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The bifurcation threshold value of the chaos detection system for a weak signal*

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Recently, it has become an important problem to confirm the bifurcation threshold value of a chaos detection system for a weak signal in the fields of chaos detection. It is directly related to whether the results of chaos detection are correct or not. In this paper, the discrimination system for the dynamic behaviour of a chaos detection system for a weak signal is established by using the theory of linear differential equation with periodic coefficients and computing the Lyapunov exponents of the chaos detection system; and then, the movement state of the chaos detection system is defined. The simulation experiments show that this method can exactly confirm the bifurcation threshold value of the chaos detection system.

Keywords: chaos detection, bifurcation threshold value, weak signal, phase plane, Lyapunov exponent
PACC: 0545

1. Introduction

It is a frontier problem in the fields of chaotic theory application to apply chaos theory in nonlinear science to the weak signal detection. In recent years, many domestic or foreign scholars have made many significant discussions on this problem. For example, using the method of minimal phase space cubage to estimate the multinomial parameters in an embedded chaotic system for detecting chaos signals;[1,2] as for signals under the background of chaotic noises, using the method of chaotic temporal serial prediction to establish the prediction model to separate the chaotic signal from the signal to be detected;[3,4] putting forward the method that is based on the intrinsic geometrical characteristics of chaotic attractors to separate the chaotic interference from the weak signal by the conception of differential flow cut phase space.[5] Lately, using the chaotic control theory and chaotic synchronization theory,[6−10] the method[11−14] based on the changes of the dynamic behaviours of a chaotic system (chaotic state, periodic state) has been proposed for recognizing whether there exists a signal to be detected in a system. This method has a very low threshold of SNR (signal-to-noise ratio), and it is greatly immune to the colour noise of arbitrary zero average value with unknown probability distribution; yet it is easy to bring about such questions as: making a wrong estimate on the phase orbit of a chaotic system, a low efficiency of simulation, bigger errors, and so on. This paper is the continuation of Ref.[14]: for concreteness, it consists of: 1) advancing questions after discussing the characteristics of chaos detection for weak signals,which has been shown in Ref.[14]; 2) computing the Lyapunov exponents of the chaos detection system (shown in Ref.[14]); 3) working out the bifurcation threshold values of the two main phase state regions(chaotic state, large-scale periodic state) in the course of chaos detection, and estimating the dynamic behaviours of the chaos detection system; 4) making simulation experiments and analysing the results.

2. Advancing questions

An outstanding characteristic of a chaotic system is that it is very sensitive to the initial conditions. With this characteristic, when the signal to be detected as an initial condition of the system is input to the chaos detection system which is in the chaotic crit-

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ical state (phase change from the chaotic state to the large-scale periodic state), the movement state of the system will transit immediately from the chaotic state to the large-scale periodic state. Since the change is brought about by the small signal, the weak signal which is submerged in the noise will be checked out by this change of behaviours of nonlinear dynamics. The following is the chaos detection system for a sine signal in Ref.[14]:

$$\ddot{x} = \varepsilon \cdot k\dot{x} - x^3 + x^5 = \varepsilon \cdot A\cos(\omega t) + \varepsilon(\text{input}). \quad (1)$$

By adjusting the amplitude A of the internal signal of the system to make the phase state of it become chaotic critical state, the temporal waveforms of the system's output and phase plane orbits are shown in Fig.1(a).

First, a pure noise is added into the system, the noise being strong; but there is no influence on the state of the system. The results are only to increase the disorder degree of the system and to make the dynamic behaviours of the system more chaotic. The temporal waveforms and phase plane orbits of the system output are shown in Fig.1(b). When adding the noise mixed with a weak periodic signal to the system, the dynamic behaviours of the system immediately turn to those of the large-scale periodic states, and the temporal waveforms and the phase orbits of the system output are shown in Fig.1(c).

