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Lie symmetries and conserved quantities of controllable nonholonomic dynamical systems

Fu Jing-Li(傅景礼)[a)b)], Chen Li-Qun(陈立群)[b)],
Bai Jing-Hua(白景华)[c)], and Yang Xiao-Dong(杨晓东)[b)]

[a)] *Institute of Mathematical Mechanics and Mathematical Physics, Shangqiu Teachers College, Shangqiu 476000, China*
[b)] *Shanghai Institute of Applied Mathematics and Mechanics, Shanghai University, Shanghai 200072, China*
[c)] *Department of Mathematics, Kaifeng University, Kaifeng 475000, China*

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This paper concentrates on studying the Lie symmetries and conserved quantities of controllable nonholonomic dynamical systems. Based on the infinitesimal transformation, we establish the Lie symmetric determining equations and restrictive equations and give three definitions of Lie symmetries before the structure equations and conserved quantities of the Lie symmetries are obtained. Then we make a study of the inverse problems. Finally, an example is presented for illustrating the results.

Keywords: control, nonholonomic dynamical systems, Lie symmetry, conserved law
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1. Introduction

The symmetric principle is one of the key issues in mathematics and physics. Research on the conserved quantities of mechanical and physical systems not only has important mathematical significance, but also show profound physical essence. Two of the modern methods of studying on the conserved quantities of mechanical systems and physical systems are the Noether method[1] and the Lie method. Li[2] studied Noether symmetries of the classical and the quantum constrained systems. Mei[3] studied Noether symmetries of Birkhoffian systems. Noether symmetries of nonholonomic systems have been studied by many authors.[4−9] The approach of Lie symmetries was set up in the 19th century, but its applications in mechanics had not started until 1979.[10] In recent years, many results have been obtained in the studies of the Lie method.[11−22] Lately, Mei[23] studied Noether symmetries as well as Lie symmetries of a variety of constrained mechanical systems, and obtained relationships between the Noether symmetries and Lie symmetries, and also between the symmetries and the conserved quantities. Along with the development of sciences and technologies, the control system theory becomes more and more important.[24,25] This paper studies the Lie symmetries of the controllable nonholonomic systems, including both direct and inverse problems.

2. Equations of motion of the controllable nonholonomic dynamical systems

Consider a system composed of N particles, and the mass of a particle is $m_i (i = 1, \cdots, N)$. The configuration of the system can be expressed by n generalized coordinates $q_s (s = 1, \cdots, n)$. We assume that this system is subjected to m holonomic constraints of the form

$$f_\rho(q_s, u_r, t) = 0$$

$$(\rho = 1, \cdots, m; i = 1, \cdots, N; r = 1, \cdots, p), \quad (1)$$

and g nonholonomic constraints of the form

$$\psi_\beta(q_s, \dot{q}_s, u_r, \dot{u}_r, t) = 0 \quad (\beta = 1, \cdots, g), \quad (2)$$

where $u_r (r = 1, \cdots, p)$ are controlled parameters. The motion of the mechanical system is controlled by using the controlling parameters u_r, which are important for a mechanical system. In the case of ideal constraints, the d'Alembert–Lagrangian principle of the constrained system can be expressed in the following

form in generalized coordinates

$$\left(Q_s - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_s} + \frac{\partial T}{\partial q_s}\right)\delta q_s = 0 \quad (s = 1, \cdots, n). \tag{3}$$

For nonholonomic systems, δq_s are not independent of each other, and they have to be constrained by the Chetaev conditions in Eq.(3), which are expressed as follows:

$$\frac{\partial \psi_\beta}{\partial \dot{q}_s}\delta q_s = 0 \quad (s = 1, \cdots, n; \beta = 1, \cdots, g). \tag{4}$$

From Eqs.(3) and (4), we can obtain the Routh equations of the controllable nonholonomic system with the usual Lagrange multiplier method as follows:[26,27]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} = Q_s + \lambda_\beta \frac{\partial \psi_\beta}{\partial \dot{q}_s}. \tag{5}$$

