{\mu_2\nu_2}\cdots P^{\mu_n\nu_n}. \qquad (21)
\end{aligned}
$$

After these simplifications, the projection operators can be expressed in a form that is suitable for the calculation of the propagators. For example, in the cases of spins 3/2, 5/2 and 7/2, the projection operators become respectively

$$
P_\pm^{\mu\nu}\left(\frac{3}{2},p\right)=\Lambda_\pm\left[\delta_{\mu\nu}-\frac{1}{3}\gamma_\mu\gamma_\nu\pm\frac{\mathrm{i}}{3W}(\gamma_\mu p_\nu-\gamma_\nu p_\mu)+\frac{2}{3W^2}p_\mu p_\nu\right], \qquad (22)
$$

$$
\begin{aligned}
P_\pm^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2},p\right)=&\Lambda_\pm\left\{\frac{1}{2}(P^{\mu_1\nu_1}P^{\mu_2\nu_2}+P^{\mu_1\nu_2}P^{\mu_2\nu_1})-\frac{1}{5}P^{\mu_1\mu_2}P^{\nu_1\nu_2}\right.\\
&\left.-\frac{1}{10}\sum_{\substack{P(\mu_1\mu_2)\\P(\nu_1\nu_2)}}\left[\gamma_{\mu_1}\gamma_{\nu_1}\mp\frac{\mathrm{i}}{W}(\gamma_{\mu_1}p_{\nu_1}-\gamma_{\nu_1}p_{\mu_1})+\frac{1}{W^2}p_{\mu_1}p_{\nu_1}\right]P^{\mu_2\nu_2}\right\}, \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
P_\pm^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\left(\frac{7}{2},p\right)=&\Lambda_\pm\left\{\frac{1}{6}\sum_{P(\nu_1\nu_2\nu_3)}P^{\mu_1\nu_1}P^{\mu_2\nu_2}P^{\mu_3\mu_3}-\frac{1}{280}\sum_{\substack{P(\mu_1\mu_2\mu_3)\\P(\nu_1\nu_2\nu_3)}}P^{\mu_1\mu_2}P^{\nu_1\nu_2}P^{\mu_3\nu_3}\right.\\
&\left.-\frac{1}{84}\sum_{\substack{P(\mu_1\mu_2\mu_3)\\P(\nu_1\nu_2\nu_3)}}\left[\left(\gamma_{\mu_1}\gamma_{\nu_1}\mp\frac{\mathrm{i}}{W}(\gamma_{\mu_1}p_{\nu_1}-\gamma_{\nu_1}p_{\mu_1})+\frac{1}{W^2}p_{\mu_1}p_{\nu_1}\right)P^{\mu_2\nu_2}P^{\mu_3\nu_3}\right.\right.
\end{aligned}
$$

$$+\frac{2}{9}\left(\gamma_{\mu_1}\gamma_{\mu_2}\mp\frac{\mathrm{i}}{W}(\gamma_{\mu_1}p_{\mu_2}-\gamma_{\mu_2}p_{\mu_1})+\frac{1}{W^2}p_{\mu_1}p_{\mu_2}\right)P^{\nu_1\nu_2}P^{\mu_3\nu_3}$$
$$+\frac{1}{5}\left(\gamma_{\nu_1}\gamma_{\mu_1}\mp\frac{\mathrm{i}}{W}(\gamma_{\nu_1}p_{\mu_1}-\gamma_{\mu_1}p_{\nu_1})+\frac{1}{W^2}p_{\mu_1}p_{\nu_1}\right)P^{\mu_2\mu_3}P^{\nu_2\nu_3}\Big]\Big\}, \tag{24}$$

where $P(\mu_1,\mu_2,\cdots,\mu_n)$ denotes permutations of $(\mu_1,\mu_2,\cdots,\mu_n)$.

3. The Feynman propagator for an arbitrary half-integral spin

We now proceed to derive the Feynman propagator for a particle with arbitrary half-integral spin. We begin with working out the Feynman propagators for spins 3/2 and 5/2 in detail in order to demonstrate the method employed in this section. After that, the procedure is generalized to the case for an arbitrary half-integral spin $n+1/2$.

3.1. The Feynman propagator for spin 3/2

In the case of spin 3/2, armed with the wavefunctions

$$\Psi^\nu(x)=\frac{1}{\sqrt{V}}\sum_p\sum_{m=-3/2}^{3/2}[a_m(\boldsymbol{p})U_m^\nu(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}+b_m^+(\boldsymbol{p})V_m^\nu(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px}], \tag{25}$$

the projection operator $P^{\mu\nu}\left(\frac{3}{2},p\right)$ given by Eq.(22), and the field quantization conditions listed below

$$\{a_m(\boldsymbol{p}),a_{m'}^+(\boldsymbol{p}')\}=\delta_{p,p'}\delta_{mm'}$$
$$=\{b_m(\boldsymbol{p}),b_{m'}^+(\boldsymbol{p}')\}, \tag{26a}$$

$$\{a_m(\boldsymbol{p}),a_{m'}(\boldsymbol{p}')\}=\{b_m(\boldsymbol{p}),b_{m'}(\boldsymbol{p}')\}$$
$$=\{a_m(\boldsymbol{p}),b_{m'}^+(\boldsymbol{p}')\}=\{a_m(\boldsymbol{p}),b_{m'}(\boldsymbol{p}')\}=0, \tag{26b}$$

