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Closed orbits and limit cycles of second-order autonomous Birkhoff systems*

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In this paper, the existence of periodic orbits and the non-existence of limit cycles for the second-order autonomous Birkhoff system are studied. Further the existence of algebraic limit cycles for a generalized second-order autonomous Birkhoff system is studied.

Keywords: Birkhoff system, equation of motion, singular point, closed orbit, limit cycle
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1. Introduction

The dynamics of Birkhoff systems[1] is an important branch of classical mechanics in modern time. Stantilli[2] made a primary research on Birkhoff systems and obtained some results. Later, we have constructed the frame of Birkhoff dynamics.[3] In recent years, the study on the Birkhoff dynamics has become a popular subject, and many important results have been obtained.[4−12] But there are still not a global analysis for Birkhoff systems. In Ref.[13] we for the first time studied on this work and gained some meaningful results. In this paper, the existence of periodic orbits for the second-order autonomous Birkhoff system is studied by using Lyapunov centre theorem, and then the existence theorem of periodic orbits is obtained. Secondly, the non-existence of limit cycles for the second-order autonomous Birkhoff system is studied. Thirdly, the existence of algebraic limit cycles for a generalized second-order autonomous Birkhoff system is studied. Finally, an example is presented to illustrate the results.

2. Singular points of the Birkhoff systems

A second-order autonomous Birkhoff system[2,3]

$$\dot{a}^1 = \omega^{12}\frac{\partial B}{\partial a^2},$$

$$\dot{a}^2 = \omega^{21}\frac{\partial B}{\partial a^1}, \tag{1}$$

where

$$\omega^{\alpha\beta} = \left(\left\|\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right\|^{-1}\right)^{\alpha\beta},$$

$$(\mu,\nu,\alpha,\beta = 1,2), \tag{2}$$

is called Birkhoff contravariant tensor. $B = B(a)$ and $R_\mu = R_\mu(a)$ are the Birkhoffian and the Birkhoff functions respectively.

Suppose that Eq.(1) have a singular point (a_0^1, a_0^2), i.e.,

$$\omega^{12}(a_0^1,a_0^2)\frac{\partial B}{\partial a^2}(a_0^1,a_0^2) = 0,$$

$$\omega^{21}(a_0^1,a_0^2)\frac{\partial B}{\partial a^1}(a_0^1,a_0^2) = 0. \tag{3}$$

Noticing that $\dfrac{\partial R_2}{\partial a^1} - \dfrac{\partial R_1}{\partial a^2} \neq 0$, we have

$$\frac{\partial B}{\partial a^1}(a_0^1,a_0^2) = 0,$$

$$\frac{\partial B}{\partial a^2}(a_0^1,a_0^2) = 0. \tag{4}$$

If Eq.(4) have solutions, the system has singular points. The number of solutions is the number of the singular points.

3. Existence of periodic orbits and non-existence of limit cycles

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The linear system of the second-order autonomous Birkhoff system (1) at the singular point (a_0^1, a_0^2) is

$$\begin{pmatrix} \dot{a}^1 \\ \dot{a}^2 \end{pmatrix} = C \begin{pmatrix} a^1 - a_0^1 \\ a^2 - a_0^2 \end{pmatrix}, \tag{5}$$

where the subscript 0 expresses the result obtained by a_0^1 and a_0^2 instead of a^1 and a^2 respectively, and where

$$C = \begin{pmatrix} \dfrac{\partial}{\partial a^1}\left(\omega^{12}\dfrac{\partial B}{\partial a^2}\right) & \dfrac{\partial}{\partial a^2}\left(\omega^{12}\dfrac{\partial B}{\partial a^2}\right) \\ \dfrac{\partial}{\partial a^1}\left(\omega^{21}\dfrac{\partial B}{\partial a^1}\right) & \dfrac{\partial}{\partial a^2}\left(\omega^{21}\dfrac{\partial B}{\partial a^1}\right) \end{pmatrix}_0 \tag{6}$$

is the Fréchet derivative at the singular point (a_0^1, a_0^2). Considering Eq.(4), we obtain

$$C = \begin{pmatrix} (\omega^{12})_0\left(\dfrac{\partial^2 B}{\partial a^1 \partial a^2}\right)_0 & (\omega^{12})_0\left(\dfrac{\partial^2 B}{\partial a^{22}}\right)_0 \\ (\omega^{21})_0\left(\dfrac{\partial^2 B}{\partial a^{12}}\right)_0 & (\omega^{21})_0\left(\dfrac{\partial^2 B}{\partial a^1 \partial a^2}\right)_0 \end{pmatrix}. \tag{7}$$

The characteristic equation of C is

$$\lambda^2 + (\omega^{12})_0^2\left[\left(\frac{\partial^2 B}{\partial a^{12}}\right)_0\left(\frac{\partial^2 B}{\partial a^{22}}\right)_0 - \left(\frac{\partial^2 B}{\partial a^1 \partial a^2}\right)_0^2\right] = 0; \tag{8}$$

and the roots must be in pair and in opposite signs. We have

Proposition 1 For the autonomous Birkhoff system (1), the characteristic roots of the Fréchet derivative C at the singular point (a_0^1, a_0^2) appear in pair and in opposite signs. If λ is a characteristic root, $-\lambda$ must be a root too.