Equations (5) are different from the usual equations of non-controllable nonholonomic systems, because there are control parameters and their derivatives with respect to time in the opposing force of constraint. To control the motions of the systems by controlling the opposing forces of the constraint in the controlled theory, we can control $u_r(t)$ for the purpose of practical realization. We first obtain the constrained multiplier λ_β by using Eqs.(2), (4) and (5), and then substitute it into Eq.(5) to obtain an explicit form[27]

$$\begin{aligned}&\ddot{q}_l + A_{sl}^{-1}(k, m; s)\dot{q}_k\dot{q}_m \\ =&A_{sl}^{-1}\left\{\left(\frac{\partial B_k}{\partial q_s} - \frac{\partial B_s}{\partial q_k}\right)\dot{q}_k + Q_s \right. \\ &\left. - \frac{\partial B_s}{\partial t} + \frac{\partial T_0}{\partial q_s} - \frac{\partial A_{ks}}{\partial t}\dot{q}_k + \lambda_\beta \frac{\partial \psi_\beta}{\partial \dot{q}_l}\right\},\end{aligned} \tag{6}$$

where

$$[k, m, s] = \frac{1}{2}\left(\frac{\partial A_{ks}}{\partial q_m} + \frac{A_{ms}}{\partial q_k} - \frac{\partial A_{km}}{\partial q_s}\right). \tag{7}$$

Suppose that the controllable parameters u_r are dependent only on time, i.e., $u_r = u_r(t)(r = 1, \cdots, p)$, then differentiating Eq.(2) with respect to t yields

$$\frac{\partial \psi_\gamma}{\partial q_l}\dot{q}_l + \frac{\partial \psi_\gamma}{\partial \dot{q}_l}\ddot{q}_l + \frac{\partial \psi_\gamma}{\partial u_r}\dot{u}_r + \frac{\partial \psi_\gamma}{\partial \dot{u}_r}\ddot{u}_r + \frac{\partial \psi_\gamma}{\partial t} = 0. \tag{8}$$

We obtain the constrained multiplier λ_β by substituting Eq.(6) into Eq.(8). Substituting λ_β into Eq.(5) leads to

$$\ddot{q}_s = \alpha_s(\boldsymbol{q}, \dot{\boldsymbol{q}}, u_r, \dot{u}_r, \ddot{u}_r, t). \tag{9}$$

3. Direct problems of Lie symmetry in controllable nonholonomic dynamical systems

3.1. Infinitesimal transformations and determining equations of Lie symmetries in controllable nonholonomic dynamical systems

We first introduce infinitesimal transformations in terms of time and generalized coordinates

$$\begin{cases} t^\bullet = t + \varepsilon\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \\ q_s^\bullet(t) = q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \end{cases} \tag{10}$$

where ε is a small parameter, and ξ_0 and ξ_s are infinitesimal generators. We again introduce the vector of generators $\boldsymbol{X}^{(0)}$, $\boldsymbol{X}^{(1)}$, $\boldsymbol{X}^{(2)}$.[23] Then based on the invariance of the differential Eqs. (9) under the infinitesimal transformation (10), if and only if

$$\boldsymbol{X}^{(2)}[\ddot{q}_s - \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, u_r, \dot{u}_r, \ddot{u}_r)]|_{\ddot{q}_s = \alpha_s} = 0, \tag{11}$$

we can have

$$\ddot{\xi}_s - \dot{q}_s\ddot{\xi}_0 - 2\dot{\xi}_0\alpha_s = X^{(1)}(\alpha_s) \quad (s = 1, \cdots, n). \tag{12}$$

Eqs. (12) are called the determining equations of Lie symmetries when the generators ξ_0 and $\xi_s(s = 1, \cdots, n)$ satisfy the infinitesimal transformation (10).

3.2. Three definitions of Lie symmetries for controllable nonholonomic dynamical systems

Definition 1 Transformations (10) are called Lie symmetric transformations corresponding to the controllable holonomic systems when the infinitesimal generators ξ_0 and ξ_s satisfy the determining Eqs. (12).

For the nonholonomic systems, the constrained Eqs. (2) also retain the form of invariance after the infinitesimal transformation (10), i.e.

$$X^{(1)}(\psi_\beta(\boldsymbol{q}, \dot{\boldsymbol{q}}, u_r, \dot{u}_r, t)) = 0$$

$$(\beta = 1, \cdots, g; r = 1, \cdot, p). \tag{13}$$

Eqs. (13) are called restrictive equations of controllable nonholonomic constraints in Lie symmetries.

