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Canonical symmetry properties of the constrained singular generalized mechanical system

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Based on generalized Apell–Chetaev constraint conditions and to take the inherent constrains for singular Lagrangian into account, the generalized canonical equations for a general mechanical system with a singular higher-order Lagrangian and subsidiary constrains are formulated. The canonical symmetries in phase space for such a system are studied and Noether theorem and its inversion theorem in the generalized canonical formalism have been established.

Keywords: generalized mechanics, subsidiary constraints, singular Lagrangian, canonical Noether theorem

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1. Introduction

As is well known, the symmetries of a system have close relation with conservation laws. There are many methods to study the symmetry of the system, such as Noether theorem,[1,2] Lie symmetry[2] and form invariance,[3−6], etc. In general, the invariance of the system under the action of finite continuous group implies the conservation laws. However, Noether theorem and its generalization for constrained and nonholomonic mechanical systems are usually based on an examination of the Lagrange's variables in configuration space.[7−10] A system with a singular Lagrangian (for example, the gauge invariance system) is subject to some inherent phase space constraints even if there is no subsidiary constraint and is called constrained Hamiltonian system. The classical and quantal canonical symmetry properties for such a system have been investigated recently.[11]

Dynamical system described in terms of higher-order derivatives have been discussed for a long time in connection with gauge theories, gravity theory, modified Korteweg–de Vries (KdV) equations, supersymmetry, string models and so on. Recently, higher-order-derivative theories obtained by many authors have exhibited a lot of interesting aspects. The dynamical equations and symmetries in configuration space for a generalized mechanical system with a regular Lagrangian[12] and with subsidiary constraints[1] have been studied.

The canonical symmetries in phase space for a system with a singular higher-order Lagrangian (without subsidiary constraint) have been also discussed.[11] The approach presented here is a more general case: studying the symmetry properties in canonical formalism for a generalized mechanical system with a singular higher-order Lagrangian and with subsidiary constraints [called a higher-order constrained singular generalized mechanical system (HOCSGMS)].

In this paper, based on generalized Apell–Chetaev conditions, the generalized canonical equations for a HOCSGMS are obtained. The canonical symmetries in phase space are investigated and the generalized canonical Noether theorem and its inverse theorem are derived for such a system.

2. Generalized canonical equations of constrained generalized dynamical system

Consider a generalized dynamical system with higher-order derivatives described by a singular Lagrangian, whose Lagrangian is given by

$$L = L(t; q^i_{(0)}(t), q^i_{(1)}(t), \cdots, q^i_{(N)}(t)),$$

where

$$q^i_{(s)} = \frac{\mathrm{d}^s}{\mathrm{d}t^s} q^i = D^s q^i \quad (i = 1, 2, \cdots, n),$$

and the system whose motion is subject to subsidiary higher-order nonholonomic constraints[13]

$$G_w(t; q^i_{(0)}(t), q^i_{(1)}(t), \cdots, q^i_{(M)}(t)) = 0$$
$$(w = 1, 2, \cdots, l; M \leq N). \tag{1}$$

Using Ostrogradsky transformation, the generalized canonical momenta is defined by[1]

$$p_i^{(N-1)} = \frac{\partial L}{\partial q^i_{(N)}}, \tag{2a}$$

$$p_i^{(s-1)} = \frac{\partial L}{\partial q^i_{(s)}} - \dot{p}_i^{(s)} \quad (s = 1, 2, \cdots, N-1). \tag{2b}$$

The generalized canonical Hamiltonian of the system is given by[1]

$$H_{\rm c} = \sum_{i=1}^{n} \sum_{s=0}^{N-1} p_i^{(s)} q^i_{(s+1)} - L(t; q^i_{(s)}, \cdots, q^i_{(N)}). \tag{3}$$

For a system with a singular higher-order Lagrangian, its generalized Hess matrix $[\partial^2 L/\partial q^i_{(N)} \partial q^j_{(N)}]$ is degenerate, from Eq.(2a) one cannot solve all $q^i_{(N)}$ as functions of $t, q^i_{(N)}, p_i^{(N)}$. This implies that there are inherent constraints.[11] We assume that the equations of motion of the system is determined by inherent constraints in phase space

$$\phi_a(q^i_{(s)}, p_i^{(s)}) = 0 \quad (a = 1, 2, \cdots, k). \tag{4}$$

Let us suppose that we have taken into account the compatibility between the constraints $\phi_a = 0$ $(a = 1, 2, \cdots, k)$ arising from the singularity of the Lagrangian and the given subsidiary constraints $G_w = 0$ $(w = 1, 2, \cdots, l)$.

