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Global stabilization of a Lorenz system*

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In this paper, using feedback linearizing technique, we show that a Lorenz system can be considered as a cascade system. Moreover, this system satisfies the assumptions of global stabilization of cascade systems. Thus continuous state feedback control laws are proposed to globally stabilize the Lorenz system at the equilibrium point. Simulation results are presented to verify our method. This method can be further generalized to other chaotic systems such as Chen system,coupled dynamos system, etc.

Keywords: control of chaos, Lorenz system, global stabilization
PACC: 0545

1. Introduction

Chaos control and synchronization have become an active area of research and development over the last few years (see for instance Refs.[1–20] and references therein). Various approaches have been proposed for the control of chaotic systems. These approaches can be divided into two broad categories: non-feedback control and feedback control. For the feedback control, nonlinear and linear state feedback controllers have been designed to stabilize the closed loop system at a desired periodic orbit (including equilibrium point).

Since the chaotic systems are nonlinear systems, the methods using linear state feedback usually cannot guarantee the stability in the global sense.[7] To obtain the global stability, many analysis and synthesis methods for nonlinear control systems have been introduced, such as backstepping control,[7] sliding mode control,[8] feedback linearization control,[9] active control,[10], etc.

In this paper, combining the feedback linearizing technique and the technique of stabilization of cascade systems together, we propose a method to control and globally stabilize the Lorenz system at any one of its three equilibrium points. Moreover, this method can be further generalized to the stabilization of other chaotic systems such as the Chen system,[21] the coupled dynamos system,[22] etc.

The paper is organized as follows. In section 2, a mathematical description of the stabilization problem of Lorenz system is presented. In section 3, we discuss some results on the global stabilization of a class of cascade systems and propose a globally stabilizing control law for Lorenz system. Numerical simulation results are also given. Concluding remarks are given in section 4.

2.Problem description

Now, let us consider the Lorenz chaotic system[8] which is given by

$$\begin{aligned} \dot{x}_1 &= \sigma(y_1 - x_1), \\ \dot{y}_1 &= r x_1 - x_1 z_1 - y_1, \\ \dot{z}_1 &= x_1 y_1 - b z_1, \end{aligned} \tag{1}$$

where σ, r, b are positive parameters. When r is greater than $\dfrac{\sigma(\sigma+b+3)}{(\sigma-b-1)}$, the Lorenz system will have a chaotic attractor and exhibit a two-lobed pattern.

Putting the right-hand side of Eq.(1) equal to zero,

$$\begin{aligned} & x_0 = y_0, \\ & r x_0 - x_0 z_0 - y_0 = 0, \\ & b z_0 = x_0 y_0, \end{aligned} \tag{2}$$

one obtains three equilibrium points of the system located at $(0,0,0)$ and $(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$.

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They are all unstable equilibrium points. To control the Lorenz chaotic system using feedback control laws, at least one and at most three control inputs are needed. Here adding one control input u to the second equation of system (1), we obtain the following control system:

$$\begin{aligned}\dot{x}_1 =&\sigma(y_1 - x_1),\\ \dot{y}_1 =&rx_1 - x_1z_1 - y_1 + u,\\ \dot{z}_1 =&x_1y_1 - bz_1.\end{aligned} \tag{3}$$

The objective here is to drive system (1) to any one of the three equilibrium points using control input u.

To rewrite system (3) in a more convenient form, we define the coordinate transformation

$$x = x_1 - x_0, \quad y = y_1 - y_0, \quad z = z_1 - z_0, \tag{4}$$

where (x_0, y_0, z_0) is an equilibrium point of the system. Based on this coordinate transformation and Eq.(2), we arrive at the following new system:

$$\begin{aligned}\dot{x} =&\sigma(y - x),\\ \dot{y} =&(r - z_0)x - y - (x + x_0)z + u,\\ \dot{z} =&(x + x_0)y + y_0x - bz.\end{aligned} \tag{5}$$

Hence, the object turns to be a problem of stabilizing system (5) to the origin.

Observing system (5) one will find that if we let state y be zero, then the (x, z)-subsystem has the following form:

$$\begin{aligned}\dot{x} =&-\sigma x,\\ \dot{z} =&-bz + y_0x.\end{aligned} \tag{6}$$

The characteristic equation of system (6) has the form

$$\lambda^2 + (b + \sigma)\lambda + \sigma b = 0$$

with solutions

$$\lambda_1 = -b, \quad \lambda_2 = -\sigma.$$

The two eigenvalues of system (6) are negative. Based on control theory, one can easily verify that no matter which value of y_0 is taken, 0, or $\pm\sqrt{b(r-1)}$, system (6) will be asymptotically stable, i.e., states x and z will converge to zero.

So we should first design a controller to drive y to zero. The next step, which is just as important, is to guarantee that the (x, z)-subsystem is bounded during the converging process of $y \to 0$. Thus we can obtain the global asymptotic stability of system (5).

