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Analytic description of tokamak equilibrium sustained by high fraction bootstrap current*

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By using an expansion technique based on the tokamak ordering, the Grad–Shafranov equation is analytically solved for tokamak equilibrium sustained by a full bootstrap current. This approximate approach is suitable for equilibria with non-circular cross-sections of modest ellipse and triangular deformations. As an input parameter set, the plasma pressure profile and the electron and the ion temperature profiles can be selected as arbitrary functions of the averaged minor radius. Equilibrium properties of this plasma are discussed.

Keywords: bootstrap current, tokamak ordering, full bootstrap sustained equilibrium
PACC: 5255, 5255G, 5255P

1. Introduction

Tokamak plasma equilibrium is the fundamental basis for further studies of magneto-hydrodynamic stability and also for various kinetic analyses. It is well known that this equilibrium has a free boundary in the sense that the realistic plasma boundary is determined self-consistently by the combined plasma current and currents flowing in the external poloidal coil systems. The final closed magnetic flux surface should be tangentially attached to a limiter in the limiter configuration or constituting an x-point separatrix in the divertor configuration. Several numerical codes have been developed and the determination of the tokamak equilibrium is already a routine task nowadays.[1−5] However, as far as the internal stability is concerned, we are usually interested in the internal structure of the magnetic configuration only, and an analytic or relatively simpler solution of the Grad–Shafranov equation (GSh-E) has been a topic pursued for a long time.[6−15] Recently, to save the current-drive power and to obtain more favourable confinement merit for a tokamak reactor, a large fraction of bootstrap current-sustained equilibrium has attracted great interest both theoretically and experimentally.[16−22] We use the tokamak ordering to expand the GSh-E to obtain a series of ordinary differential equations which allow for different sets of input profiles.[13] The tokamak equilibrium sustained by a full bootstrap current is then solved analytically. Our method is to some extent similar to the recently developed perturbed equilibrium method,[15], however, our analysis is more simple and physical.

2. Grad–Shafranov equation in different forms

We discuss the toroidally axisymmetric equilibrium, and assume an isotropic plasma pressure, correspondingly, then the GSh-E in cylindrical coordinates can be written as

$$\Delta^*\psi = R^2\nabla\cdot\left(\frac{\nabla\psi}{R^2}\right) = \mu_0 R j_\phi(R,\psi), \qquad (1)$$

where ψ is the poloidal magnetic flux, while the total magnetic field is defined as

$$\boldsymbol{B} = \nabla\psi\times\nabla\phi + F\,\nabla\phi. \qquad (2)$$

The toroidal current j_ϕ is related to the plasma pressure and the poloidal current flux F by

$$j_\phi = -R\frac{\mathrm{d}p}{\mathrm{d}\psi} - \frac{F}{\mu_0 R}\frac{\mathrm{d}F}{\mathrm{d}\psi}. \qquad (3)$$

Introducing the parallel current averaged over the magnetic surface, i.e., the term

$$<\boldsymbol{j}\cdot\boldsymbol{B}>=<\boldsymbol{j}\cdot B>_\Omega + <\boldsymbol{j}\cdot\boldsymbol{B}>_{\rm cd} + <\boldsymbol{j}\cdot\boldsymbol{B}>_{\rm bs}, \qquad (4)$$

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we can relate the toroidal current to realistic driven mechanisms. In Eq.(4), the subscripts Ω, cd, bs denote the ohmic drive, wave or beam drive and the bootstrap current, respectively. In this way, the j_ϕ component can be written as

$$j_\phi = -R\frac{\mathrm{d}p}{\mathrm{d}\psi}\left[1-\frac{F^2}{R^2<B^2>}\right]+\frac{F}{R<B^2>}<\boldsymbol{j}\cdot\boldsymbol{B}>. \tag{5}$$

This current can be divided into two parts further

$$j_\phi = \frac{R}{R_0}j_1(\psi)+\frac{R_0}{R}j_2(\psi), \tag{6}$$

where

$$\begin{aligned} j_1(\psi) =& -R_0\frac{\mathrm{d}p}{\mathrm{d}\psi},\\ j_2(\psi) =& \frac{F}{R_0}\left[\frac{F}{<B^2>}\frac{\mathrm{d}p}{\mathrm{d}\psi}+\frac{<\boldsymbol{j}\cdot\boldsymbol{B}>}{<B^2>}\right], \end{aligned} \tag{7}$$

$$<B^2> = F^2\left\langle\frac{1}{R^2}\right\rangle+\left\langle\left(\frac{\nabla\psi}{R}\right)^2\right\rangle. \tag{8}$$

The bracket $<\cdot>$ means an averaging over a magnetic surface and R_0 is the major radius of the magnetic axis.