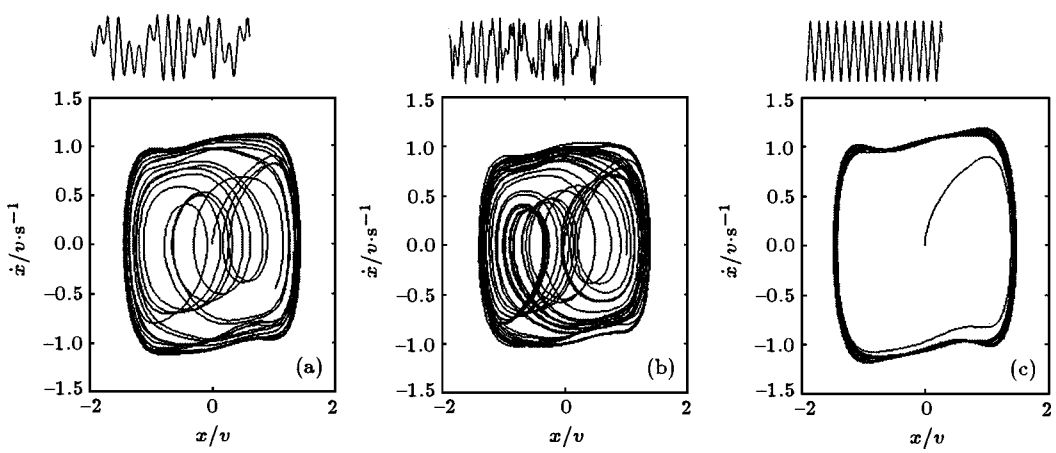


Fig.1. The temporal waveforms and the phase plane orbits changing the dynamic behaviours of the chaos detection system for a sine signal. (a) chaotic critical phase state; (b) chaotic phase state; (c) large-scale periodic state.

The change of the dynamic behaviours of the system can be seen in Fig.1: a pure noise has no influence on the state of the system, but once there is a signal of a specific period in the input signals, even if the amplitude is small, it will lead to the phase change of the system. We can estimate whether there is a periodic signal in the system or not, based on whether the phase orbit is in chaotic states or steady periodic states. It is a problem difficult to solve when we have to make certain the bifurcation threshold value of the chaotic critical state. For example, two problems have been found in simulation experiments: firstly, if the output of the system is chaotic, yet the time of the simulation is not long enough, it is easy to make a wrong estimation, namely to mistake the apparent periodic movement state at the beginning of the simulation for the periodic states. Actually, the chaos will come out with increasing the time of simulation. Therefore, if the state, which is really chaotic, is estimated to be steady periodic, the results of the detection will be wrong. Secondly, if the output of the system is initially steady periodic movement, yet it cannot be known in advance, then the only way to prove the output is a steady state is by intuitive observation and the simulation of very long time; thus the efficiency of this method is very low. It is obvious that it is not enough only to observe intuitively the changes of the phase orbit, it is needed to make a theoretical analysis on the change of the system dynamic behaviours, as well as to work out the bifurcation threshold value of the chaos detection system and to study the formation of the phase change of the system.

3. The bifurcation threshold value of a chaos detection system

3.1. Algorithm for computing Lyapunov exponents

To confirm the existence of the weak signal to be detected and the amplitude of the signal, we need to define a proper index for denoting the change in states of the chaos detection system. The index should be sensitive to a weak signal, but insensitive to the noise. From the viewpoint of statistical characteristics, Lyapunov exponents reflect the dynamic characteristics of a system, and play an important role in the criterion of chaotic states. This method is more accurate than other methods, and it gives a quantitative criterion for the analysis. For a two-dimensional (2D) mapping, we have

$$\begin{cases} x_{n+1} = f_1(x_n, y_n), \\ y_{n+1} = f_2(x_n, y_n), \end{cases} \tag{2}$$

then the system state can be denoted by Lyapunov exponents. The Jacobi matrix of Eq.(2) is

$$\boldsymbol{J} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_n} & \dfrac{\partial f_1}{\partial y_n} \\ \dfrac{\partial f_2}{\partial x_n} & \dfrac{\partial f_2}{\partial y_n} \end{bmatrix}.$$

The two Lyapunov exponents of the mapping are, respectively,

$$l_1 = \frac{1}{n}\ln|\lambda_1|, \quad l_2 = \frac{1}{n}\ln|\lambda_2|. \tag{3}$$

Equation (3) is related to the eigenvalue problem of the matrix $\boldsymbol{J}$. Thus, we can apply the theory on the differential equation with periodical coefficients to the discussion of the Lyapunov exponents to the chaos detection system when the system is in the state of large-scale period.