From the variation of the generalized canonical Hamiltonian, one obtains [11]

$$\sum_{i=1}^{n} \sum_{s=0}^{N-1} \left(\dot{q}^i_{(s)} - \frac{\partial H_{\rm c}}{\partial p_i^{(s)}} \right) \delta p_i^{(s)} + \sum_{i=1}^{n} \sum_{s=0}^{N-1} \left(-\dot{p}_i^{(s)} - \frac{\partial H_{\rm c}}{\partial q^i_{(s)}} \right) \delta q^i_{(s)} = 0. \tag{5}$$

For a regular Lagrange system without subsidiary constraints, $\delta q^i_{(s)}$ and $\delta p_i^{(s)}$ are mutually independent. From Eq.(5) one can obtain the generalized canonical equations. But, for a singular Lagrangian system, $q^i_{(s)}$ and $p_i^{(s)}$ are not independent, the variations of the canonical variables satisfy the conditions

$$\sum_{i=1}^{n} \sum_{s=0}^{N-1} \left(\frac{\partial \phi_a}{\partial q^i_{(s)}} \delta q^i_{(s)} + \frac{\partial \phi_a}{\partial p_i^{(s)}} \delta p_i^{(s)} \right) = 0. \tag{6}$$

For a generalized dynamical system with subsidiary constraints, the Mth-order simultaneous variations $\delta q^i_{(M)}$ must satisfy the Apell–Chetaev conditions[13]

$$\delta G_w = \sum_{i=1}^{n} \frac{\partial G_w}{\partial q^i_{(M)}} \delta q^i_{(M-1)} = 0$$
$$(w = 1, 2, \cdots, l; M \leq N). \tag{7}$$

Introducing the Lagrange multipliers $\mu^a(t)$, and $\lambda^w(t)$ multiplying (6) by μ^a and multiplying (7) by λ^w, and combining the expression (5), one obtains

$$\begin{aligned} &\sum_{i=1}^{n} \sum_{s=0}^{N-1} \left(\dot{q}^i_{(s)} - \frac{\partial H_{\rm c}}{\partial p_i^{(s)}} - \mu^a \frac{\partial \phi_a}{\partial p_i^{(s)}} \right) \delta p_i^{(s)} \\ &+ \sum_{i=1}^{n} \sum_{s=0}^{N-1}{}' \left(-\dot{p}_i^{(s)} - \frac{\partial H_{\rm c}}{\partial q^i_{(s)}} - \mu^a \frac{\partial \phi_a}{\partial q^i_{(s)}} \right) \delta q^i_{(s)} \\ &+ \sum_{i=1}^{n} \left(-\dot{p}_i^{(M-1)} - \frac{\partial H_{\rm c}}{\partial q^i_{(M-1)}} - \mu^a \frac{\partial \phi_a}{\partial q^i_{(M-1)}} \right. \\ &\left. - \lambda^w \frac{\partial G_w}{\partial q^i_{(M)}} \right) \delta q^i_{(M-1)} = 0, \end{aligned} \tag{8}$$

where $\sum'$ does not include $s = M - 1$. According to Lagrange multiplier method, from Eq.(8), one finds

$$\dot{q}^i_{(s)} = \frac{\partial H_{\rm c}}{\partial p_i^{(s)}} + \mu^a \frac{\partial \phi_a}{\partial p_i^{(s)}}$$
$$(i = 1, 2, 3, \cdots, n; s = 0, 1, 2, 3, \cdots, N-1), \tag{9a}$$

$$\dot{p}_i^{(s)} = -\frac{\partial H_{\rm c}}{\partial q^i_{(s)}} - \mu^a \frac{\partial \phi_a}{\partial q^i_{(s)}}$$
$$(i = 1, 2, 3, \cdots, n; s = 1, 2, 3, \cdots, N-1; s \neq M-1), \tag{9b}$$

$$\dot{p}_i^{(M-1)} = -\frac{\partial H_{\rm c}}{\partial q^i_{(M-1)}} - \mu^a \frac{\partial \phi_a}{\partial q^i_{(M-1)}} - \lambda^w \frac{\partial G_w}{\partial q^i_{(M)}}$$
$$(i = 1, 2, 3, \cdots, n; M \leq N). \tag{9c}$$

The expression (9) is the generalized canonical equations of HOCSGMS.