3. Coordinate transformation and expansion technique

We select a flux surface-related radial coordinate system. It is well known that this selection is not unique. Using the technique developed in Refs.[11,13], we make the following coordinate transformation:

$$\begin{aligned} R =& R_0 + r(\rho,\theta)\cos\theta,\\ Z =& r(\rho,\theta)\sin\theta, \end{aligned} \tag{9}$$

$$r = \rho+\sum_{n=1}a_n(\rho)\cos n\theta, \tag{10}$$

so that

$$\rho = \frac{1}{2\pi}\int_0^{2\pi} r\,\mathrm{d}\,\theta. \tag{11}$$

The coefficients $a_n(\rho)$ correspond to an elliptic deformation for n=2 and a triangular one for n=3, etc; however, the usual elongation $e=(\rho+a_2(\rho))/(\rho-a_2(\rho))$ can be relatively larger than the coefficient a_2 itself. This might be a merit of this expansion. Besides, the coefficient a_1 is equivalent to the Shafranov shift. In the following, we assume the following tokamak ordering:

$$\begin{aligned} \rho/R_0 \approx& a_n/\rho \approx o(\varepsilon),\\ (\nabla\psi)^2/F^2 \approx& (2\mu_0 P/B^2)\approx o(\varepsilon^2). \end{aligned} \tag{12}$$

In the (ρ,θ,ϕ) coordinates, the Gsh-E becomes

$$\begin{aligned} \Delta^*\psi =& \frac{1}{(\partial r/\partial\rho)^2}\left[1+\left(\frac{1}{r}\frac{\partial r}{\partial\theta}\right)^2\right]\frac{\mathrm{d}^2\psi}{\mathrm{d}\rho^2}\\ &+\left\{\frac{1}{2}\frac{\partial}{\partial\rho}\left[\left(1+\frac{1}{r^2}\left(\frac{\partial r}{\partial\theta}\right)^2\right)\Big/\left(\frac{\partial r}{\partial\rho}\right)^2\right]\right.\\ &+\left(\frac{1}{r\partial r/\partial\rho}\right)-\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial r/\partial\theta}{r\partial r/\partial\rho}\right)\\ &\left.-\frac{\cos\theta+\frac{1}{r}\frac{\partial r}{\partial\theta}\sin\theta}{R\partial r/\partial\rho}\right\}\frac{\mathrm{d}\psi}{\mathrm{d}\rho}\\ =&\mu_0 R\left[\frac{R}{R_0}j_1(\psi)+\frac{R_0}{R}j_2(\psi)\right]. \end{aligned} \tag{13}$$

Multiplying Eq.(13) by $\left(\frac{(\partial r/\partial\rho)^2}{1+(\partial r/r\partial\theta)^2}\right)\cos n\theta$, then averaging it over θ, i.e., carrying out the integration $\int_0^{2\pi}\mathrm{d}\theta$ for n=0, 1, 2, 3 $\cdots$, we can obtain a series of coupled nonlinear ordinary differential equations about variables. $a_1, a_2, a_3, \cdots$ The first equation serves to determine $\psi(\rho)$, which reads

$$\begin{aligned} &\frac{\mathrm{d}^2\psi}{\mathrm{d}\rho^2}+A_0(\rho,a_n,a_n',a_n'')\frac{\mathrm{d}\psi}{\mathrm{d}\rho}\\ =&B_1(\rho,a_n,a_n')j_1(\psi)+B_2(\rho,a_n,a_n')j_2(\psi). \end{aligned} \tag{14}$$