Definition 2 Transformations (10) are called weak Lie symmetric transformations corresponding to the controllable nonholonomic systems when the infinitesimal generators ξ_0 and ξ_s satisfy both the determining Eqs. (12) and the restrictive Eqs. (13).

To obtain Eqs.(5), we exploit the Chetaev condition (7) which applies restrictions to δq_s,

$$\delta q_s = \Delta q_s - \dot{q}_s\Delta t = \varepsilon(\xi_s - \dot{q}_s\xi_0), \tag{14}$$

and substitute Eq.(14) into Eq.(4), then

$$\frac{\partial \psi_\beta}{\partial \dot{q}_s}(\xi_s - \dot{q}_s \xi_0) = 0 \quad (\beta = 1, \cdots, g; s = 1, \cdots, n). \tag{15}$$

Eqs. (15) are called supplementary restrictive equations.

Definition 3 Transformations (10) are called strong Lie symmetric transformations corresponding to the controllable nonholonomic systems when the infinitesimal generators ξ_0 and ξ_s satisfy the determining Eqs. (12), the restrictive Eqs. (13) and the supplementary restrictive Eqs. (15).

3.3. Lie symmetrical structural equations and conserved quantities in the controllable nonholonomic dynamical systems

Theorem 1 If the infinitesimal transformation generators ξ_0 and ξ_s satisfy the determining Eqs. (12), and if the function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, u_r, \dot{u}_r)$ satisfies the following equation

$$L\dot{\xi}_0 + X^{(1)}(L) + \frac{\partial L}{\partial u_r}\dot{u}_r \xi_0 + \frac{\partial L}{\partial \dot{u}_r}\ddot{u}_r \xi_0 + \dot{G} = 0, \tag{16}$$

there will be a conserved quantity corresponding to the Lie symmetries for the controllable holonomic system, i.e.

$$I = L\xi_0 + \frac{\partial L}{\partial \dot{q}_s}(\xi_s - \dot{q}_s \xi_0) + G = \text{constant.}, \tag{17}$$

where $L = T - V$ is the Lagrangian of the system.

Proof:

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t} =& \dot{L}\xi_0 + L\dot{\xi}_0 + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) \\ &+ \frac{\partial L}{\partial \dot{q}_s}\left(\dot{\xi}_s - \ddot{q}_s\xi_0 - \dot{q}_s\dot{\xi}_0\right) \\ &- L\dot{\xi}_0 - \frac{\partial L}{\partial t}\xi_0 - \frac{\partial L}{\partial q_s}\xi_s - \frac{\partial L}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0) \\ &- \frac{\partial L}{\partial u_r}\dot{u}_r\xi_0 - \frac{\partial L}{\partial \dot{u}_r}\ddot{u}_r\xi_0 - Q''_s(\xi_s - \dot{q}_s\xi_0) \\ =& (\xi_s - \dot{q}_s\xi_0)\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} - Q''_s\right) = 0.\end{aligned}$$

Eq. (16) is called the structural equation for the controllable holonomic system.

Theorem 2 If the infinitesimal transformation generators ξ_0 and ξ_s satisfy both the determining Eqs. (12) and restrictive Eqs. (13), and if there exists a function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, u_r, \dot{u}_r)$ satisfying the structural Eq. (16), there will be a conserved quantity (17) corresponding to the weak Lie symmetries for the controllable nonholonomic system.

Theorem 3 If the infinitesimal transformation generators ξ_0 and ξ_s satisfy the determining Eqs. (12), restrictive Eqs. (13) and supplementary restrictive Eqs. (15), and if there is a function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, u_r, \dot{u}_r)$ satisfying the structural Eq. (16), then there will be a conserved quantity (17) corresponding to the strong Lie symmetries for the controllable nonholonomic system.

4. The inverse problem of the Lie symmetries for controllable nonholonomic dynamical systems

Finding Lie symmetries from the given conserved quantities is called the inverse problem, which can be solved by the following steps.