Noticing the antisymmetry of $\omega^{\alpha\beta}$, we can obtain the derivative of the Birkhoffian B with respect to time t by Eq.(1)

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \frac{\partial B}{\partial a^1}\omega^{12}\frac{\partial B}{\partial a^2} + \frac{\partial B}{\partial a^2}\omega^{21}\frac{\partial B}{\partial a^1} \equiv 0. \tag{9}$$

Thus Eq.(1) have a first integral

$$B(a) = \text{constant}. \tag{10}$$

Choosing the Birkhoffian as follows:

$$B = \frac{1}{2}\sum_{\mu=1}^{2}\sum_{\nu=1}^{2} b_{\mu\nu}a^\mu a^\nu + W(a^\mu), \tag{11}$$

where $b_{\mu\nu}$ are constants with $\det(b_{\mu\nu}) \neq 0$, and W is the third- and higher-order terms of a^μ. In this situation, we have

$$\mathrm{D}B(a_0^1, a_0^2) = 0, \quad S = \mathrm{D}^2 B(a_0^1, a_0^2) \neq 0.$$

From proposition 1 we know that the characteristic roots of C is in pair. Suppose that C have a couple of pure imaginary characteristic roots $\pm\omega$i, using the Lyapunov centre theorem[14] we have the proposition as follows:

Proposition 2 For the autonomous Birkhoff system (1), there exists a C^2 first integral $B(a) : R^2 \to R$, if $\mathrm{D}B(a_0^1, a_0^2)$=0, $S = \mathrm{D}^2 B(a_0^1, a_0^2)$ is invertible, and the Fréchet derivative C have a couple of pure imaginary characteristic roots $\pm\omega$i. Then the system (1) has a one-parameter family of periodic orbits with period $2\pi/\omega$ in the neighbourhood of (a_0^1, a_0^2).

Because there exists periodic orbits $B(a_1, a_2) =$ constant in the neighbourhood of (a_0^1, a_0^2), we have

$$\begin{aligned} &\mathrm{div}\left(\omega^{12}\frac{\partial B}{\partial a^2}, \omega^{21}\frac{\partial B}{\partial a^1}\right) \\ =&\frac{\partial \omega^{12}}{\partial a^1}\frac{\partial B}{\partial a^2} + \omega^{12}\frac{\partial^2 B}{\partial a^1 \partial a^2} \\ &+ \frac{\partial \omega^{21}}{\partial a^2}\frac{\partial B}{\partial a^1} + \omega^{21}\frac{\partial^2 B}{\partial a^1 \partial a^2} \\ =&\frac{\partial \omega^{12}}{\partial a^1}\frac{\partial B}{\partial a^2} + \frac{\partial \omega^{21}}{\partial a^2}\frac{\partial B}{\partial a^1} \end{aligned}$$

is always equal to zero in the neighbourhood of (a_0^1, a_0^2), and these periodic orbits are not limit cycles. We obtain:

Proposition 3 For the autonomous Birkhoff system (1), there exist no limit cycles in the neighbourhood of (a_0^1, a_0^2).

4. Existence of algebraic limit cycles of a generalized second-order autonomous Birkhoff system

We study the algebraic limit cycles of a generalized second-order autonomous Birkhoff system in the form

$$\begin{aligned} \dot{a}^1 =& \omega^{12}\frac{\partial B}{\partial a^2} = P(a^1, a^2), \\ \dot{a}^2 =& \omega^{21}\frac{\partial B}{\partial a^1} + B(a^1, a^2) = Q(a^1, a^2), \end{aligned} \tag{12}$$

where $B(a^1, a^2) = 0$ is an n-th degree algebraic curve.

Proposition 4 If the number of closed branches of the n-th degree algebraic curve $B(a^1, a^2)$=0 reaches the maximum M, where[15]

$$M = \begin{cases} \dfrac{1}{2}(n-1)(n-2) + 1, & n \text{ is even number;} \\ \dfrac{1}{2}(n-1)(n-2), & n \text{ is odd number;} \end{cases}$$

and $\dfrac{\partial\omega^{21}}{\partial a^2}$ is not equal to zero at all the singular points, then the system (12) has M algebraic limit cycles.

Proof

$$\begin{aligned}
&\left.\frac{\mathrm{d}B}{\mathrm{d}t}\right|_{B(a^1,a^2)=0}\\
=&\left.\left(\frac{\partial B}{\partial a^1}\dot{a}^1+\frac{\partial B}{\partial a^2}\dot{a}^2\right)\right|_{B(a^1,a^2)=0}\\
=&\left.\left\{\frac{\partial B}{\partial a^1}\left(\omega^{12}\frac{\partial B}{\partial a^2}\right)\right.\right.\\
&\left.\left.+\frac{\partial B}{\partial a^2}\left[\omega^{21}\frac{\partial B}{\partial a^1}+B(a^1,a^2)\right]\right\}\right|_{B(a^1,a^2)=0}\\
=&0,
\end{aligned}$$

hence $B(a^1,a^2)=0$ is a solution of the system (12).