For the generalized dynamical system described by the singular Lagrangian with subsidiary holonomic constraints $G_w(t; q^i) = 0$, the simultaneous variations δq^i must satisfy[13]

$$\delta G_w = \frac{\partial G_w}{\partial q^i} \delta q^i = 0. \tag{7a}$$

Introducing the Lagrange multipliers $\mu^a(t)$ and $\lambda^w(t)$, and multiplying (6) by μ^a and multiplying (7a) by λ^w,

and combining (5), one obtains

$$\sum_{i=1}^{n}\sum_{s=0}^{N-1}\left(\dot{q}^i_{(s)}-\frac{\partial H_{\rm c}}{\partial p^{(s)}_i}-\mu^a\frac{\partial\phi_a}{\partial p^{(s)}_i}\right)\delta p^{(s)}_i + \sum_{i=1}^{n}\sum_{s=0}^{N-1}\left(-\dot{p}^{(s)}_i-\frac{\partial H_{\rm c}}{\partial q^i_{(s)}} - \mu^a\frac{\partial\phi_a}{\partial q^i_{(s)}}-\lambda^w\frac{\partial G_w}{\partial q^i_{(s)}}\right)\delta q^i_{(s)}=0.$$

Thus, we can still proceed in the same way to obtain the following generalized canonical equations

$$\dot{q}^i_{(s)}=\frac{\partial H_{\rm c}}{\partial p^{(s)}_i}+\mu^a\frac{\partial\phi_a}{\partial p^{(s)}_i}$$

$$(i=1,2,3,\cdots,n;\ s=0,1,2,3,\cdots,N-1), \tag{10a}$$

$$\dot{p}^{(s)}_i=-\frac{\partial H_{\rm c}}{\partial q^i_{(s)}}-\mu^a\frac{\partial\phi_a}{\partial q^i_{(s)}}-\lambda^w\frac{\partial G_w}{\partial q^i_{(s)}}$$

$$(i=1,2,3,\cdots,n;\ s=0,1,2,3,\cdots,N-1). \tag{10b}$$

For a system without additional subsidiary constraints, expressions (9) and (10) can be deduced to the generalized canonical equations of a generalized dynamical system described by a singular Lagrangian.[14]

3. Generalized canonical Noether theorem

Let us first discuss the classical canonical symmetry properties of a system described by a singular higher-order Lagrangian and with non-holonomic additional constraints. Consider the finite continuous group with the infinitesimal transformation

$$\begin{cases} t'=t+\Delta t=t+\varepsilon_\sigma\tau^\sigma(t,q^i_{(s)},p^{(s)}_i),\\ {q^i_{(s)}}'(t')=q^i_{(s)}(t)+\Delta q^i_{(s)}(t)\\ \qquad\quad =q^i_{(s)}(t)+\varepsilon_\sigma\xi^{i\sigma}_{(s)}(t,q^i_{(s)},p^{(s)}_i),\\ {p^{(s)}_i}'(t')=p^{(s)}_i(t)+\Delta p^{(s)}_i(t)\\ \qquad\quad =p^{(s)}_i(t)+\varepsilon_\sigma\eta^{(s)\sigma}_i(t,q^i_{(s)},p^{(s)}_i), \end{cases} \tag{11}$$

where ε_σ $(\sigma=1,2,\cdots,r)$ are infinitesimal arbitrary parameters, and $\tau^\sigma,\xi^{i\sigma}_{(s)},\eta^{(s)\sigma}_i$ are functions of canonical variables, the summation is taken over repeated indices. Let it be supposed that the generalized canonical Lagrangian $L_p=p^{(s)}_i q^i_{(s+1)}-H_{\rm c}(q^i_{(s)},p^{(s)}_i)$ is invariant up to an exact differential term, $\delta L^p=\varepsilon_\sigma D\Lambda^\sigma$, $\Lambda^\sigma=\Lambda^\sigma(t;q^i_{(s)},p^{(s)}_i)$ under the transformation (11). It is supposed that the Mth-order simultaneous variations determined by Eq.(11) satisfy Apell–Chetaev conditions, and the total variations of the inherent constraint conditions (4) are invariant under Eq.(11), i.e.