Here A_0, B_1, B_2 not only depend on ρ, but also on $a_n(\rho)$ (n=1,2,3$\cdots$), as well as their first and second derivatives. The subsequent equations do not include the $\mathrm{d}^2\psi/\mathrm{d}\rho^2$ term and serve to determine the $a_n(\rho)$s. They usually have the following form:

$$\begin{aligned} &A_n(\rho,a_n,a_n',a_n'')\frac{\mathrm{d}\psi}{\mathrm{d}\rho}\\ =&B_{1n}(\rho,a_n,a_n')j_1+B_{2n}(\rho,a_n,a_n')j_2. \end{aligned} \tag{15}$$

Though the above sets of equations are ordinary differential in nature, they are coupled and nonlinear so that they do not describe the equilibrium in a simple way. Sometimes, we can use this set of equations to describe some special equilibrium in a numerical way, in general. However, we do not think it is a proper way. Instead of doing this way, in what follows, we use the tokamak ordering (12) to simplify this series of equations to a linear one. Now we expand

$$\psi = \psi_0+\varepsilon\psi_1+\varepsilon^2\psi_2+\cdots, \tag{16}$$

$$a_n = \varepsilon a_{n0}+\varepsilon^2 a_{n1}+\cdots. \tag{17}$$

Substituting them into Eqs.(14) and (15), collecting terms of the same order, we obtain a linear set of equations. This procedure is rather tedious but straightforward, and the results take the following form:[13]

$$\frac{\mathrm{d}^2\psi_0}{\mathrm{d}\rho^2}+\frac{1}{\rho}\frac{\mathrm{d}\psi_0}{\mathrm{d}\rho}=\mu_0 R_0(j_1+j_2), \tag{18}$$

$$\frac{\mathrm{d}^2 a_{10}}{\mathrm{d}\rho^2}+\left(\frac{1}{\rho}+\frac{2\psi_0''}{\psi_0'}\right)\frac{\mathrm{d}a_{10}}{\mathrm{d}\rho}=-\frac{1}{R_0}+2\mu_0\rho\frac{j_1}{\psi_0'}, \tag{19}$$

$$\frac{\mathrm{d}^2 a_{n0}}{\mathrm{d}\rho^2}+\left(\frac{1}{\rho}+\frac{2\psi_0''}{\psi_0'}\right)\frac{\mathrm{d}a_{n0}}{\mathrm{d}\rho}+\frac{1-n^2}{\rho^2}a_{n0}=0, \quad (n=2,3,\cdots). \tag{20}$$

It is noted that Eq.(18) is same for non-circular and circular tokamak plasmas, and the non-circular deformations are determined by the homogeneous Eq. (20) so that its solution is proportional to the deformation at the boundary: $a_{n0}(\rho_\mathrm{b})=\varepsilon_n$. The Shafranov shift a_{10} is determined by Eq.(19) that is inhomogeneous so that its solution is related to the j_1 part, i.e., the pressure gradient. Then the lowest-order quantities of ψ and a_1 are the same for both circular and non-circular tokamaks. The more complicated higher-order equations are no use for the present purpose and we do not write their explicit forms. The boundary conditions are

$$\psi_0(0)=\psi_0'(0)=0,$$
$$a_{10}(0)=a_{10}'(0)=0,$$
$$a_{n0}(0)=0,$$
$$a_{n0}(\rho_0)=\delta_n, \quad \text{for } n\le 2,$$

with ρ_0 being the averaged minor radius of the boundary magnetic surface. Condition (21) implies that we are treating a fixed boundary problem.

4. Lowest-order solution for high fraction bootstrap current equilibria

We note that there is a degree of freedom in the choice of the input parameter forms. In the case of high fraction of bootstrap current, we assume that all density and temperature profiles are known function of the minor radius.

4.1. Tokamak equilibrium sustained by a pure bootstrap current

First, we consider the case of no other current driving scheme. As can be seen later, in this case the safety factor near the magnetic axis is divergent and the magnetic shear is negative. This means that in the near magnetic axis region the rotational transform of the field line approaches zero, similar to a stellarator solution with a helical winding $l\ge 3$. This has been the reason to seek for a seed current in this region. However, from the point of view of the establishment of equilibrium, this situation is possible and its properties are of particularly interest at least for theoretical study to compare this tokamak equilibrium with the stellarator counterpart. As a matter of fact, during the initial stage of tokamak discharge when the toroidal current does not penetrate into the near magnetic axis region, the rotational transform in this region is very small. The direct demonstration of the stable tokamak equilibrium with central current hole was also reported recently in JT-60U and the JET experiments where in a considerable central region the toroidal current was almost zero.[23,24]