together with the definition of the adjoint field

$$\overline{\Psi}^\mu(x)=(\Psi^\mu(x))^+\gamma_4$$
$$=\frac{1}{\sqrt{V}}\sum_p\sum_{m=-3/2}^{3/2}[a_m^+(\boldsymbol{p})\overline{U}_m^\nu(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px}+b_m(\boldsymbol{p})\overline{V}_m^\nu(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}], \tag{27}$$

we find that the general commutation relations take the following form

$$\{\Psi^\mu(x),\overline{\Psi}^\nu(x')\}=\mathrm{i}\hat{P}^{\mu\nu}\left(\frac{3}{2}\right)\Delta(x-x'), \tag{28}$$

or equivalently

$$\{\Psi^\mu(x)^{(-)},\overline{\Psi}^\nu(x')^{(+)}\}=\mathrm{i}\hat{P}^{\mu\nu}\left(\frac{3}{2}\right)\Delta^{(+)}(x-x'), \tag{29a}$$

$$\{\Psi^\mu(x)^{(+)},\overline{\Psi}^\nu(x')^{(-)}\}=\mathrm{i}\hat{P}^{\mu\nu}\left(\frac{3}{2}\right)\Delta^{(-)}(x-x'), \tag{29b}$$

where

$$\Psi^\mu(x)^{(-)}=\frac{1}{\sqrt{V}}\sum_{pm}a_m(\boldsymbol{p})U_m^\mu(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px},$$
$$\Psi^\mu(x)^{(+)}=\frac{1}{\sqrt{V}}\sum_{pm}b_m^+(\boldsymbol{p})V_m^\mu(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px},$$
$$\overline{\Psi}^\mu(x)^{(+)}=\frac{1}{\sqrt{V}}\sum_{p}a_m^+(\boldsymbol{p})\overline{U}_m^\mu(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px},$$
$$\overline{\Psi}^\mu(x)^{(-)}=\frac{1}{\sqrt{V}}\sum_{p}b_m(\boldsymbol{p})\overline{V}_m^\mu(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px},$$

$$\hat{P}^{\mu\nu}\left(\frac{3}{2}\right)=-(\partial\!\!\!/-W)\hat{R}^{\mu\nu}\left(\frac{3}{2}\right), \tag{30a}$$

$$\hat{R}^{\mu\nu}\left(\frac{3}{2}\right)=\hat{P}_{\mu\nu}-\frac{1}{3}\Big[\gamma_\mu\gamma_\nu-\frac{1}{W}(\gamma_\mu\partial_\nu-\gamma_\nu\partial_\mu)-\frac{1}{W^2}\partial_\mu\partial_\nu\Big], \tag{30b}$$

$$\hat{P}^{\mu\nu}=\delta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{W^2}, \tag{30c}$$

$$\mathrm{i}\Delta(x-x')=\mathrm{i}\Delta^{(+)}(x-x')+\mathrm{i}\Delta^{(-)}(x-x'), \tag{30d}$$

$$\mathrm{i}\Delta^{(+)}(x-x')=\frac{1}{V}\sum_p\frac{1}{2E}\mathrm{e}^{\mathrm{i}p(x-x')},$$
$$\mathrm{i}\Delta^{(-)}(x-x')=-\frac{1}{V}\sum_p\frac{1}{2E}\mathrm{e}^{-\mathrm{i}p(x-x')}. \tag{30e}$$

The Feynman propagator for spin 3/2 is defined, in the usual fashion, by

$$S_F^{\mu\nu}(x-x')\equiv\langle 0|T\,\Psi^\mu(x)\overline{\Psi}^\nu(x')|0\rangle$$
$$=\begin{cases}\langle 0|\,\Psi^\mu(x)\overline{\Psi}^\nu(x')|0\rangle, & t>t';\\ -\langle 0|\overline{\Psi}^\nu(x')\,\Psi^\mu(x)|0\rangle, & t<t';\end{cases} \tag{31}$$

and it can be rewritten as, with the aid of the commutation rules given by Eq.(29),

$$S_F^{\mu\nu}(x-x')=\mathrm{i}\theta(t-t')\left[\hat{P}^{\mu\nu}\left(\frac{3}{2}\right)\Delta^{(+)}(x-x')\right]-\mathrm{i}\theta(t-t')\left[\hat{P}^{\mu\nu}\left(\frac{3}{2}\right)\Delta^{(-)}(x-x')\right]. \tag{32}$$

By virtue of

$$\partial_t\theta(t)=\delta(t),\quad \delta(t)[\Delta^{(+)}(x)+\Delta^{(-)}(x)]=0,$$

$$\dot{\delta}(t)[\Delta^{(+)}(x)+\Delta^{(-)}(x)]=\delta^{(4)}(x), \tag{33}$$

and performing a direct calculation that makes the θ-functions commuted past the differential operator $\hat{P}^{\mu\nu}\left(\frac{3}{2}\right)$, one finds that the Feynman propagator for spin 3/2 in coordinate representation takes the form

$$\begin{aligned}S_F^{\mu\nu}(x-x')=&\hat{P}^{\mu\nu}\left(\frac{3}{2}\right)\Delta_F(x-x')\\&+\hat{K}^{\mu\nu}\left(\frac{3}{2}\right)\delta^{(4)}(x-x'),\end{aligned} \tag{34}$$

where

$$\begin{aligned}\Delta_F(x-x')=&\mathrm{i}\theta(t-t')\Delta^{(+)}(x-x')\\&-\mathrm{i}\theta(t'-t')\Delta^{(-)}(x-x'),\end{aligned} \tag{35a}$$

$$\begin{aligned}\hat{K}^{\mu\nu}\left(\frac{3}{2}\right)=&-\mathrm{i}\left[\frac{1}{3W}\gamma_4(\delta_{\mu4}\gamma_\nu-\gamma_\mu\delta_{\nu4})\right.\\&\left.+\frac{2}{3W^2}(\not{\partial}-W)\delta_{\mu4}\delta_{\nu4}\right].\end{aligned} \tag{35b}$$