We use $x + \delta x$ and x to denote the two points with different initial locations, where δx is the deviation between the two points. The following equation of deviation will be obtained when $x+\delta x$ is substituted in Eq.(1):

$$\delta\ddot{x} + k\delta\dot{x} - c(t)\delta x = 0, \tag{4}$$

where $c(t) = 3x^2 - 5x^4$. Transform Eq.(4) into the form of equation of state. Let $\delta\dot{x} = \delta y$, then

$$\begin{cases} \delta\dot{x} = \delta y, \\ \delta\dot{y} = -k\delta y + c(t)\delta x. \end{cases} \tag{5}$$

Transform Eq.(5) into the form of vector differential equation:

$$\dot{\boldsymbol{X}}(t) = \boldsymbol{H}(t)\boldsymbol{X}(t), \tag{6}$$

where

$$\boldsymbol{X}(t) = \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \begin{bmatrix} \delta x \\ \delta\dot{x} \end{bmatrix},$$

$$\boldsymbol{H}(t) = \begin{bmatrix} 0 & 1 \\ c(t) & -k \end{bmatrix}.$$

$x(t)$ in Eq.(4) is a periodic function (its period is the same as that of $\cos(t)$, i.e., 2π), so $\boldsymbol{H}(t)$ is a continuous periodic real matrix with rank 2×2, and its minimum period T is 2π. Therefore, the following equation holds

$$\boldsymbol{H}(t+T) = \boldsymbol{H}(t), \ \ (-\infty < t < +\infty). \tag{7}$$

Since $\boldsymbol{H}(t)$ is a periodic matrix, Eq.(6) is a linear differential equation of periodic parameters. For computing the Lyapunov exponents of the system, Eq.(6) should be changed into a nonautonomous system.

Based on the theory of linear differential equation with periodic coefficients,[15] if $\boldsymbol{D}(t)$ is a standard fundamental solution matrix of Eq.(6), then $\boldsymbol{D}(t+T)$ should also be a basic solution matrix.

Thus, there exists a constant matrix $\boldsymbol{C}$, so that $\boldsymbol{D}(t+T) = \boldsymbol{D}(t)\boldsymbol{C}$ holds.

When $t = 0$, $\boldsymbol{D}(0) = \boldsymbol{I}$. And we have

$$\boldsymbol{D}(t+T) = \boldsymbol{D}(t)\boldsymbol{D}(T).$$

Since $\boldsymbol{D}(T)$ is nonsingular, there exists a matrix denoted by $\ln \boldsymbol{D}(T)$ such that $\mathrm{e}^{\ln D(T)} = \boldsymbol{D}(T)$ holds. The standard fundamental solution matrix of Eq.(6) can then be expressed as

$$\boldsymbol{D}(t) = \boldsymbol{F}(t)\mathrm{e}^{T^{-1}t\ln D(T)}, \tag{8}$$

where $F(t)$ is a nonsingular periodic matrix

$$\boldsymbol{F}(t) = \boldsymbol{D}(t)\mathrm{e}^{-T^{-1}t\ln D(T)}. \tag{9}$$

Making linear transform on Eq.(6),we have

$$\boldsymbol{X} = \boldsymbol{F}(t)\boldsymbol{Z}, \tag{10}$$

then the periodic system expressed by Eqs.(6) and (7) becomes a noninteracting system

$$\dot{\boldsymbol{Z}} = T^{-1}\ln \boldsymbol{D}(T)\boldsymbol{Z}. \tag{11}$$

If the two eigenvalues of the matrix $\boldsymbol{D}(T)$ are λ_1 and λ_2, then the two Lyapunov exponents of Eq.(6) can be written as

$$l_1 = \frac{1}{T}\ln|\lambda_1|, \text{and } l_2 = \frac{1}{T}\ln|\lambda_2|. \qquad (12)$$

From the above analysis, it follows that the Lyapunov exponents can be obtained by finding the two eigenvalues of the matrix $\boldsymbol{D}(T)$.

3.2. Bifurcation threshold of chaos detection system

The instantaneous state of dynamic movement of the chaos detection system can be estimated by analysing the Lyapunov exponents of the system sequentially, the existence of the signal to be detected can be confirmed; and the relationship between the change of the system state and the amplitude parameters of the signal to be detected can be found more precisely. A system for distinguishing dynamic behaviours of the chaos detection system is established on the basis of Lyapunov exponents (the model of simulation is shown in Fig.2), where scope 1 and scope 2 show the temporal waveform of the system output, X-Y Graph shows the relationship between x and $\dot{x}$ of the system output from the work space, namely the phase planar orbit of the chaos detection system. After the outputs of the system are transmitted into the work space, the Lyapunov exponents will be computed by putting the output from the work space into a function arithmetic unit. Then the results should be input to the comparison unit for estimating the state of the system. Finally, the estimated results of the state of the chaos detection system will be displayed.

