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Two expanding forms of a Lie algebra and their application*

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With the help of a known Lie algebra, two new high order Lie algebras are constructed. It is remarkable that they have different constructing approaches. The first Lie algebra is constructed by the definition of integrable couplings, the second one by an extension of Lie algebra. Then by making use of Tu scheme, a generalized AKNS hierarchy and another new hierarchy are obtained. As a reduction case of the first hierarchy, a kind of coupled KdV equation is presented. As a reduction case of the second one, a new coupled Schrödinger equation is given.

Keywords: loop algebra, integrable coupling, Tu scheme, AKNS hierarchy
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1. Introduction

Seeking for integrable system (referring to Lax integrability or Liouville integrability) has been an important problem in solitary theory.[1] Tu found a straightforward, efficient method which is called Tu-model to obtain integrable hierarchies of equations, such as the AKNS hierarchy, AC hierarchy, KN hierarchy, etc. In order that the hierarchies with multiple potential functions be obtained, Guo obtained a few interesting results by definition of gradation for the bases of Lie algebra $\tilde{A}_1$.[2,3] In this paper, we expand a basis of an existing Lie algebra $\tilde{A}_1$ to two new high order Lie algebras. The first Lie algebra is established by the method of integrable couplings, the second one by a direct extension method of Lie algebra. As their application, a generalized AKNS hierarchy and another new integrable hierarchy are obtained, respectively. As reduction cases of the two hierarchies, a coupled KdV system and a new coupled Schrödinger equation are presented, respectively.

2. The method of integrable couplings

The definition of integrable couplings appeared in Ref.[4]. By making use of it, expanding integrable systems of some known systems were obtained.[5−7] Let us consider a known Lie algebra $\tilde{A}_1$ [1]

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h. \tag{1}$$

Now we expand it into the following set of 3×3 matrices:

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

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$$e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix};$$

$$\begin{aligned}
&[e_1, e_2] = 2e_2, \\
&[e_1, e_3] = -2e_3, \\
&[e_2, e_3] = e_1, \\
&[e_1, e_4] = e_4, \\
&[e_1, e_5] = -e_5, \\
&[e_2, e_4] = 0, \\
&[e_2, e_5] = e_4, \\
&[e_3, e_4] = e_5, \\
&[e_3, e_5] = 0, \\
&[e_4, e_5] = 0.
\end{aligned} \tag{2}$$

Let

$$\begin{aligned}
a &= \sum_{i=1}^{5} a_i e_i, \\
b &= \sum_{i=1}^{5} b_i e_i, \\
c &= \sum_{i=1}^{5} c_i e_i,
\end{aligned}$$

where a_i, b_i, c_i are all constants or functions, then we have $[[a,b],c] + [[b,c],a] + [[c,a],b] = 0$, i.e., Jacobi identity holds. Therefore, Eq.(2) is a Lie algebra.

Furthermore, let

$$\begin{aligned}
&e_i(n) = e_i \lambda^n, \\
&[e_i(m), e_j(n)] = [e_i, e_j]\lambda^{m+n}, \\
&\deg e_i(n) = n, \quad 1 \le i,\ j \le 5,
\end{aligned} \tag{3}$$

then a loop algebra is obtained.

In terms of Eq.(3), we consider

$$\begin{aligned}
\phi_x =& U\phi, \\
\phi =& (\phi_1, \phi_2)^{\mathrm{T}}, \\
\lambda_t =& 0, \\
U =& -e_1(1) + u_1 e_2(0) + u_2 e_3(0) \\
&+ u_3 e_4(0) + u_4 e_5(0).
\end{aligned} \tag{4}$$