Utilizing

$$\Delta_F(x)=\frac{1}{(2\pi)^4}\int \mathrm{d}^4p\mathrm{e}^{\mathrm{i}px}\Delta_F(p),$$

$$\Delta_F(p)=\frac{-\mathrm{i}}{p^2+W^2-\mathrm{i}\varepsilon}, \tag{36}$$

the Fourier representation for $S_F^{\mu\nu}(x)$ is found to be

$$S_F^{\mu\nu}(x)=\frac{1}{(2\pi)^4}\int \mathrm{d}^4p\mathrm{e}^{\mathrm{i}px}S_F^{\mu\nu}\left(\frac{3}{2},p\right), \tag{37}$$

$$\begin{aligned}S_F^{\mu\nu}\left(\frac{3}{2},p\right)=&\frac{-1}{\not{p}-\mathrm{i}W+\mathrm{i}\varepsilon}R^{\mu\nu}\left(\frac{3}{2},p\right)\\&+K^{\mu\nu}\left(\frac{3}{2},p\right),\end{aligned} \tag{38}$$

with

$$\begin{aligned}R^{\mu\nu}\left(\frac{3}{2},p\right)=&P_{\mu\nu}-\frac{1}{3}\left[\gamma_\mu\gamma_\nu-\frac{\mathrm{i}}{W}(\gamma_\mu p_\nu-\gamma_\nu p_\mu)\right.\\&\left.-\frac{1}{W^2}p_\mu p_\nu\right],\end{aligned} \tag{39a}$$

$$\begin{aligned}K^{\mu\nu}\left(\frac{3}{2},p\right)=&-\mathrm{i}\left[\frac{1}{3W}\gamma_4(\delta_{\mu4}\gamma_\nu-\gamma_\mu\delta_{\nu4})\right.\\&\left.+\frac{2}{3W^2}(\mathrm{i}\not{p}-W)\delta_{\mu4}\delta_{\nu4}\right],\end{aligned} \tag{39b}$$

where $S_F^{\mu\nu}\left(\frac{3}{2},p\right)$ is the Feynman propagator for spin 3/2 in momentum representation. We emphasize here that in the theory of free field for spin larger than 1/2, an additional non-covariant term such as that in Eq.(34) or (38) inevitably appears in the expression of the Feynman propagator.

3.2. The Feynman propagator for spin 5/2

In a similar manner as in the case of spin 3/2, by using the wavefunctions for spin 5/2

$$\begin{aligned}\Psi^{\nu_1\nu_2}(x)=&\sum_p\sum_{m=-5/2}^{5/2}\frac{1}{\sqrt{V}}[a_m(\boldsymbol{p})U_m^{\nu_1\nu_2}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}\\&+b_m^+(\boldsymbol{p})V_m^{\nu_1\nu_2}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px}],\end{aligned} \tag{40}$$

the projection operator $P^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2},p\right)$ given by Eq.(23), the field quantization conditions Eq.(26), and the definition of the adjoint field

$$\begin{aligned}\overline{\Psi}^{\nu_1\nu_2}(x)=&(\Psi^{\nu_1\nu_2}(x))^+\gamma_4\\=&\sum_{p,m}\frac{1}{\sqrt{V}}[a_m^+(\boldsymbol{p})\overline{U}_m^{\nu_1\nu_2}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px}\\&+b_m(\boldsymbol{p})\overline{V}_m^{\nu_1\nu_2}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}],\end{aligned} \tag{41}$$

we find that the general commutation rules are

$$\{\Psi^{\mu_1\mu_2}(x),\overline{\Psi}^{\nu_1\nu_2}(x')\}=\mathrm{i}\hat{P}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}\right)\Delta(x-x'), \tag{42}$$

or equivalently

$$\begin{aligned}&\{\Psi^{\mu_1\mu_2}(x)^{(-)},\overline{\Psi}^{\nu_1\nu_2}(x')^{(+)}\}\\=&\mathrm{i}\hat{P}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}\right)\Delta^{(+)}(x-x'),\end{aligned} \tag{43a}$$

$$\begin{aligned}&\{\Psi^{\mu_1\mu_2}(x)^{(+)},\overline{\Psi}^{\nu_1\nu_2}(x')^{(-)}\}\\=&\mathrm{i}\hat{P}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}\right)\Delta^{(-)}(x-x'),\end{aligned} \tag{43b}$$

where

$$\Psi^{\mu_1\mu_2}(x)^{(-)}=\frac{1}{\sqrt{V}}\sum_{pm}a_m(\boldsymbol{p})U_m^{\mu_1\mu_2}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px},$$

$$\Psi^{\mu_1\mu_2}(x)^{(+)}=\frac{1}{\sqrt{V}}\sum_{pm}b_m^+(\boldsymbol{p})V_m^{\mu_1\mu_2}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px},$$