Lyapunov exponents are applied to estimate the state of the chaos detection system. When the chaos detection system is in the chaotic critical state, one of the two Lyapunov exponents of the system should be zero.

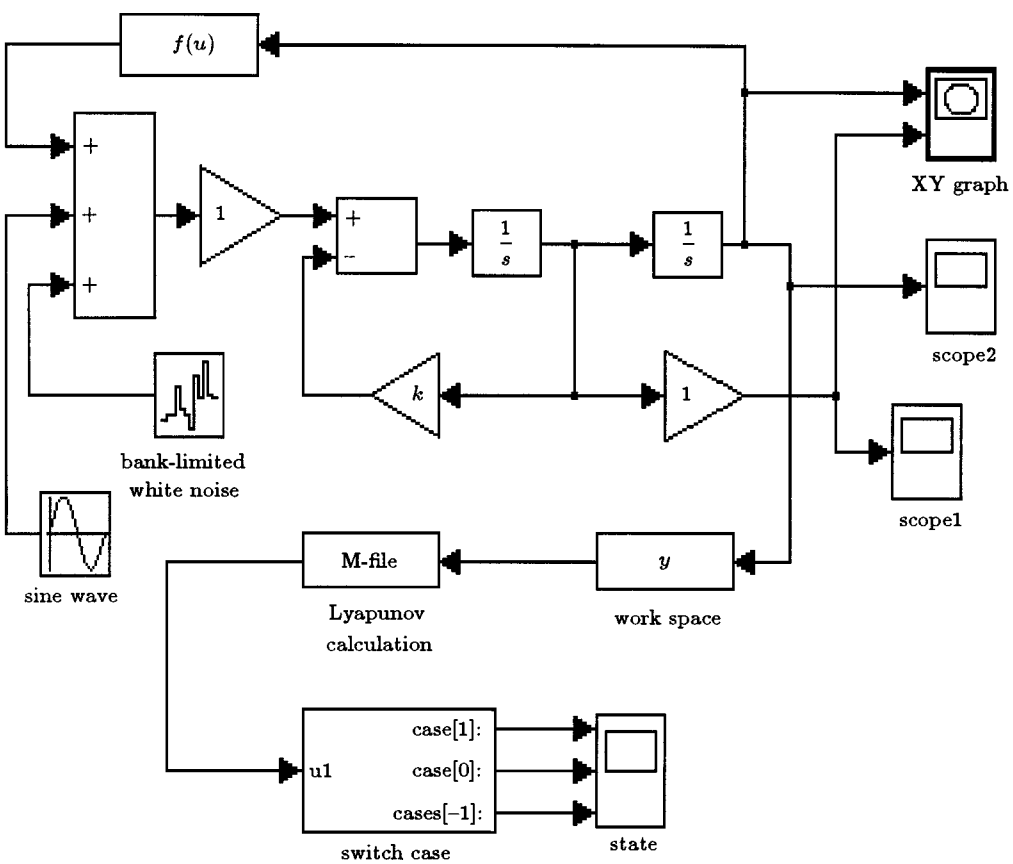


Fig.2. System model for estimating dynamics behaviours of chaos detection system.

When the system is in the large-scale periodic state, both of the two Lyapunov exponents are negative. When the system is in the chaotic state, at least one of the two Lyapunov exponents of the system is positive.

The way for determining the bifurcation threshold value when the state of the system changes from chaos to the large-scale period state is described as follows:

if the signal frequency is given, by adjusting the amplitude of the system's internal signal to the point at which one of the two Lyapunov exponents of the chaos detection system is zero, then we can unambiguously decide that the system is in a chaotic critical state. Under the condition of the chaotic critical state, the amplitude of the internal signal gives the bifurcation threshold value.

4. Simulation experiment

Rewrite Eq.(6) in the form of the state equation

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -0.5y + x^3 - x^5 + A\cos(t) + (\text{input}). \end{cases} \tag{13}$$

Since $\boldsymbol{D}(t)$ is a standard fundamental solution matrix of Eq.(6), it should satisfy the following equation

$$\frac{\mathrm{d}\boldsymbol{D}(t)}{\mathrm{d}t} = \boldsymbol{H}(t)\boldsymbol{D}(t). \tag{14}$$