Again set $V = \sum_{m=0}^{\infty}(a_m e_1(-m) + b_m e_2(-m) + c_m e_3(-m) + d_m e_4(-m) + f_m e_5(-m))$, solving the adjoint equation

$$V_x = [U, V], \tag{5}$$

yields that

$$\begin{aligned}
b_{m+1} =& -\frac{1}{2} b_{mx} - u_1 a_m, \\
c_{m+1} =& \frac{1}{2} c_{mx} - u_2 a_m, \\
a_{mx} =& u_1 c_m - u_2 b_m, \\
d_{m+1} =& -d_{mx} + u_1 f_m - u_3 a_m - u_4 b_m, \\
f_{m+1} =& f_{mx} - u_2 d_m + u_3 c_m - u_4 a_m, \\
a_0 =& \beta = \text{constant} \neq 0, \\
b_0 =& c_0 = d_0 = f_0 = 0.
\end{aligned} \tag{6}$$

Set $V_+^{(n)} = \sum_{m=0}^{n}(a_m e_1(n-m) + b_m e_2(n-m) + c_m e_3(n-m) + d_m e_4(n-m) + f_m e_5(n-m))$, $V_-^{(n)} = \lambda^n V - V_+^{(n)}$, then Eq.(5) can be written as

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = V_{-x}^{(n)} - [U, V_-^{(n)}]. \tag{7}$$

It is easy to calculate that

$$\begin{aligned}
&-V_{+x}^{(n)} + [U, V_+^{(n)}] \\
=& 2b_{n+1} e_2(0) - 2c_{n+1} e_3(0) + d_{n+1} e_4(0) - f_{n+1} e_5(0).
\end{aligned}$$

Note $V^{(n)} = V_+^{(n)}$, then the zero-curvature equation

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0 \tag{8}$$

determines the following system:

$$u_{t_n} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_{t_n} = \begin{pmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_{n+1} \\ b_{n+1} \\ f_{n+1} \\ d_{n+1} \end{pmatrix} = JG_{n+1}. \tag{9}$$

From Eq.(6), we find that

$$G_{n+1} = \begin{pmatrix} -\frac{1}{2}\partial + u_2\partial^{-1}u_1 & -u_2\partial^{-1}u_2 & 0 & 0 \\ u_1\partial^{-1}u_1 & \frac{1}{2}\partial - u_1\partial^{-1}u_2 & 0 & 0 \\ -u_3 + u_4\partial^{-1}u_1 & -u_4\partial^{-1}u_2 & -\partial & u_2 \\ u_3\partial^{-1}u_1 & u_4 - u_3\partial^{-1}u_2 & -u_1 & \partial \end{pmatrix} G_n = LG_n.$$

Thus Eq.(9) can be rewritten as

$$u_{t_n} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_{t_n} = JL^nG_1 = JL^n \begin{pmatrix} \beta u_2 \\ \beta u_1 \\ \beta u_4 \\ \beta u_3 \end{pmatrix}. \quad (10)$$

Taking $u_3 = u_4 = 0$ in Eq.(10) gives rise to the well-known AKNS hierarchy,

$$u_{t_n} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{t_n} = JL^n \begin{pmatrix} \beta u_2 \\ \beta u_1 \end{pmatrix}, \quad (11)$$

where

$$J = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix},$$

$$L = \begin{pmatrix} -\frac{1}{2}\partial + u_2\partial^{-1}u_1 & -u_2\partial^{-1}u_2 \\ u_1\partial^{-1}u_1 & \frac{1}{2}\partial - u_1\partial^{-1}u_2 \end{pmatrix}.$$

In particular, taking $n = 3,\ u_1 = u,\ u_2 = 1,\ \beta = 4$ in Eq.(10), a coupled KdV equation is obtained as follows:

$$\begin{aligned} u_t =& u_{xxx} - 6uu_x, \\ u_{3t} =& 4u_{3xxx} - 6uu_{3x} - 3u_xu_3 + 3u_{xx}u_4 + 6u_xu_{4x}, \\ u_{4t} =& 4u_{4xxx} - u_xu_4 - 2uu_{4x}. \end{aligned}$$