3. Global stabilization of a Lorenz system

Defining the following feedback linearizing input v:

$$v = (r - z_0)x - y - (x + x_0)z + u, \tag{7}$$

one obtains the following form:

$$\begin{pmatrix}\dot{x}\\ \dot{z}\end{pmatrix} = \begin{pmatrix}-\sigma & 0\\ y_0 & -b\end{pmatrix}\begin{pmatrix}x\\ z\end{pmatrix} + \begin{pmatrix}\sigma\\ x + x_0\end{pmatrix}y,$$

$$\dot{y} = v. \tag{8}$$

System (8) can now be considered as a cascade system. So let us first review some results of global stabilization of cascade systems.[23]

Consider a system of the form

$$\begin{aligned}&\dot{\eta} = f(\eta) + \psi(\eta, \xi), && \eta \in R^{n-m},\\ &\dot{\xi} = a(\xi, u), && \xi \in R^m,\end{aligned} \tag{9}$$

where $f(\cdot), \psi(\cdot,\cdot), a(\cdot,\cdot)$ is C^1 and $f(0) = 0$, $\psi(\eta, 0) = 0$, $a(0, 0) = 0$.

Assumption 1 The function $\psi(\eta, \xi)$ has linear growth in η, differentiable at $\xi = 0$, such that

$$||\psi(\eta, \xi)|| \le \gamma_1(||\xi||)||\eta|| + \gamma_2(||\xi||). \tag{10}$$

Assumption 2 The Jacobian linearization (A, B) of $\dot{\xi} = a(\xi, u)$ at $\xi = 0$ is stabilizable.

Theorem 1[23] Suppose that assumptions 1 and 2 hold and let $u = k(\xi)$ be any C^1 partial-state feedback such that the equilibrium $\xi = 0$ of $\dot{\xi} = a(\xi, k(\xi))$ is globally asymptotically stable (GAS) and locally exponentially stable(LES). If there exists a positive semidefinite radially unbounded function $W(\eta)$ and positive constants c and M such that for $\|\eta\| > M$,

$$\begin{aligned}&\text{(i)} \quad L_fW(\eta) \le 0,\\ &\text{(ii)} \quad \left\|\frac{\partial W}{\partial \eta}\right\|\|\eta\| \le cW(\eta);\end{aligned} \tag{11}$$

then the feedback $u = k(\xi)$ guarantees boundedness of all the solutions of system (9). If, in addition, $\dot{\eta} = f(\eta)$ is GAS, then the feedback $u = k(\xi)$ achieves GAS of the equilibrium $(\eta, \xi) = (0, 0)$.

Thus we obtain the following theorem:

Theorem 2 System (5) is GAS around the origin with the following control law:

$$\begin{aligned}u =&-(r - z_0)x + y + (x + x_0)z + v,\\ v =&-ky, \qquad k > 0,\end{aligned} \tag{12}$$

which means that any one of the three equilibrium points of system (3) is GAS with the following control law:

$$u = -rx_1 + y_1 + x_1 z_1 - k(y_1 - y_0), \qquad (13)$$

where $k > 0$.

Proof According to the previous discussion, system (5) has been transformed into system (8) which is in the form of cascade system (9). Considering system (8), assumption 2 are satisfied because $\dot{y} = v$ is controllable and the feedback control law (12) has made the equilibrium $y = 0$ of $\dot{y} = v$ GAS and LES. Let X denote $(x\, z)^{\mathrm{T}}$. It can be easily verified that system (8) satisfies assumption 1 because the interconnection term $\psi(X, y)$ is linear growth in X.

From the previous discussion, we know that the linear system

$$\dot{X} = AX, \qquad A = \begin{pmatrix} -\sigma & 0 \\ y_0 & -b \end{pmatrix} \qquad (14)$$

is globally asymptotically stable with respect to the origin. From the Lyapunov criterion for the linear time-invariant system,[24] the global asymptotic stability of system (14) means that for any given symmetric positive-definite matrix Q, there exists a symmetric positive-definite matrix P that satisfies the matrix equation

$$A^{\mathrm{T}}P + PA = -Q.$$

Thus we obtain the following Lyapunov function for system (14):

$$W(X) = X^{\mathrm{T}}PX = X^{\mathrm{T}}SSX = (SX)^{\mathrm{T}}(SX), \quad (15)$$

where S is also a symmetric positive-definite matrix such that $P = S^2$. Taking the time derivative of Eq.(15), we obtain

$$\begin{aligned} \dot{W} =& \dot{X}^{\mathrm{T}}PX + X^{\mathrm{T}}X\dot{Y} = X^{\mathrm{T}}(A^{\mathrm{T}}P + PA)X \\ =& - X^{\mathrm{T}}QX \le 0 \end{aligned}$$

and

$$\begin{aligned} \left\| \frac{\partial W}{\partial X} \right\| \|X\| =& \|2PX\|\|X\| \\ =& \|2SSX\|\|S^{-1}SX\| \le cW(X), \end{aligned} \qquad (16)$$

where

$$c = 2\|S\|\|S^{-1}\| > 0.$$

Thus $W(x, z)$ is radially unbounded and satisfies conditions (i) and (ii) of theorem 1. Thus, global asymptotic stability of the equilibrium $(x, y, z) = (0, 0, 0)$ of system (8), i.e., $(x, y, z) = (x_0, y_0, z_0)$ of system (5), follows from theorem 1 with control law $v = -ky$. According to Eqs.(4), (7), we can easily obtain the control law (12) for system (5) and the control law (13) for system (3).