$$\overline{\Psi}^{\nu_1\nu_2}(x)^{(+)}=\frac{1}{\sqrt{V}}\sum_{p}a_m^+(\boldsymbol{p})\overline{U}_m^{\nu_1\nu_2}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px},$$

$$\overline{\Psi}^{\nu_1\nu_2}(x)^{(-)}=\frac{1}{\sqrt{V}}\sum_{p}b_m(\boldsymbol{p})\overline{V}_m^{\nu_1\nu_2}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px},$$

$$\hat{P}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}\right) = -(\not{\partial} - W)\hat{R}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}\right), \quad (44a)$$

$$\begin{aligned}
&\hat{R}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}\right)\\
=&\frac{1}{2}[\hat{P}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2} + \hat{P}^{\mu_1\nu_2}\hat{P}^{\mu_2\nu_1}] - \frac{1}{5}[\hat{P}^{\mu_1\mu_2}\hat{P}^{\nu_1\nu_2}]\\
&- \frac{1}{10}\sum_{\substack{P(\mu_1\mu_2)\\P(\nu_1\nu_2)}}\left[\gamma_{\mu_1}\gamma_{\nu_1} - \frac{1}{W}(\gamma_{\mu_1}\partial_{\nu_1} - \gamma_{\nu_1}\partial_{\mu_1})\right.\\
&\left. - \frac{1}{W^2}\partial_{\mu_1}\partial_{\nu_1}\right]\hat{P}^{\mu_2\nu_2}. \quad (44b)
\end{aligned}$$

The Feynman propagator for spin 5/2 is defined by

$$\begin{aligned}
&S_F^{\mu_1\mu_2\nu_1\nu_2}(x-x')\\
\equiv&\langle 0|T\,\Psi^{\mu_1\mu_2}(x)\overline{\Psi}^{\nu_1\nu_2}(x')|0\rangle\\
=&\begin{cases}\langle 0|\,\Psi^{\mu_1\mu_2}(x)\overline{\Psi}^{\nu_1\nu_2}(x')|0\rangle, & t>t',\\ -\langle 0|\,\overline{\Psi}^{\nu_1\nu_2}(x')\,\Psi^{\mu_1\mu_2}(x)|0\rangle, & t<t',\end{cases} \quad (45)
\end{aligned}$$

and can be re-expressed, with the aid of Eq.(43), as

$$\begin{aligned}
&S_F^{\mu_1\mu_2\nu_1\nu_2}(x-x')\\
=&\mathrm{i}\theta(t-t')\hat{P}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}\right)\Delta^{(+)}(x-x')\\
&-\mathrm{i}\theta(t'-t)\hat{P}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}\right)\Delta^{(-)}(x-x'). \quad (46)
\end{aligned}$$

Performing a direct calculation that makes the θ-functions commuted past the differential operator $\hat{P}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}\right)$, using Eq.(33), we obtain

$$\begin{aligned}
&S_F^{\mu_1\mu_2\nu_1\nu_2}(x-x')\\
=&\hat{P}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}\right)\Delta_F(x-x')\\
&+\hat{K}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}\right)\delta^{(4)}(x-x'), \quad (47)
\end{aligned}$$

where

$$\begin{aligned}
&\hat{K}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}\right)\\
=&-\frac{\mathrm{i}(\not{\partial}-W)}{2W^2}[\delta_{\mu_1\nu_1}\delta_{\mu_2 4}\delta_{\nu_2 4} + \delta_{\mu_1 4}\delta_{\nu_1 4}\hat{P}^{\mu_2\nu_2}]\\
&-\frac{\mathrm{i}(\not{\partial}-W)}{2W^2}[\delta_{\mu_1\nu_2}\delta_{\mu_2 4}\delta_{\nu_1 4} + \delta_{\mu_1 4}\delta_{\nu_2 4}\hat{P}^{\mu_2\nu_1}]\\
&+\frac{\mathrm{i}(\not{\partial}-W)}{5W^2}[\delta_{\mu_1\mu_2}\delta_{\nu_1 4}\delta_{\nu_2 4} + \delta_{\mu_1 4}\delta_{\mu_2 4}\hat{P}^{\nu_1\nu_2}]\\
&-\frac{1}{10}\sum_{\substack{P(\mu_1\mu_2)\\P(\nu_1\nu_2)}}\left\{\left[\frac{\mathrm{i}}{W}\gamma_4(\delta_{\mu_1 4}\gamma_{\nu_1} - \gamma_{\mu_1}\delta_{\nu_1 4})\right.\right.\\
&\left.-\frac{\mathrm{i}}{W^2}(\not{\partial}-W)\delta_{\mu_1 4}\delta_{\nu_1 4}\right]\hat{P}^{\mu_2\nu_2}\\
&\left.+(\not{\partial}-W)\gamma_{\mu_1}\gamma_{\nu_1}\left(\frac{\delta_{\mu_2 4}\delta_{\nu_2 4}}{W^2}\right)\right\}. \quad (48)
\end{aligned}$$