3. A direct extension method

Let $A^*_{n-1} = gl(n, C) = \{(x_{ij})_{n\times n} | x_{ij} \in C\}$, where C denotes a set of complex numbers. For a given $Q \in A^*_{n-1}$, we define an operation

$$[X, Y] = XQY - YQX, \quad \forall X, \quad Y \in A^*_{n-1}. \quad (12)$$

It is easy to verify that $[X, Y]$ satisfies

1) shew-symmetry:

$$[X, Y] = -[Y, X];$$

2) bilinear:

$$[aX + bY, Z] = a[X, Z] + b[Y, Z];$$

3) Jacobi identity:

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

Therefore, A^*_{n-1} is a Lie algebra. When Q is taken as a unit matrix, Eq.(12) reduces to $[X, Y] = XY - YX$. Hence, the Lie algebra A^*_{n-1} is an extension of the Lie algebra A_{n-1}.

For the sake of convenience, we only consider the following case:

Taking

$$Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we expand Eq.(1) into

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (13)$$

In terms of Eq.(12), we have

$$\begin{aligned} &[h, e] = p, \\ &[h, f] = p, \\ &[h, p] = 2(e + f), \\ &[e, f] = 0, \\ &[e, p] = -h, \\ &[f, p] = -h. \end{aligned} \quad (14)$$

Obviously, Eqs.(13), (14) consist of a Lie algebra, which is an extension form of Lie algebra Eq.(1).

Let

$$\begin{aligned}
h(n) =& h\lambda^{2n+1},\\
e(n) =& e\lambda^{2n},\\
p(n) =& p\lambda^{2n+1},\\
f(n) =& f\lambda^{2n},\\
[h(m), e(n)] =& p(m+n),\\
[h(m), f(n)] =& p(m+n),\\
[h(m), p(n)] =& 2(e(m+n+1) + f(m+n+1)),\\
[e(m), f(n)] =& 0,\\
[e(m), p(n)] =& -h(m+n),\\
[f(m), p(n)] =& -h(m+n),\\
\deg h(n) =& \deg p(n) = 2n+1,\\
\deg e(n) =& \deg f(n) = 2n. \qquad (15)
\end{aligned}$$

then Eq.(15) is a loop algebra.

Let

$$\begin{aligned}
\varphi_x =& UQ\varphi, \quad \lambda_t = 0,\\
\varphi_t =& VQ\varphi,
\end{aligned}$$

then $\varphi_{xt} = \varphi_{tx}$ gives rise to

$$\begin{aligned}
&U_tQ\varphi + UQVQ\varphi = V_xQ\varphi + VQUQ\varphi,\\
&[U_t - V_x + (UQV - VQU)]Q\varphi = 0.
\end{aligned}$$

Since φ is arbitrary, we have

$$U_t - V_x + [U, V] = 0. \qquad (16)$$

We also call Eq.(16) the zero-curvature equation. Evolution equation determined by Eq.(16) is also called Lax integrable.

Specially, we take

$$\begin{aligned}
U =& \begin{pmatrix} (q+r)\lambda & \lambda^2 + s\\ s & (-q+r)\lambda \end{pmatrix}\\
=& e(1) + qh(0) + rp(0) + s(e(0) + f(0)),\\
V =& \begin{pmatrix} (a+c)\lambda & b\\ 0 & (-a+c)\lambda \end{pmatrix}\\
=& ah(0) + be(0) + cp(0),
\end{aligned}$$