Now we consider the solution of Eq.(18) first. To the lowest order of the tokamak ordering, we find that the first term containing $\mathrm{d}p/\mathrm{d}\psi$ on the right-hand side (the so-called P-S current) in Eq.(5) is a higher-order quantity. By neglecting higher-order terms will retain only the driving current term. Finally, we have (in the following we neglect the subscript 0)

$$j_1+j_2=j_\mathrm{bs}. \tag{22}$$

Because our main purpose is to extract the method of solving the GSh-E, we will use the simplest model for the bootstrap current. A suitable form can be written as[25]

$$j_\mathrm{bs}=\frac{p_\mathrm{e}}{B_\mathrm{p}}\left(\frac{\rho}{R_0}\right)^{1/2}\left[-2.44\left(1+\frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\frac{\mathrm{d}}{\mathrm{d}\rho}(\ln n)-0.69\frac{\mathrm{d}}{\mathrm{d}\rho}(\ln T_\mathrm{e})+\frac{T_\mathrm{i}}{T_\mathrm{e}}\frac{\mathrm{d}}{\mathrm{d}\rho}(\ln T_\mathrm{i})\right], \tag{23}$$

where $B_\mathrm{p}=\psi'/R_0$ is the poloidal magnetic field. Now we introduce a normalized radius $x=\rho/\rho_0$, with ρ_0 being the averaged radius of the last closed magnetic surface. Rewrite Eq.(23) as

$$j_\mathrm{bs}=(R_0\rho_0)^{1/2}p_{\mathrm{e}0}\sqrt{x}f(x)/\psi'(x), \tag{24}$$

where

$$f(x)=(p_\mathrm{e}/p_{\mathrm{e}0})[-2.44(1+T_\mathrm{i}/T_\mathrm{e})\mathrm{d}(\ln n)/\mathrm{d}x-0.69\mathrm{d}(\ln T_\mathrm{e})/\mathrm{d}x+(T_\mathrm{i}/T_\mathrm{e})\mathrm{d}(\ln T_\mathrm{i})/\mathrm{d}x]. \tag{25}$$

This can be selected as an arbitrary function depending on the density and the temperature profiles. Now Eq.(18) has the following form:

$$\psi'(x)\psi''(x)+\psi'(x)^2/x = \mu_0 R_0^{3/2}\rho_0^{5/2}p_{e0}x^{1/2}f(x). \quad (26)$$

Then its general solution can be written as

$$\psi'(x)=\left[2\mu_0 p_{e0}R_0^{3/2}\rho_0^{5/2}\int_0^x \mathrm{d}x' x'^{5/2}f(x')\right]^{1/2}/x. \quad (27)$$

If we assume that near the magnetic axis the profiles of the density and the temperatures all have a parabolic type, then the function $f(x)$ will be proportional to $x^{3/2}$, this gives $\psi'(x)\propto x^{5/4}$ and the first non-zero term of the bootstrap current j_{bs} is proportional to $\rho^{1/4}$. Generally, we can assume

$$f(x)=\sum_{n=1}\alpha_n x^n, \quad (28)$$

then we have

$$\psi'(x)=(2\mu_0 p_{e0}R_0^{3/2}\rho_0^{5/2})^{1/2}x^{5/4}\left[\sum_{i=1}\frac{\alpha_i}{i+7/2}x^{i-1}\right]^{1/2}. \quad (29)$$

With this solution, we can express the terms appearing in Eqs.(19) and (20) as

$$\frac{1}{\rho}+\frac{2\psi''}{\psi'}=\frac{\rho^2 f(\rho)}{\int_0^\rho \mathrm{d}x x^2 f(x)}-\frac{1}{\rho} = \frac{1}{\rho}\left(1+\frac{\sum_{i=1}(3+2i)\alpha_i x^{i+3/2}/(7+2i)}{\sum_{i=1}2\alpha_i x^{i+3/2}/(7+2i)}\right). \quad (30)$$