This is the Feynman propagator for spin 5/2 in coordinate representation. The Fourier representation for $S_F^{\mu_1\mu_2\nu_1\nu_2}(x)$ can be easily deduced, using Eq.(36),

$$S_F^{\mu_1\mu_2\nu_1\nu_2}(x) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4 p \mathrm{e}^{\mathrm{i}px}\, S_F^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}, p\right), \quad (49)$$

$$\begin{aligned}
S_F^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}, p\right) =&\frac{-1}{\not{p} - \mathrm{i}W + \mathrm{i}\varepsilon}R^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}, p\right)\\
&+ K^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}, p\right), \quad (50)
\end{aligned}$$

with

$$\begin{aligned}
&R^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}, p\right)\\
=&\frac{1}{2}[P^{\mu_1\nu_1}P^{\mu_2\nu_2} + P^{\mu_1\nu_2}P^{\mu_2\nu_1}] - \frac{1}{5}[P^{\mu_1\mu_2}P^{\nu_1\nu_2}]\\
&-\frac{1}{10}\sum_{\substack{P(\mu_1\mu_2)\\P(\nu_1\nu_2)}}\left[\gamma_{\mu_1}\gamma_{\nu_1} - \frac{\mathrm{i}}{W}(\gamma_{\mu_1}p_{\nu_1} - \gamma_{\nu_1}p_{\mu_1})\right.\\
&\left.-\frac{1}{W^2}p_{\mu_1}p_{\nu_1}\right]P^{\mu_2\nu_2}. \quad (51a)
\end{aligned}$$

$$\begin{aligned}
&K^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}, p\right)\\
=&-\frac{\mathrm{i}(\mathrm{i}\not{p}-W)}{2W^2}[\delta_{\mu_1\nu_1}\delta_{\mu_2 4}\delta_{\nu_2 4} + \delta_{\mu_1 4}\delta_{\nu_1 4}P^{\mu_2\nu_2}]\\
&-\frac{\mathrm{i}(\mathrm{i}\not{p}-W)}{2W^2}[\delta_{\mu_1\nu_2}\delta_{\mu_2 4}\delta_{\nu_1 4} + \delta_{\mu_1 4}\delta_{\nu_2 4}P^{\mu_2\nu_1}]\\
&+\frac{\mathrm{i}(\mathrm{i}\not{p}-W)}{5W^2}[\delta_{\mu_1\mu_2}\delta_{\nu_1 4}\delta_{\nu_2 4} + \delta_{\mu_1 4}\delta_{\mu_2 4}P^{\nu_1\nu_2}]\\
&-\frac{1}{10}\sum_{\substack{P(\mu_1\mu_2)\\P(\nu_1\nu_2)}}\left\{\left[\frac{\mathrm{i}}{W}\gamma_4(\delta_{\mu_1 4}\gamma_{\nu_1} - \gamma_{\mu_1}\delta_{\nu_1 4})\right.\right.\\
&\left.-\frac{\mathrm{i}}{W^2}(\mathrm{i}\not{p}-W)\delta_{\mu_1 4}\delta_{\nu_1 4}\right]P^{\mu_2\nu_2}\\
&\left.+(\mathrm{i}\not{p}-W)\gamma_{\mu_1}\gamma_{\nu_1}\left(\frac{\delta_{\mu_2 4}\delta_{\nu_2 4}}{W^2}\right)\right\}. \quad (51b)
\end{aligned}$$

Similar to the previous case, $S_F^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}, p\right)$ in the above is the Feynman propagator for spin 5/2 in momentum representation.

3.3. The Feynman propagator for an arbitrary half-integral spin $n + 1/2$

The above procedure is now extended to the general case of an arbitrary half-integral spin $n+1/2$. We begin by working out the general commutation rules. From Eqs.(1), (5), (8), (16) and (26), we have, by following the same steps as in the cases of spin 3/2 and 5/2,

$$\{\Psi^{\mu_1\mu_2\cdots\mu_n}(x), \overline{\Psi}^{\nu_1\nu_2\cdots\nu_n}(x')\}$$
$$=\mathrm{i}\hat{P}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2}\right)\Delta(x-x'), \quad (52)$$

or equivalently

$$\{\Psi^{\mu_1\mu_2\cdots\mu_n}(x)^{(-)}, \overline{\Psi}^{\nu_1\nu_2\cdots\nu_n}(x')^{(+)}\}$$
$$=\mathrm{i}\hat{P}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2}\right)\Delta^{(+)}(x-x'), \quad (53a)$$

$$\{\Psi^{\mu_1\mu_2\cdots\mu_n}(x)^{(+)}, \overline{\Psi}^{\nu_1\nu_2\cdots\nu_n}(x')^{(-)}\}$$
$$=\mathrm{i}\hat{P}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2}\right)\Delta^{(-)}(x-x'), \quad (53b)$$

with

$$\Psi^{\mu_1\mu_2\cdots\mu_n}(x)^{(-)}=\frac{1}{\sqrt{V}}\sum_{pm}a_m(\boldsymbol{p})U_m^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px},$$
$$\Psi^{\mu_1\mu_2\cdots\mu_n}(x)^{(+)}=\frac{1}{\sqrt{V}}\sum_{pm}b_m^{+}(\boldsymbol{p})V_m^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px}, \quad (54a)$$

$$\overline{\Psi}^{\mu_1\mu_2\cdots\mu_n}(x)^{(+)}=\frac{1}{\sqrt{V}}\sum_{p}a_m^{+}(\boldsymbol{p})\overline{U}_m^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px},$$
$$\overline{\Psi}^{\mu_1\mu_2\cdots\mu_n}(x)^{(-)}=\frac{1}{\sqrt{V}}\sum_{p}b_m(\boldsymbol{p})\overline{V}_m^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}, \quad (54b)$$

and

$$\hat{P}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2}\right)$$
$$=-(\not{\partial}-W)\hat{R}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2}\right), \quad (55)$$