Firstly, we find out the corresponding Lie symmetries from the known conserved quantities. Suppose that there exist conserved quantities in the controllable nonholonomic system

$$I = I(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, u_r, \dot{u}_r) = \text{constant.}, \tag{18}$$

then we have

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\mathrm{d}t} + \frac{\partial I}{\partial q_s}\dot{q}_s + \frac{\partial I}{\partial \dot{q}_s}\ddot{q}_s + \frac{\partial I}{\partial u_r}\dot{u}_r + \frac{\partial I}{\partial \dot{u}_r}\ddot{u}_r = 0. \tag{19}$$

Multiply Eqs.(5) by

$$\overline{\xi}_s = \xi_s - \dot{q}_s\xi_0, \tag{20}$$

then summing up for s leads to

$$\begin{aligned}&\overline{\xi}_s\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} - Q''_s\right) = 0, \\ &\left(Q''_s = Q_s + \lambda_\beta \frac{\partial \psi_\beta}{\partial \dot{q}_s}\right).\end{aligned} \tag{21}$$

Adding the two sides of Eqs.(20) and (21), expanding the resulting equation, and letting the coefficients of the terms including $\ddot{q}_k$ to be zero, we have

$$\overline{\xi}_s = \overline{h}_{sk}\frac{\partial I}{\partial \dot{q}_k}, \tag{22}$$

where

$$\overline{h}_{sk}h_{kr} = \delta_{sr}, h_{sk} = \frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k}. \tag{23}$$

Let the conserved quantity in Eq.(17) equal to Eq.(18), i.e.,

$$L\xi_0 + \frac{\partial L}{\partial \dot{q}_s}\overline{\xi}_s + G = I, \tag{24}$$

and then we can obtain generators ξ_0 and ξ_s from Eqs.(22) and (24). ξ_0 and ξ_s are Noether symmetries of Eqs.(5).

Secondly, substituting generators ξ_0 and ξ_s into Eqs.(12), (17) and (15), we can determine the Lie symmetries.

Theorem 4 If the infinitesimal transformation generators ξ_0 and ξ_s determined by Eqs.(22) and (24) satisfy the determining Eqs. (12), the symmetry is called Lie symmetries for the controllable holonomic system; if generators ξ_0 and ξ_s also satisfy restrictive Eqs. (13), then the symmetry is called weak Lie symmetries for the controllable nonholonomic system. Furthermore, if generators ξ_0 and ξ_s satisfy the supplementary restrictive Eqs. (15), then the symmetry is called strong Lie symmetries for the controllable nonholonomic system.

5. Example

Supposing that the configuration of the system can be specified with the generalized coordinates q_1 and q_2, its Lagrange function is

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - kq_2, \tag{25}$$

and the constrained equation is

$$f = \dot{q}_2 - u(t)\dot{q}_1 = 0, \tag{26}$$

where $u(t)$ is the controllable parameter. The system is a holonomic one when the controllable parameter $u(t)$ does not include time t, and $u(t)$ is neither a linear function with respect to time t. If the momentum along the generalized coordinate q_2 of the system is conservational when the controllable parameter $u(t) \to 0$, i.e., it is a conservative holonomic system, which is very important in physics. The system expressed by Eqs.(25) and (26) is a nonholonomic system in the ordinary case.

To study the Lie symmetry of the system, we first study the direct problem of the Lie symmetries. The equations of motion of the system are as follows

$$\ddot{q}_1 = -\lambda u, \quad \ddot{q}_2 = \lambda - k. \tag{27}$$

From Eqs.(26) and (27), we obtain

$$\lambda = (k + u\dot{u}\dot{q}_1)/(1 + u^2). \tag{28}$$

Substituting it into Eqs.(27) yields

$$\ddot{q}_1 = -\frac{ku + u\dot{u}\dot{q}_1}{1 + u^2}, \quad \ddot{q}_2 = \frac{u\dot{u}\dot{q}_1 - ku^2}{1 + u^2}. \tag{29}$$