solving

$$V_x = [U, V], \qquad (17)$$

yields that

$$\begin{aligned}
c_{m+1} =& -a_{mx} + rb_m - 2sc_m,\\
a_{m+1} =& -c_{mx} + qb_m - 2sa_m,\\
b_{mx} =& 2qc_{m+1} - 2ra_{m+1} = -2qa_{mx} + 2rc_{mx} - 4qsc_m + 4rsa_m,\\
a_0 =& c_0 = 0, \quad b_0 = \alpha, \quad c_1 = \alpha r, \quad a_1 = \alpha q, \quad b_1 = 0,\\
c_2 =& -\alpha q_x - 2\alpha sr, \quad a_2 = -\alpha r_x - 2\alpha qs,\\
b_2 =& 2\alpha(qr_x - q_xr) + 2\alpha s(q^2 - r^2),\\
a_3 =& \alpha q_{xx} + \alpha(sr)_x + 2\alpha q^2 r_x - 2\alpha qq_x r + 2\alpha qs(q^2 - r^2) + 2\alpha sr_x + 2\alpha qs^2,\\
c_3 =& \alpha r_{xx} + \alpha(qs)_x + 2\alpha qrr_x - 2\alpha r^2 q_x + 2\alpha rs(q^2 - r^2) + 2\alpha sq_x + 2\alpha s^2 r. \qquad (18)
\end{aligned}$$

A direct calculation gives that

$$-V^{(n)}_{+x} + [U, V^{(n)}_+] = a_{n+1}p(0) + c_{n+1}h(0) + (2ra_{n+1} - 2qc_{n+1})(e(0) + f(0)).$$

Taking $V^{(n)} = V^{(n)}_+$, then from Eq.(16), we have

$$u_{t_n} = \begin{pmatrix} q\\ r\\ s \end{pmatrix}_{t_n} = \begin{pmatrix} -c_{n+1}\\ -a_{n+1}\\ 2qc_{n+1} - 2ra_{n+1} \end{pmatrix} = \begin{pmatrix} -c_{n+1}\\ -a_{n+1}\\ b_{nx} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2} & 0\\ -\frac{1}{2} & 0 & 0\\ 0 & 0 & \partial \end{pmatrix} \begin{pmatrix} 2a_{n+1}\\ 2c_{n+1}\\ b_n \end{pmatrix} = JG_n. \qquad (19)$$

From Eq.(18), we find that

$$G_n = \begin{pmatrix} -2q\partial^{-1}q\partial - 2s & -\partial + 2q\partial^{-1}r\partial & -2q\\ -\partial - 2r\partial^{-1}q\partial & 2r\partial^{-1}r\partial - 2s & 2r\partial^{-1}r\partial\\ -2\partial^{-1}q\partial & 2\partial^{-1}r\partial & -2 \end{pmatrix} G_{n-1} = LG_{n-1}.$$

Thus, Eq.(19) can be written as

$$u_{t_n} = \begin{pmatrix} q \\ r \\ s \end{pmatrix}_{t_n} = JL^n G_0 = JL^n \begin{pmatrix} 2\alpha q \\ 2\alpha r \\ \alpha \end{pmatrix}. \quad (20)$$

Taking $n = 2$ in Eq.(20), we obtain a new coupled Schrödinger equation as

$$\begin{aligned} q_{t_2} =& -\alpha r_{xx} - \alpha(qs)_x - 2\alpha q r r_x + 2\alpha r^2 q_x \\ & - 2\alpha r s(q^2 - r^2) - 2\alpha s q_x - 2\alpha s^2 r, \\ r_{t_2} =& -\alpha q_{xx} - \alpha(sr)_x - 2\alpha q^2 r_x + 2\alpha q q_x r \\ & - 2\alpha q s(q^2 - r^2) - 2\alpha s r_x - 2\alpha q s^2, \\ s_{t_2} =& 2\alpha(q r_{xx} - q_{xx} r) + 2\alpha s_x(q^2 - r^2) \\ & + 4\alpha s(q q_x - r r_x). \end{aligned}$$

Remarks

In this paper, two methods for expanding known Lie algebra are presented. Specially, the second method is proposed first. But there exist open problems. The two hierarchies (10) and (20) obtained in this paper are integrable only in Lax sense. How do we verify they are integrable in the Liouville sense? How do we obtain their Hamiltonian structures? These questions are worth while studying in the future.

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