where $\hat{R}^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}\left(n+\frac{1}{2}\right)$ is a differential operator constructed from $Q^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}\left(n+\frac{1}{2},p\right)$ in the following way:

$$3P^{\mu_1\nu_1}P^{\mu_2\nu_2}\cdots P^{\mu_n\nu_n}\Rightarrow 3\hat{P}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2}\cdots\hat{P}^{\mu_n\nu_n}, \quad (56a)$$

$$\left[\gamma_{\mu_1}\gamma_{\nu_1}\mp\frac{\mathrm{i}}{W}(\gamma_{\mu_1}p_{\nu_1}-\gamma_{\nu_1}p_{\mu_1})+\frac{1}{W^2}p_{\mu_1}p_{\nu_1}\right]P^{\mu_2\nu_2}\cdots P^{\mu_n\nu_n}$$
$$\Rightarrow\left[\gamma_{\mu_1}\gamma_{\nu_1}-\frac{1}{W}(\gamma_{\mu_1}\partial_{\nu_1}-\gamma_{\nu_1}\partial_{\mu_1})-\frac{1}{W^2}\partial_{\mu_1}\partial_{\nu_1}\right]\hat{P}^{\mu_2\nu_2}\cdots\hat{P}^{\mu_n\nu_n}. \quad (56b)$$

The Feynman propagator for spin $n+1/2$ is defined by

$$S_F^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(x-x')$$
$$\equiv\langle 0|T\,\Psi^{\mu_1\mu_2\cdots\mu_n}(x)\overline{\Psi}^{\nu_1\nu_2\cdots\nu_n}(x')|0\rangle$$
$$=\begin{cases}\langle 0|\,\Psi^{\mu_1\mu_2\cdots\mu_n}(x)\overline{\Psi}^{\nu_1\nu_2\cdots\nu_n}(x')|0\rangle, & t>t',\\ -\langle 0|\,\overline{\Psi}^{\nu_1\nu_2\cdots\nu_n}(x')\,\Psi^{\mu_1\mu_2\cdots\mu_n}(x)|0\rangle, & t<t',\end{cases} \quad (57)$$

and can be rewritten as, using Eqs.(53),

$$S_F^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(x-x')$$
$$=\mathrm{i}\theta(t-t')\hat{P}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2}\right)\Delta^{(+)}(x-x')$$
$$-\mathrm{i}\theta(t-t')\hat{P}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2}\right)\Delta^{(-)}(x-x'). \quad (58)$$

Performing a step-by-step calculation that makes the θ-functions commuted past the differential operator $P^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2}\right)$, we find that the general expression for the Feynman propagator for spin $n+1/2$ in coordinate representation is

$$S^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(x-x')$$
$$=\hat{P}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2}\right)\Delta_F(x-x')$$
$$+\hat{K}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2}\right)\delta^{(4)}(x-x'), \quad (59)$$

where $\hat{K}^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}\left(n+\frac{1}{2}\right)$ results from the commutation between the θ-functions and $\hat{P}^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}\left(n+\frac{1}{2}\right)$, and consists of the two kinds of terms [refer to Eqs.(56a) and (56b)]: one originates from the terms such as $(\not{\partial}-W)3\hat{P}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2}\cdots\hat{P}^{\mu_n\nu_n}$, and can be expressed as $\frac{3\mathrm{i}(\not{\partial}-W)}{W^2}[\hat{B}^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n}]$, because of

$$\mathrm{i}\theta(t)(\not{\partial}-W)3\hat{P}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2}\cdots\hat{P}^{\mu_n\nu_n}\Delta^{(+)}(x)$$
$$-\mathrm{i}\theta(-t)(\not{\partial}-W)3\hat{P}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2}\cdots\hat{P}^{\mu_n\nu_n}\Delta^{(-)}(x)$$
$$=(\not{\partial}-W)3\hat{P}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2}\cdots\hat{P}^{\mu_n\nu_n}\Delta_F(x)$$
$$+\frac{3\mathrm{i}(\not{\partial}-W)}{W^2}[\hat{B}^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n}]\delta^{(4)}(x), \quad (60a)$$

where

$$\hat{B}^{\mu_1\nu_1}=\delta_{\mu_1 4}\delta_{\nu_1 4},$$
$$\hat{B}^{\mu_1\nu_1\mu_2\nu_2}=\hat{B}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2}+\delta_{\mu_1\nu_1}\delta_{\mu_2 4}\delta_{\nu_2 4},$$
$$\hat{B}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}=\hat{B}^{\mu_1\nu_1\mu_2\nu_2}\hat{P}^{\mu_3\nu_3}+\delta_{\mu_1\nu_1}\delta_{\mu_2\nu_2}\delta_{\mu_3 4}\delta_{\nu_3 4},\cdots,$$
$$\hat{B}^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n}=\hat{B}^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_{n-1}\nu_{n-1}}\hat{P}^{\mu_n\nu_n}+\delta_{\mu_1\nu_1}\delta_{\mu_2\nu_2}\cdots\delta_{\mu_{n-1}\nu_{n-1}}\delta_{\mu_n 4}\delta_{\nu_n 4}. \quad (60b)$$