$$\Delta\phi_a=\frac{\partial\phi_a}{\partial t}\Delta t+\frac{\partial\phi_a}{\partial q^i_{(s)}}\Delta q^i_{(s)}+\frac{\partial\phi_a}{\partial p^{(s)}_i}\Delta p^{(s)}_i=0. \tag{12}$$

The relation between the total variation and the simultaneous variation can be expressed as

$$\Delta q^i_{(s)}=\delta q^i_{(s)}+\dot{q}^i_{(s)}\Delta t,\quad \Delta p^{(s)}_i=\delta p^{(s)}_i+\dot{p}^{(s)}_i\Delta t. \tag{13}$$

Owing to the consistency conditions of constraints, one obtains

$$\dot{\phi}_a=\frac{\partial\phi_a}{\partial t}+\frac{\partial\phi_a}{\partial q^i_{(s)}}\dot{q}^i_{(s)}+\frac{\partial\phi_a}{\partial p^{(s)}_i}\dot{p}^{(s)}_i=0. \tag{14}$$

From Eqs.(12)–(14), one obtains

$$\delta\phi_a=\frac{\partial\phi_a}{\partial q^i_{(s)}}\delta q^i_{(s)}+\frac{\partial\phi_a}{\partial p^{(s)}_i}\delta p^{(s)}_i=0. \tag{15}$$

Conversely, one can also deduce Eq.(12) from Eq.(15). The variation of the canonical action $I^p=\int_{t_1}^{t_2}(p^{(s)}_i q^i_{(s+1)}-H_{\rm c})\mathrm{d}t$ under the transformation (11) is given by[11]

$$\int_{t_1}^{t_2}\left\{\left(\dot{q}^i_{(s)}-\frac{\partial H_{\rm c}}{\partial p^{(s)}_i}\right)\delta p^{(s)}_i+\left(-\dot{p}^{(s)}_i-\frac{\partial H_{\rm c}}{\partial q^i_{(s)}}\right)\delta q^i_{(s)} + D[p^{(s)}_i\delta q^i_{(s)}+(p^{(s)}_i q^i_{(s+1)}-H_{\rm c})\Delta t]\right\}\mathrm{d}t = \int_{t_1}^{t_2}\varepsilon_\sigma D\Lambda^\sigma\,\mathrm{d}t, \tag{16a}$$

then one has

$$\left(\dot{q}^i_{(s)}-\frac{\partial H_{\rm c}}{\partial p^{(s)}_i}\right)\delta p^{(s)}_i+\left(-\dot{p}^{(s)}_i-\frac{\partial H_{\rm c}}{\partial q^i_{(s)}}\right)\delta q^i_{(s)} + D[p^{(s)}_i\delta q^i_{(s)}+(p^{(s)}_i q^i_{(s+1)}-H_{\rm c})\Delta t]=\varepsilon_\sigma D\Lambda^\sigma. \tag{16b}$$

Multiplying Eq.(15) by $\mu^a(t)$ and multiplying Eq.(7) by $\lambda^w(t)$, and using the equation of motion (9), i.e. along the trajectory of motion of a singular constrained system, one obtains

$$D[p^{(s)}_i\delta q^i_{(s)}+(p^{(s)}_i q^i_{(s+1)}-H_{\rm c})\Delta t]=\varepsilon_\sigma D\Lambda^\sigma. \tag{17}$$

Using the expressions (11) and (12), one obtains

$$p^{(s)}_i\xi^{i\sigma}_{(s)}-H_{\rm c}\tau_\sigma-\Lambda^\sigma=\text{constant}\quad(\sigma=1,2,\cdots,r). \tag{18}$$

Therefore, we have the following Noether theorem in the generalized canonical formalism for a constrained singular generalized dynamical system: if, the canonical Lagrangian is invariant up to an exact differential

term under the transformation (11), the total variations of inherent constraints $\phi_a = 0$ are invariant and the simultaneous variations of additional constraints G_w=0 satisfy generalized Apell–Chetaev conditions (7), then, the expressions (18) give constants of motion for a HOCSGMS in the phase space.