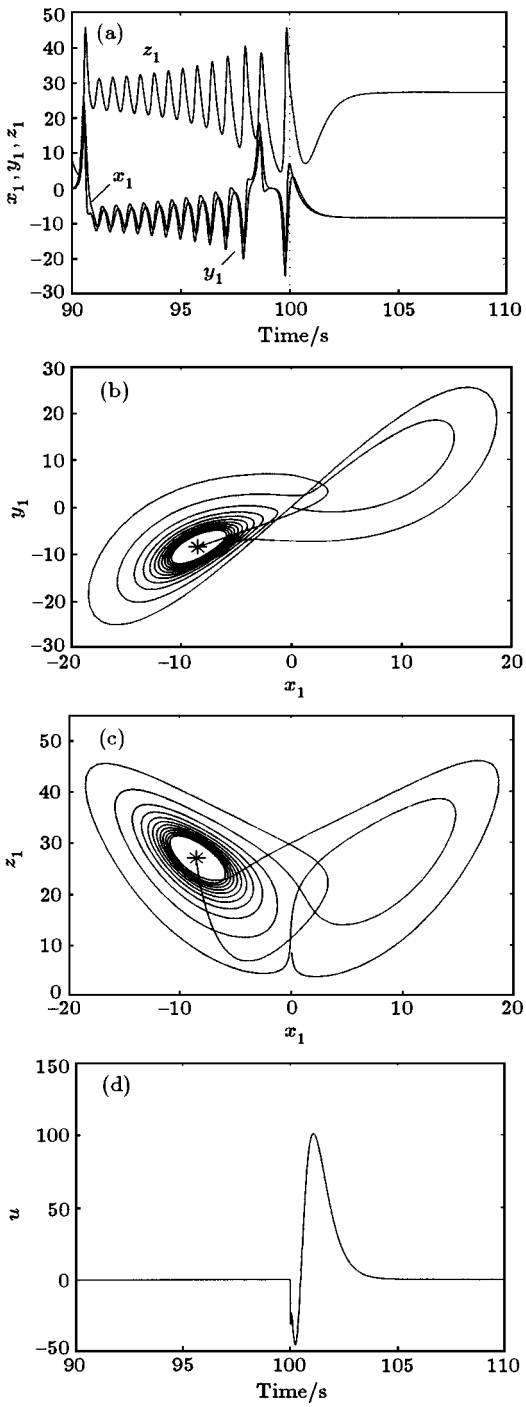


Fig.1. Stabilization of a Lorenz system after t=100. (a) States x_1, y_1, z_1, (b) phase plot at x_1-y_1 plane, (c) phase plot at x_1-z_1 plane, (d) control signal u.

Remark 1 If some states of physical chaotic system are not available through direct measurements, our state feedback stabilization method cannot be used directly. However, we can use state observers to estimate these states of chaotic systems. Detailed descriptions about the observer method can be found in relevant articles (see Refs.[25,26] for instance).

Simulation results are presented as follows. The parameters are $\sigma = 10, b = 8/3$, and $r = 28$. The initial values are $x_1(0)$=–5, $y_1(0)$=2, $z_1(0)$=0.8. The gain constant k is chosen to be $k = 2$. The desired equilibrium point is assumed to be (x_0, y_0, z_0)=$(-6\sqrt{2}, -6\sqrt{2}, 27)$. After a time $t = 100$, the Lorenz system (3) is to be controlled with the control law (13). The simulation results are shown in Fig.1: (a) shows the time response of states x_1, y_1, z_1, (b) displays the phase plot at x_1-y_1 plane from $t = 90$ to $t = 110$, (c) displays the phase plot at x_1-z_1 plane from $t = 90$ to $t = 110$, (d) shows the time history of control input u. Note that the symbol $*$ in Fig.1 denotes the desired equilibrium point. The simulation results show that the control law (13) proposed in this paper can successfully globally stabilize system (3) at the desired equilibrium point.

4. Conclusion

In this paper, using feedback linearizing technique,we show that the Lorenz system can be considered as a cascade system. Moreover, this system satisfies the assumptions of global stabilization of cascade systems. Thus continuous state feedback control laws are proposed to globally stabilize the Lorenz chaotic system at a desired equilibrium point. This method can be further generalized to other chaotic systems such as Chen system, coupled dynamos system, etc.

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