In the near magnetic axis region, this term becomes $7/2\rho$, the nonhomogenous Eq. (19) has a solution of the following form:

$$a_1'(\rho)=\rho^{-7/2}\int_0^\rho \mathrm{d}x x^{7/2}\left[-\frac{1}{R_0}+2\mu_0 x\frac{j_1(x)}{\psi'(x)}\right] = \frac{1}{R_0\rho^{7/2}}\int_0^1 \mathrm{d}x x^{7/2}(-1+\alpha(x)), \quad (31)$$

where the parameter $\alpha=2\mu_0 R_0\rho p'(\rho)/\psi'^2(\rho)$ is useful in ballooning mode analysis. A peculiar point is that this parameter does not depend on the absolute value of the pressure (see below). Besides, α is divergent near the magnetic axis; however, the Shafranov shift is regular. The elliptic and the triangular deformations then are determined by Eq.(20) which is homogeneous in nature, in the near magnetic axis region, its solution can be expressed as

$$a_n(\rho)=\delta_n(\rho/\rho_0)^s. \quad (32)$$

The power index s in Eq.(32) is determined by

$$s(s-1)+7s/2-n^2+1=0. \quad (33)$$

For more general density and temperature profiles, the $a_1'(\rho)$ can still be expressed as an integral form like Eq.(31), while $a_n(\rho)$ can be expressed as a power series

$$a_n(\rho)=\delta_n x^s\sum_{i=0}\beta_i x^i. \quad (34)$$

The coefficients $\beta_i' s$ are related to $\alpha_i' s$.

The safety factor profile can be determined from its definition as

$$q(\rho)=\frac{F(\rho)}{2\pi\psi'(\rho)}\oint\frac{\mathrm{d}l}{R|\nabla\rho|} =\frac{F\rho}{R_0\psi'(\rho)}(1+o(\varepsilon^2)) \approx\frac{B_0}{\sqrt{2\mu_0 P_{e0}}}\left(\frac{\rho_0}{R_0}\right)^{3/4} \times x^{-1/4}\left[\sum_{i=1}\frac{\alpha_i x^{i-1}}{i+7/2}\right]^{-1/2}. \quad (35)$$

The edge safety factor that is related to the total current I_{p} is then determined by

$$q(1)=\frac{B_0}{\left(2\mu_0 P_{e0}\sum_{i=1}\alpha_i/(i+7/2)\right)^{1/2}}\left(\frac{\rho_0}{R_0}\right)^{3/4}. \quad (36)$$

In the case discussed above, the input set consists of the density, the electron temperature and the ion temperature profiles. This gives, in fact, all the needed information about equilibrium. For example, we can determine the toroidal current density from Eqs.(6), (7), (22) and (23). We have

$$j_\phi=\frac{R_0}{R}j_{\mathrm{bs}}+\frac{1}{\psi'}\frac{\mathrm{d}p}{\mathrm{d}\rho}\left(\frac{R_0^2}{R}-R\right). \quad (37)$$

From Eqs.(3), (22) and (24), we obtain the expression for function F

$$F^2=F^2(0)+2\mu_0 R_0^2[p(0)-p(x)] -2\mu_0(R_0\rho_0)^{3/2}p_{e0}\int_0^x \mathrm{d}x'\sqrt{x'}f(x'). \quad (38)$$

4.2. Large fraction of bootstrap current with other drive scenarios

Physically, for a large fraction bootstrap current reactor, it is thought that in the region near the magnetic axis, some kind of the seed current is needed. More generally, bootstrap current with combined ohmic drive or other drive schemes is more realistic. In all these cases, to analytically solve Eqs.(18)–(20), we still suppose the input profile has the following form:

$$j_1 + j_2 = g(x)/\psi'(x) + j_{\rm bs}(\rho). \tag{39}$$

Here the first term represents the seed current and the bootstrap current density is still expressed by Eq.(24). Now the function $g(x)$ can also take a power series form as

$$g(x) = \sum_{i=1} \gamma_i x^i. \tag{40}$$

In this way, the solution of Eq.(18) can be written as

$$\begin{aligned}\psi'(x) =&(2\mu_0 R_0)^{1/2}\rho_0 x \\ &\times \left[\sum_{i=1}\left(\frac{\gamma_i x^{i-1}}{i+3}\right.\right. \\ &\left.\left. + (R_0\rho_0)^{1/2} p_{\rm e0}\frac{\alpha_i x^{i-1/2}}{i+7/2}\right)\right]^{1/2}. \end{aligned}\tag{41}$$

Near the magnetic axis the seed current is dominant, then we have $\psi'(\rho) \to (2\mu_0 R_0)^{1/2}\rho_0\gamma_1 x/4$, and the seed current density in this region is $j_{\rm seed} = 2\left(\frac{2\gamma_1}{\mu_0 R_0 \rho_0^2}\right)^{1/2}$. This defines the value and dimension of the coefficient $\gamma_i' s$.