While the other originates from the terms such as

$$(\partial\!\!\!/ - W)\left[\gamma_{\mu_1}\gamma_{\nu_1} - \frac{1}{W}(\gamma_{\mu_1}\partial_{\nu_1} - \gamma_{\nu_1}\partial_{\mu_1}) - \frac{1}{W^2}\partial_{\mu_1}\partial_{\nu_1}\right]\hat{P}^{\mu_2\nu_2}\cdots\hat{P}^{\mu_n\nu_n}$$

and can be expressed by $\hat{C}^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n}$, because of

$$\begin{aligned}
&\mathrm{i}\theta(t)(\partial\!\!\!/ - W)\left[\gamma_{\mu_1}\gamma_{\nu_1} - \frac{1}{W}(\gamma_{\mu_1}\partial_{\nu_1} - \gamma_{\nu_1}\partial_{\mu_1}) - \frac{1}{W^2}\partial_{\mu_1}\partial_{\nu_1}\right]\hat{P}^{\mu_2\nu_2}\cdots\hat{P}^{\mu_n\nu_n}\Delta^{(+)}(x)\\
&-\mathrm{i}\theta(-t)(\partial\!\!\!/ - W)\left[\gamma_{\mu_1}\gamma_{\nu_1} - \frac{1}{W}(\gamma_{\mu_1}\partial_{\nu_1} - \gamma_{\nu_1}\partial_{\mu_1}) - \frac{1}{W^2}\partial_{\mu_1}\partial_{\nu_1}\right]\hat{P}^{\mu_2\nu_2}\cdots\hat{P}^{\mu_n\nu_n}\Delta^{(-)}(x)\\
=&(\partial\!\!\!/ - W)\left[\gamma_{\mu_1}\gamma_{\nu_1} - \frac{1}{W}(\gamma_{\mu_1}\partial_{\nu_1} - \gamma_{\nu_1}\partial_{\mu_1}) - \frac{1}{W^2}\partial_{\mu_1}\partial_{\nu_1}\right]\hat{P}^{\mu_2\nu_2}\cdots\hat{P}^{\mu_n\nu_n}\Delta_F(x) + C^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n}\delta^{(4)}(x), \quad (61a)
\end{aligned}$$

with

$$\begin{aligned}
\hat{C}^{\mu_1\nu_1} =&\mathrm{i}\left[\frac{1}{W}\gamma_4(\delta_{\mu_1 4}\gamma_{\nu_1} - \gamma_{\mu_1}\delta_{\nu_1 4}) - \frac{1}{W^2}(\partial\!\!\!/ - W)\delta_{\mu_1 4}\delta_{\nu_1 4}\right],\\
\hat{C}^{\mu_1\nu_1\mu_2\nu_2} =&\hat{C}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2} + \mathrm{i}(\partial\!\!\!/ - W)\gamma_{\mu_1}\gamma_{\nu_1}\left(\frac{\delta_{\mu_2 4}\delta_{\nu_2 4}}{W^2}\right),\\
\hat{C}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} =&\hat{C}^{\mu_1\nu_1\mu_2\nu_2}\hat{P}^{\mu_3\nu_3} + \mathrm{i}(\partial\!\!\!/ - W)\gamma_{\mu_1}\gamma_{\nu_1}\delta_{\mu_2\nu_2}\left(\frac{\delta_{\mu_3 4}\delta_{\nu_3 4}}{W^2}\right),\cdots,\\
\hat{C}^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n} =&\hat{C}^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_{n-1}\nu_{n-1}}\hat{P}^{\mu_n\nu_n} + \mathrm{i}(\partial\!\!\!/ - W)\gamma_{\mu_1}\gamma_{\nu_1}\delta_{\mu_2\nu_2}\delta_{\mu_3\nu_3}\cdots\delta_{\mu_{n-1}\nu_{n-1}}\left(\frac{\delta_{\mu_n 4}\delta_{\nu_n 4}}{W^2}\right). \quad (61b)
\end{aligned}$$

The Fourier representation for $S_F^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(x)$ can be easily derived, using Eq.(36),

$$\begin{aligned}
&S_F^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(x)\\
=&\frac{1}{(2\pi)^4}\int \mathrm{d}^4p\mathrm{e}^{\mathrm{i}px}S_F^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2},p\right), \quad (62a)
\end{aligned}$$

$$\begin{aligned}
&S_F^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2},p\right)\\
=&\frac{-1}{p\!\!\!/ - \mathrm{i}W + \mathrm{i}\varepsilon}R^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2},p\right)\\
&+K^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2},p\right), \quad (62b)
\end{aligned}$$

with

$$\begin{aligned}
&R^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2},p\right)\\
=&\hat{R}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2}\right)\Big|_{\partial_\mu=\mathrm{i}p_\mu}, \quad (64a)\\
&K^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2},p\right)\\
=&\hat{K}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2}\right)\Big|_{\partial_\mu=\mathrm{i}p_\mu}. \quad (64b)
\end{aligned}$$