Now let us prove that each closed branch of $B(a^1,a^2)=0$ is an algebraic limit cycle of the system (12). For this purpose, it is only needed to prove that there exist no singular points on each closed branch of $B(a^1,a^2)=0$, and the system (12) does not have centres.

Suppose that (a_0^1,a_0^2) is a singular point on the closed branch of the system (12), i.e.,

$$B(a_0^1,a_0^2)=0.$$

Consider

$$\begin{aligned}
&\omega^{12}(a_0^1,a_0^2)\frac{\partial B(a_0^1,a_0^2)}{\partial a^2}=0,\\
&\omega^{21}(a_0^1,a_0^2)\frac{\partial B(a_0^1,a_0^2)}{\partial a^1}+B(a_0^1,a_0^2)=0, \qquad (13)
\end{aligned}$$

and

$$\omega^{12}=-\omega^{21}\neq 0,$$

thus we obtain

$$\frac{\partial B(a_0^1,a_0^2)}{\partial a^2}=\frac{\partial B(a_0^1,a_0^2)}{\partial a^1}=0. \qquad (14)$$

But when the number of closed branches of $B(a^1,a^2)=0$ reaches the maximum M, each closed branch of $B(a^1,a^2)=0$ is a smooth curve,[15] that is, $\partial B/\partial a^1$ and $\partial B/\partial a^2$ cannot be equal to zero at the same time at the points of the isolated closed branch of $B(a^1,a^2)=0$, which is a contradiction with Eq.(14). So it follows that $B(a_0^1,a_0^2)\neq 0$, that is, there exist no singular points on each closed branch of $B(a^1,a^2)=0$.

In addition, we prove that the system (12) has no centre. For this purpose, consider the divergence of the system (12)

$$\begin{aligned}
\mathrm{div}(P,Q)=&\frac{\partial\omega^{12}}{\partial a^1}\frac{\partial B}{\partial a^2}+\omega^{12}\frac{\partial^2 B}{\partial a^1\partial a^2}\\
&+\frac{\partial\omega^{21}}{\partial a^2}\frac{\partial B}{\partial a^1}+\omega^{21}\frac{\partial^2 B}{\partial a^1\partial a^2}+\frac{\partial B}{\partial a^2}\\
=&\frac{\partial\omega^{12}}{\partial a^1}\frac{\partial B}{\partial a^2}+\frac{\partial\omega^{21}}{\partial a^2}\frac{\partial B}{\partial a^1}+\frac{\partial B}{\partial a^2}.
\end{aligned}$$

If (a_0^1,a_0^2) is a singular point of the system (12), then Eq.(13) are satisfied. From Eq.(13) we know

$$\frac{\partial B(a_0^1,a_0^2)}{\partial a^2}=0, \quad \frac{\partial B(a_0^1,a_0^2)}{\partial a^1}\neq 0;$$

hence

$$\mathrm{div}(P,Q)|_{(a_0^1,a_0^2)}=\frac{\partial\omega^{21}(a_0^1,a_0^2)}{\partial a^2}\frac{\partial B(a_0^1,a_0^2)}{\partial a^1}.$$

So if

$$\frac{\partial\omega^{21}(a_0^1,a_0^2)}{\partial a^2}\neq 0,$$

we obtain

$$\mathrm{div}(P,Q)|_{(a_0^1,a_0^2)}\neq 0.$$

Here, the singular points of the system (12) do not pass the curve $\mathrm{div}(P,Q)=0$, so all the singular points of the system (12) are not centres. And because there exist no singular points in the closed branches of $B(a^1,a^2)=0$, each closed branch is a limit cycle of the system (12).

5. An example

Consider a generalized Birkhoff system

$$B=(a^1)^2+(a^2)^2-1, \quad \omega^{12}=-\omega^{21}=a^2+2.$$

Using Eq.(12), we obtain the generalized Birkhoff equations

$$\begin{aligned}
&\dot{a}^1=2(a^2+2)a^2,\\
&\dot{a}^2=-2(a^2+2)a^1+(a^1)^2+(a^2)^2-1. \qquad (15)
\end{aligned}$$

Because

$$B=(a^1)^2+(a^2)^2-1=0$$

is a quadratic algebraic curve, and its number of closed branches reaches the maximum M=1, and because

$$\frac{\partial\omega^{21}}{\partial a^2}=-1\neq 0$$

by proposition 4, we know that the closed branch $B=(a^1)^2+(a^2)^2-1=0$ is an algebraic limit cycle of the generalized Birkhoff system (15).

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