As $\boldsymbol{H}(t)$ changes with the time, we cannot find any analytic formulas for $\boldsymbol{D}(t)$. But the values of $\boldsymbol{D}(t)$ can be obtained by using the numerical method. The values of $x(t)$ in a period can be obtained by solving the above equation by the fourth-order Runge-kutta method. Then $x(t)$ is sampled with a very small value Δt as the step in order to gain the value $x(t + n \cdot \Delta t)$ at each sample point $(t+n\cdot\Delta t)$, by which the value of $\boldsymbol{H}(t+n\cdot\Delta t)$ at the same sample time can be confirmed. Equation (14) can be solved by iteration. Since the sampling interval Δt is small, $\boldsymbol{H}(t)$ can be considered constant within Δt, i.e.,

$$\boldsymbol{H}(n\Delta t + \Delta t) = \boldsymbol{H}(n\Delta t).$$

We now transform Eq.(14) into a different form. Since $(\ln \boldsymbol{D}(t))' = \dfrac{1}{\boldsymbol{D}(t)}$, namely $\dfrac{\mathrm{d}\ln \boldsymbol{D}(t)}{\mathrm{d}\boldsymbol{D}(t)} = \dfrac{1}{\boldsymbol{D}(t)}$, thus $\mathrm{d}\ln \boldsymbol{D}(t) = \dfrac{\mathrm{d}\boldsymbol{D}(t)}{\boldsymbol{D}(t)} = \boldsymbol{H}(t)\mathrm{d}t$. Integration of the above equation on both sides gives

$$\ln \boldsymbol{D}(T) = \sum_{n=0}^{T/\Delta t} \boldsymbol{H}(n\Delta t)\cdot\Delta t,$$

$$\boldsymbol{D}(T) = \mathrm{e}^{\sum\limits_{n=0}^{T/\Delta t} H(n\Delta t)\cdot\Delta t} = \prod_{n=0}^{T/\Delta t} \mathrm{e}^{H(n\Delta t)\cdot\Delta t}. \tag{15}$$

Eq. (15) is suitable for computer evaluation. First, the fourth-order Runge–Kutta is applied for working out Eq.(14) in a period T with Δt as the step, then a series of sample values $x(n\cdot\Delta t)(n = 0, 1, 2, \cdots, T/\Delta t)$ of $x(t)$ in a period T with Δt as the step should be obtained. After $x(n\cdot\Delta t)$ is inserted into $\boldsymbol{H}(t)$, discrete $\boldsymbol{H}(n\cdot\Delta t)$ will be obtained. Expression (15) is evaluated iteratively and the number of iterations is equal to $T/\Delta t$. As for the initial $\boldsymbol{H}(0)$, $\boldsymbol{H}(0\cdot\Delta t) = \boldsymbol{I}$ can be known from the demonstration above. After $\boldsymbol{D}(T)$ is acquired, its two eigenvalues should be worked out, then put them into Eq.(12) to obtain the two Lyapunov exponents of the system. In the simulation experiment, we have sampled 100 points of the signal amplitude (denoted by A) between 0 and 2 to compute Lyapunov exponents of the system (denoted by l_1 and l_2) for each sample point. Forty-eight typical values of the results are shown in Table 1 and the curves of the Lyapunov exponents of the chaos detection system are shown in Fig.3.

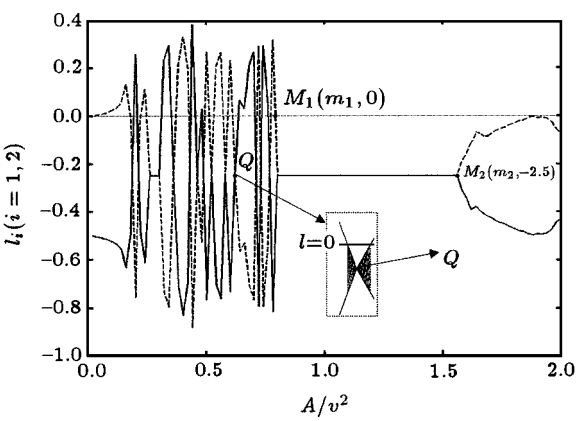


Fig.3. The distribution graph of Lyapunov exponents.

5. Discussion

(1) The curves of instantaneous Lyapunov exponents $l_i(i = 1, 2)$ shown in Fig.3 can be approximately divided into two parts by the point $M_1(m_1, 0)$. When $A < m_1$, the system is in the state of unsteady chaos with alternating l_i of positive and minus values. $A = m_1 = 0.78057$ is the bifurcation threshold value at which the system changes from a chaos state to a steady large-scale period state. When $A > m_1$ and $l_i < 0$, the system is in a state of steady large-scale period.