Equation (62b) gives the general momentum representation for the Feynman propagator for an arbitrary half-integral spin. In order to illustrate these formulae, we provide finally the explicit expression for the propagator for spin 7/2

$$\begin{aligned}
&S_F^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\left(\frac{7}{2},p\right)\\
=&\frac{-1}{p\!\!\!/ - \mathrm{i}W + \mathrm{i}\varepsilon}R^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\left(\frac{7}{2},p\right)\\
&+K^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\left(\frac{7}{2},p\right), \quad (65)
\end{aligned}$$

where

$$\begin{aligned}
&R^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\left(\frac{7}{2},p\right)\\
=&\frac{1}{6}\sum_{P(\nu_1\nu_2\nu_3)}P^{\mu_1\nu_1}P^{\mu_2\nu_2}P^{\mu_3\nu_3}\\
&-\frac{1}{280}\sum_{\substack{P(\mu_1\mu_2\mu_3)\\P(\nu_1\nu_2\nu_3)}}P^{\mu_1\mu_2}P^{\nu_1\nu_2}P^{\mu_3\nu_3}\\
&-\frac{1}{84}\sum_{\substack{P(\mu_1\mu_2\mu_3)\\P(\nu_1\nu_2\nu_3)}}\Bigg[\left(\gamma_{\mu_1}\gamma_{\nu_1} - \frac{\mathrm{i}}{W}(\gamma_{\mu_1}p_{\nu_1} - \gamma_{\nu_1}p_{\mu_1})\right.\\
&\left.+\frac{1}{W^2}p_{\mu_1}p_{\nu_1}\right)P^{\mu_2\nu_2}P^{\mu_3\nu_3}\\
&+\frac{2}{9}\left(\gamma_{\mu_1}\gamma_{\mu_2} - \frac{\mathrm{i}}{W}(\gamma_{\mu_1}p_{\mu_2} - \gamma_{\mu_2}p_{\mu_1})\right.\\
&\left.+\frac{1}{W^2}p_{\mu_1}p_{\mu_2}\right)P^{\nu_1\nu_2}P^{\mu_3\nu_3}\\
&+\frac{1}{5}\left(\gamma_{\nu_1}\gamma_{\mu_1} - \frac{\mathrm{i}}{W}(\gamma_{\nu_1}p_{\mu_1} - \gamma_{\mu_1}p_{\nu_1})\right.\\
&\left.+\frac{1}{W^2}p_{\mu_1}p_{\nu_1}\right)P^{\mu_2\mu_3}P^{\nu_2\nu_3}\Bigg]\Bigg\}, \quad (66a)
\end{aligned}$$

$$K^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\left(\frac{7}{2},p\right)$$
$$=\frac{1}{m^2}(\not{p}-\mathrm{i}W)\left[\frac{1}{6}\sum_{P(\nu_1\nu_2\nu_3)}B^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\right.$$
$$\left.-\frac{1}{280}\sum_{\substack{P(\mu_1\mu_2\mu_3)\\P(\nu_1\nu_2\nu_3)}}B^{\mu_1\mu_2\nu_1\nu_2\mu_3\nu_3}\right]$$
$$+\frac{1}{84}\sum_{\substack{P(\mu_1\mu_2\mu_3)\\P(\nu_1\nu_2\nu_3)}}\left[C^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\right.$$
$$\left.+\frac{2}{9}C^{\mu_1\mu_2\nu_1\nu_2\mu_3\nu_3}+\frac{1}{5}C^{\nu_1\mu_1\mu_2\mu_3\nu_2\nu_3}\right], \quad (66b)$$

with

$$C^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}=C^{\mu_1\nu_1\mu_2\nu_2}P^{\mu_3\nu_3}+\frac{\mathrm{i}(\mathrm{i}\not{\partial}-W)}{W^2}\times[\gamma_{\mu_1}\gamma_{\nu_1}\delta_{\mu_2\nu_2}\delta_{\mu_3 4}\delta_{\nu_3 4}], \quad (67a)$$

$$C^{\mu_1\nu_1\mu_2\nu_2}=C^{\mu_1\nu_1}P^{\mu_2\nu_2}+\frac{\mathrm{i}(\mathrm{i}\not{\partial}-W)}{W^2}\times[\gamma_{\mu_1}\gamma_{\nu_1}\delta_{\mu_2 4}\delta_{\nu_2 4}], \quad (67b)$$

$$C^{\mu_1\nu_1}=\mathrm{i}\left[\frac{1}{W}\gamma_4(\delta_{\mu_1 4}\gamma_{\nu_1}-\gamma_{\mu_1}\delta_{\nu_1 4})-\frac{(\mathrm{i}\not{\partial}-W)}{W^2}\delta_{\mu_1 4}\delta_{\nu_1 4}\right]. \quad (67c)$$

References

[1] Chung S U 1998 *Phys. Rev.* D **57** 431
[2] Chung S U 1993 *Phys. Rev.* D **48** 1225
[3] Chung S U *et al* 2002 *Phys. Rev.* D **65** 072001
[4] Filippini V *et al* 1995 *Phys. Rev.* D **51** 2247
[5] Huang S Z *et al* 2001 *Acta Phys. Sin.* **8** 1456 (in Chinese)
[6] Huang S Z *et al* 2002 *Chin. Phys.* **6** 561
[7] Huang S Z *et al* 2003 *Chin. Phys.* **1** 25
[8] Bargmann V and Wigner E P 1948 *Proc. Natl. Acad. Sci.* (USA) **34** 211
[9] Auvil P R and Brehm J J 1966 *Phys. Rev.* **145** 1152
[10] Huang S Z *et al* 2003 *Acta Phys. Sin.* **52** to be published (in Chinese)
[11] Behrends R E and Fronsdal C 1957 *Phys. Rev.* **106** 345
[12] Fronsdal C 1958 *Nuovo Cimento* Suppl. **9** 416
[13] Rarita W and Schwinger J 1941 *Phys. Rev.* **60** 61
[14] Zemach C 1965 *Phys. Rev.* B **140** 97
Zemach C 1965 *Phys. Rev.* B **140** 109
[15] Lurie D 1968 *Particles and Fields* (New York: Wiley) Chap. 4, p143