Using the above results, similar discussion can be made for the solution of the non-circular deformation and for the Shafranov shift, the total equilibrium property then being approached. Of course, for general profile sets of density and temperature, some simple numerical calculations are needed.

5. Examples

For equilibrium sustained by a full bootstrap current, we assume a very simple input set for the density and the electron and ion temperature profiles: all these profiles are assumed to be parabolic, expressed as

$$\begin{aligned} n_{\rm e} =& n_{\rm e0}(1-x^2), \\ T =& T_0(1-x^2), \end{aligned}\tag{42}$$

and also $T_{\rm e} = T_{\rm i}$. The function $f(x)$ in Eq.(25) becomes

$$f(x) = 9.14x(1-x^2). \tag{43}$$

It readily gives

$$\begin{aligned}\psi'(x) =&1.425(2\mu_0 p_{\rm e0})^{1/2} \\ &\times R_0^{3/4}\rho_0^{5/4}x^{5/4}(1-9x^2/13)^{1/2}. \end{aligned}\tag{44}$$

The bootstrap current itself then has the profile as

$$j_{\rm bs}(\rho) = \frac{6.414}{R_0}\left(\frac{p_{\rm e0}}{2\mu_0}\right)^{1/2}\left(\frac{R_0}{\rho_0}\right)^{3/4}\frac{x^{1/4}(1-x^2)}{\sqrt{1-9x^2/13}}. \tag{45}$$

The q-profile becomes

$$q(\rho) = \frac{0.702 B_0}{\sqrt{2\mu_0 p_{\rm e0}}}\left(\frac{\rho_0}{R_0}\right)^{3/4} x^{-1/4}(1-9x^2/13)^{-1/2}. \tag{46}$$

As indicated above, q_0 is divergent. This means that the rotational transformation approaches zero. Previously, people think that this situation is forbidden for tokamaks. However, recent JT-60U and JET experiments suggest that the equilibrium sustained by a full bootstrap current is possible. It is easy to obtain the magnetic surface structure by solving Eqs.(19) and (20) correspondingly. Functions $a_n(\rho)$ all have regular behaviour. In region $x \le 0.537$ the magnetic shear is negative, and the minimum value of q is $q_{\rm min} = 0.725q(1)$. Assuming

$$B_0 = 5\,{\rm T}, \quad \rho_0 = 2\,{\rm m}, \quad R_0 = 6\,{\rm m},$$

$$n_{\rm e0} = 2.0\times 10^{20}\,{\rm m}^{-3},$$

$$T_{\rm e0} = T_{\rm i0} = 15\,{\rm keV},$$

$$p_{\rm e0} = 4.8\times 10^5\,{\rm Pa},$$

we have $q(1)$=2.52, corresponding to a total bootstrap current

$$I_{\rm bs} = \frac{2\pi B_0\rho_0^2}{\mu_0 q(1) R_0} = 2\pi\rho_0^2\int_0^1 {\rm d}x' x' j_{\rm bs}(x') = 6.6\,{\rm MA}. \tag{47}$$

For given density and temperature profiles, $I_{\rm bs}$ scales as $\sqrt{p_{\rm e0}}R_0^{-1/4}\rho_0^{5/4}$.