Second, let us consider a generalized dynamical system described by a singular Lagrangian with holonomic constraints. It is supposed that the total variations of the subsidiary constraint conditions are invariant, i.e. under Eq.(11), one has

$$\Delta G_w = \frac{\partial G_w}{\partial t}\Delta t + \frac{\partial G_w}{\partial q^i}\Delta q^i = 0. \tag{19}$$

Owing to the consistency conditions of the subsidiary holonomic constraints, one obtains

$$\dot{G}_w = \frac{\partial G_w}{\partial t} + \frac{\partial G_w}{\partial q^i}\dot{q}^i = 0. \tag{20}$$

From Eqs.(13), (19) and (20), one obtains that the simultaneous variations δq^i satisfy following conditions

$$\delta G_w = \frac{\partial G_w}{\partial q^i}\delta q^i = 0. \tag{21}$$

Thus we can still proceed in the same way, and can obtain the generalized canonical Noether theorem for a generalized dynamical system described by a singular Lagrangian with holonomic constraints. If the canonical Lagrangian of a system is invariant up to an exact differential term, and the total variations of constraints (including inherent constraints and subsidiary constraints) are invariant under the transformation (11), then, the expressions (18) give constants of motion for a HOCSGMS in the phase space.

4. Inverse theorem of the canonical Noether theorem

We now find the conditions under which the inversion of the Noether theorem is possible in the canonical formalism for a HOCSGMS. Suppose that we know r independent conservative quantities in phase space for a HOCSGMS which is given by

$$D^\sigma(t; q^i_{(s)}, p^{(s)}_i) = C^\sigma \quad (\sigma = 1, 2, \cdots, r). \tag{22}$$

For this system, any dynamical trajectory of the motion satisfying Eqs. (9), it follows that

$$\begin{aligned}
&\left(\dot{q}^i_{(s)} - \frac{\partial H_{\rm c}}{\partial p^{(s)}_i} - \mu^a\frac{\partial \phi_a}{\partial p^{(s)}_i}\right)\overline{\eta}^{(s)\sigma}_i \\
&+ \left(-\dot{p}^{(s)}_i - \frac{\partial H_{\rm c}}{\partial q^i_{(s)}} - \mu^a\frac{\partial \phi_a}{\partial q^i_{(s)}}\right)\overline{\xi}^{i\sigma}_{(s)} \\
&+ \left(-\dot{p}^{(M-1)}_i - \frac{\partial H_{\rm c}}{\partial q^i_{(M-1)}}\right. \\
&\left. - \mu^a\frac{\partial \phi_a}{\partial q^i_{(M-1)}} - \lambda^w\frac{\partial G_w}{\partial q^i_{(M)}}\right)\overline{\xi}^{i\sigma}_{(M-1)} = 0,
\end{aligned} \tag{23}$$

where the second term in Eq.(23) does not include the term for $s = M-1$, and

$$\overline{\xi}^{i\sigma}_{(s)} = \xi^{i\sigma}_{(s)} - \dot{q}^i_{(s)}\tau^\sigma; \quad \overline{\eta}^{(s)\sigma}_i = \eta^{(s)\sigma}_i - \dot{p}^{(s)}_i\tau^\sigma. \tag{24}$$