Then substituting Eqs.(29) into the determining Eqs. (16), we obtain

$$\begin{cases} \ddot{\xi}_1 - \dot{q}_1\ddot{\xi}_0 - 2\dot{\xi}_0\left(-\dfrac{ku + u\dot{u}\dot{q}_1}{1 + u^2}\right) \\ = \xi_0\left(-\dfrac{k(1-u^2)\dot{u}}{(1+u^2)^2} - \dfrac{u\ddot{u}}{1+u^2}\dot{q}_1 - \dfrac{\dot{u}^2(1-u^2)}{(1+u^2)^2}\dot{q}_1\right) \\ +(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0)\dfrac{u\dot{u}}{1+u^2}, \\ \ddot{\xi}_2 - \dot{q}_2\ddot{\xi}_0 - 2\dot{\xi}_0\left(\dfrac{u\dot{u}\dot{q}_1}{1+u^2} - \dfrac{ku^2}{1+u^2}\right) \\ = \xi_0\left(\dfrac{u\ddot{u}}{1+u^2}\dot{q}_1 + \dfrac{\dot{u}^2}{(1+u^2)^2}\dot{q}_1\right. \\ \left. -\dfrac{\dot{u}^2u^2}{(1+u^2)^2}\dot{q}_1 - \dfrac{2ku\dot{u}}{(1+u^2)^2}\right) \\ +(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0)\dfrac{u\dot{u}}{1+u^2}. \end{cases} \tag{30}$$

Equation (30) has the following solutions

$$\xi_0 = 0, \quad \xi_1 = 0, \quad \xi_2 = 1, \tag{31}$$

$$\xi_0 = 0, \quad \xi_1 = 1, \quad \xi_2 = 0, \tag{32}$$

$$\xi_0 = 0, \quad \xi_1 = 1, \quad \xi_2 = 1, \tag{33}$$

$$\xi_0 = 0, \quad \xi_1 = 1, \quad \xi_2 = t, \tag{34}$$

$$\xi_0 = 0, \quad \xi_1 = 0, \quad \xi_2 = t. \tag{35}$$

They have Lie symmetries corresponding to the controllable holonomic system.

By substituting generators ξ_0 and ξ_s into Eqs.(13), we have

$$-\dot{u}\dot{q}_1\xi_0 - (\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0)u + (\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0) = 0. \tag{36}$$

The generators (34) and (35) do not satisfy restrictive Eq. (36), so there are no weak Lie symmetries in the controllable nonholonomic system. Whereas the generators (31), (32) and (34) satisfy restrictive Eqs. (36), so there are weak Lie symmetries for the nonholonomic system. From Eqs.(16) and (17) we obtain gauge functions and conserved quantities for the generators (31), (32) and (34) as follows

$$G_g = kt, \quad I_g = \dot{q}_2 + kt = \text{constant.}, \tag{37}$$

$$G_h = 0, \quad I_h = \dot{q}_1 = \text{constant.}, \tag{38}$$

$$G_j = kt, \quad I_j = \dot{q}_1 + \dot{q}_2 + kt = \text{constant.} \tag{39}$$

They are the gauge functions and conserved quantities for weak Lie symmetries of Eqs.(25) and (26), i.e., they are usual gauge functions and conserved quantities of

the Lie symmetries. Under the infinitesimal transformations (31), (32) and (33), some conclusions can be obtained: form (37) expresses that the momentum is not conservative along the generalized coordinates q_2, form (38) indicates that the momentum is conservative along the generalized coordinates q_1, and form (39) expresses that the momenta of the system is not conservative.

Substituting constraint (26) into Eqs.(15), we obtain

$$-u(t)(\xi_1 - \dot{q}_1) + (\xi_2 - \dot{q}_2) = 0. \tag{40}$$

Eq. (40) cannot be satisfied when substituting the generators (31), (32) and (34) into Eq.(40), so the symmetries are just weak Lie symmetries.

Now we study the inverse problem. Suppose the system has a conserved quantity

$$I = \dot{q}_1 + \dot{q}_2 + kt. \tag{41}$$

Substituting Eq.(41) into Eqs.(22) and (24) leads to

$$\overline{\xi}_1 = 1, \quad \overline{\xi}_2 = 1,$$

$$L\xi_0 + \dot{q}_1\overline{\xi}_1 + \dot{q}_2\overline{\xi}_2 + G = \dot{q}_1 + \dot{q}_2 + kt, \tag{42}$$

and then

$$\xi_0 = \frac{1}{L}(kt - G), \quad \xi_1 = 1 + \dot{q}_1\xi_0, \quad \xi_2 = 1 + \dot{q}_2\xi_0. \tag{43}$$

Substituting Eqs.(39) into Eqs.(43), we obtain

$$\xi_0 = 0, \quad \xi = 1, \quad \xi_2 = 1. \tag{44}$$

The generators (44) satisfy the restrictive Eqs. (36), but they do not satisfy the supplementary restrictive Eq. (40), so the symmetry (44) is weak Lie symmetries, not strong Lie symmetries.

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