In this simple illustrative case, the Shafranov shift can be determined analytically. First by substituting the assumed pressure profile and $\psi'(x)$ in Eq.(44), we have

$$a(x) = 0.985\left(\frac{R_0}{\rho_0}\right)^{1/2}\left(1+\frac{T_{\rm i0}}{T_{\rm e0}}\right)x^{-1/2}\left(1-\frac{9x^2}{13}\right)^{-1}$$

and

$$\int \mathrm{d}x \left[\frac{1}{x} + \frac{2\psi''}{\psi'}\right] = \int \mathrm{d}x \left[\frac{7}{2x} + \frac{2x}{x^2 - 13/9}\right] = \ln\left[x^{7/2}\left(1 - \frac{9x^2}{13}\right)\right].$$

Finally, we have

$$a_1(x) = \frac{\rho_0^2}{R_0}\left\{0.49\left(\frac{R_0}{\rho_0}\right)^{1/2} \int_0^x \mathrm{d}x \frac{\sqrt{x}}{1 - 9x^2/13} - \frac{4}{81}\ln(1 - 9x^2/13) + \frac{x^2}{13}\right\}.$$

The second and the third terms on the right-hand side are both due to the vacuum contribution. In the near axis region, the first term is dominant and $a_1 \approx \frac{1}{3}\rho_0^{3/2} R_0^{1/2} x^{3/2}$, which is independent of the pressure value as indicated above.

The deformations of the cross section can be obtained by a little of numerical work. They are all regular functions of the minor radius, indicating that the equilibrium is well established.

6. Conclusion

We have suggested an expanding procedure to solve the tokamak equilibrium sustained by a high-fraction of bootstrap current. With no seed current the solution in the near magnetic axis region is similar to a stellarator configuration with $l \geq 3$. The stability and confinement merits of this kind of equilibrium still need to identify. Of cause, a more accurate expression for the bootstrap current especially in the near magnetic axis region should be used for this purpose. A recent theoretical derivation by considering the finite trapping ratio at the magnetic axis indicates that there exists a finite bootstrap current density in this region.[26] A new expression for the bootstrap current is also given there. To include this modification, the only change is the form of function $f(x)$ and this does not change our procedure for seeking solutions.

References

[1] Lachner K 1976 *Comput. Phys. Commun.* **38** 13
[2] Ling K M and Jardin S C 1985 *J. Comput. Phys.* **58** 330
[3] Gruber R, Iacono R and Troyan F 1987 *J. Comput. Phys.* **73** 168
[4] Turkinton B, Lifschitz A, Eydeland A and Spruck J 1993 *J. Comput. Phys.* **106** 269
[5] Lujens H, Bondesons A and Roy A 1992 *Comput. Phys. Commun.* **69** 287
[6] Greene J M, Johnson J L and Weimer K E 1971 *Phys. Fluids* **14** 671
[7] Hass F A 1972 *Phys. Fluids* **15** 141
[8] Maschke E K 1973 *Nucl. Fusion* **15** 535
[9] Yoshikawa S 1974 *Phys. Fluids* **17** 178
[10] Okamoto M, Watakani M and Amano T 1975 *Nucl. Fusion* **15** 225
[11] Okada O 1976 *Phys. Fluids* **19** 2034
[12] Dory R A and Peng Y K M 1977 *Nucl. Fusion* **17** 21
[13] Shi B R 1981 *Kexue Tongbao* (A monthly Journal of China Science) **26** 494
[14] Ludwig G O 1997 *Plasma Phys. Contr. Fusion* **39** 2021
[15] Zahkarov L E and Pletzer A 1999 *Phys. Plasmas* **6** 4693
[16] Ehst D A *et al* 1998 *Nucl. Fusion* **38** 13
[17] Kessel G E 1994 *Nucl. Fusion* **34** 1221
[18] Zornstorff M C *et al* 1988 *Phys. Rev. Lett.* **60** 1306
[19] Cordy J D *et al* 1988 *Plasma Phys. Contr. Fusion* **30** 1625
[20] Kikuchi M *et al* 1990 *Nucl. Fusion* **30** 343
[21] Tokuda S, Takeda T and Okamoto M 1989 *J. Phys. Soc. Japan.* **58** 871
[22] Lin-Liu Y R *et al* 1999 *Phys. Plasmas* **6** 3934
[23] Fujita T *et al* 2001 *Phys. Rev. Lett.* **87** 245001
[24] Hawkes N C *et al* 2001 *Phys. Rev. Lett.* **87** 115001
[25] Hazeltine R D and Meiss J D 1992 *Plasma Confinement* (New York: Addison-Wesley) p 363
[26] Wang S J 1998 *Phys. Plasma* **5** 3310