(2) The curves of instantaneous Lyapunov exponents l_i show the course of clear change of the phase state of the system.In the area of $A < m_1$, the speciality of the two Lyapunov exponents values can be noticed. Although there is $l_i < 0$, the area of the phase state corresponding to a negative value is strictly limited, namely, in the limited area near the place where the two curves intersect, we have $l_i < 0$ (see the

shadow area near the point Q in Fig.3) which actually increases the degree of complexity even more. In the area of $A > m_1$, the curves are divided by point $M_2(m_2, -0.25)$, which is a characteristic point in the large-scale periodic state and the reason for its representation needs to be studied further.

(3) The curves of instantaneous Lyapunov exponents shown in Fig.3 are symmetrical, the equation of the symmetric axis is $2\sum_{i=1}^{2} l_i(A) = -1$. The location of the symmetric axis is determined by the parameter K=0.5, and the existence of symmetry is the common result of the periodic matrix $\boldsymbol{H}(t)$ and Eq.(12). The formation of the symmetry of the curves may come from the intrinsic nature of this chaos detection system.

(4) Analysis of errors: in the simulation experimentation, some data of the experiment have errors. The main reasons for the errors are:

(a) The high-order infinitesimal of δx is ignored when we prove Eq.(4) of deviation.

(b) The period of computation, which is regarded as an aptotic value when the computation is conducted, can be influenced by the different values of input amplitudes, which can lead some points to depart from the theoretical region.

(c) Single precision is routinely used in our computation, the finite precision may introduce some error to the results.

Table 1. The Lyapunov exponents of the chaos detection system.

A	0.0200	0.0600	0.1000	0.1400	0.1800	0.2200	0.2600	0.3000
l_1	0.0010	0.0084	0.0229	0.0550	–0.0093	–0.0197	–0.2500	–0.2500
l_2	–0.5010	–0.5084	–0.5229	–0.5550	–0.4907	–0.4803	–0.2500	–0.2500

A	0.3400	0.3800	0.4200	0.4600	0.5000	0.5400	0.5800	0.6200
l_1	–0.7942	0.2071	0.1939	–0.2500	0.2651	0.2035	–0.2500	–0.2500
l_2	0.2942	–0.7071	–0.6939	–0.2500	–0.7651	–0.7035	–0.2500	–0.2500

A	0.6600	0.7000	0.7400	0.7800	0.7806	0.8000	0.8400	0.8800
l_1	–0.5308	–0.7657	–0.7921	0.3165	0.0000	–0.2500	–0.2500	–0.2500
l_2	0.0308	0.2657	0.2921	–0.8165	–0.5000	–0.2500	–0.2500	–0.2500

A	0.9210	0.9600	1.000	1.0400	1.0800	1.1200	1.1600	1.2000
l_1	–0.2500	–0.2500	–0.2500	–0.2500	–0.2500	–0.2500	–0.2500	–0.2500
l_2	–0.2500	–0.2500	–0.2500	–0.2500	–0.2500	–0.2500	–0.2500	–0.2500

A	1.2400	1.2800	1.3200	1.3600	1.4000	1.4400	1.4800	1.5200
l_1	–0.2500	–0.2500	–0.2500	–0.2500	–0.2500	–0.2500	–0.2500	–0.2500
l_2	–0.2500	–0.2500	–0.2500	–0.2500	–0.2500	–0.2500	–0.2500	–0.2500

A	1.5600	1.6000	1.6400	1.6800	1.7200	1.7600	1.8000	1.8400
l_1	–0.2500	–0.1562	–0.1205	–0.0863	–0.0644	–0.0493	–0.0313	–0.0184
l_2	–0.2500	–0.3438	–0.3795	–0.4137	–0.4356	–0.4507	–0.4687	–0.4816

6. Conclusion

In this paper, a new method for determining the value of the bifurcation threshold is presented, based on Lyapunov exponents. The algorithm of Lyapunov exponents of chaos detection system and the course of quantitative analysis on the chaos detection system are discussed in detail. The curves of Lyapunov exponents show clearly the course of evolvement of the phase state of the chaos detection system and provide the bifurcation value at which the system changes from a chaotic state to a large-scale periodic state. Moreover, the curves are found to be symmetric about an axis parallel to the direction of the abscissa axis. The results of the simulation experiments are consistent basically with the results of theoretic analysis. We leave the problem of "band-width" (the distance between the innermost and outermost trajectories of the chaos system) of the bifurcation orbit and the stability of the system to a future work.

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