Differentiating (22) with respect to time t and combining expression (23), we obtain

$$\begin{aligned}
&\frac{\partial D^\sigma}{\partial t} + \frac{\partial D^\sigma}{\partial q^i_{(s)}}\dot{q}^i_{(s)} + \frac{\partial D^\sigma}{\partial p^{(s)}_i}\dot{p}^{(s)}_i \\
&+ \left(\dot{q}^i_{(s)} - \frac{\partial H_{\rm c}}{\partial p^{(s)}_i} - \mu^a\frac{\partial \phi_a}{\partial p^{(s)}_i}\right)\overline{\eta}^{(s)\sigma}_i \\
&+ \left(-\dot{p}^{(s)}_i - \frac{\partial H_{\rm c}}{\partial q^i_{(s)}} - \mu^a\frac{\partial \phi_a}{\partial q^i_{(s)}}\right)\overline{\xi}^{i\sigma}_{(s)} \\
&+ \left(-\dot{p}^{(M-i)}_i - \frac{\partial H_{\rm c}}{\partial q^i_{(M-1)}}\right. \\
&\left. - \mu^a\frac{\partial \phi_a}{\partial q^i_{(M-1)}} - \lambda^w\frac{\partial G_w}{\partial q^i_{(M)}}\right)\overline{\xi}^{i\sigma}_{(M-1)} = 0,
\end{aligned} \tag{25}$$

where $\sigma = 1, 2, \cdots, r$. These relations are to be fulfilled for any dynamical trajectory of motion, this leads to terms containing the time derivatives of $\dot{p}^{(s)}_i$ cancelling each other; thus

$$\overline{\xi}^{i\sigma}_{(s)} = \frac{\partial D^\sigma}{\partial p^{(s)}_i}. \tag{26}$$

Now let

$$D^\sigma = p^{(s)}_i\overline{\xi}^{i\sigma}_{(s)} - H_{\rm c}\tau^\sigma - \Lambda, \tag{27}$$

hence

$$\tau^\sigma = H_{\rm c}^{-1}(p^{(s)}_i\overline{\xi}^{i\sigma}_{(s)} - D^\sigma - \Lambda). \tag{28}$$

When $\overline{\xi}^{i\sigma}_{(s)}$, τ^σ have been found, using expression $\xi^{i\sigma}_{(s)} = \overline{\xi}^{i\sigma}_{(s)} + \dot{q}^i_{(s)}\tau^\sigma$, one can find $\xi^{i\sigma}_{(s)}$, The $\overline{p}^{(s)}_i$ are completely determined in terms of $\xi^{i\sigma}_{(s)}$, and τ^σ, the corresponding quantities $\eta^{(s)\sigma}_i$ can be obtained as functions of τ^σ, $\xi^{i\sigma}_{(s)}$.

Thus, we have obtained τ^σ, $\xi^{i\sigma}_{(s)}$, $\eta^{(s)\sigma}_i$ as generating functions of the infinitesimal transformation (11) in extend phase space. If these generators satisfy conditions

$$\frac{\partial \phi_a}{\partial q^i_{(s)}}(\xi^{i\sigma}_{(s)} - \dot{q}^i_{(s)}\tau^\sigma) + \frac{\partial \phi_a}{\partial p^{(s)}_i}(\eta^{(s)\sigma}_i - p^{(s)}_i\tau^\sigma) \approx 0, \quad (29)$$

$$\frac{\partial G_w}{\partial q^i_{(M)}}(\xi^{i\sigma}_{(M-1)} - \dot{q}^i_{(M-1)}\tau^\sigma) \approx 0$$

(higher order nonholonomic constraints), (30)

or

$$\frac{\partial G_w}{\partial q^i}(\xi^{i\sigma} - \dot{q}^i\tau^\sigma) \approx 0$$

(holonomic constraints). (31)

Then, the infinitesimal transformation (11) is generated by these generators τ^σ, $\xi^{i\sigma}_{(s)}$, and $\eta^{(s)\sigma}_i$, under which the variation of the canonical action I^p satisfies (16). In fact, substituting (27), (28), (29), and (30) into the expression (25), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{p}^{(s)}_i \overline{\xi}^{i\sigma}_{(s)} - H_\mathrm{c}\tau^\sigma) + \left(\dot{q}^i_{(s)} - \frac{\partial H_\mathrm{c}}{\partial p^{(s)}_i}\right)\overline{\eta}^{(s)\sigma}_i + \left(-\dot{p}^{(s)}_i - \frac{\partial H_\mathrm{c}}{\partial q^i_{(s)}}\right)\overline{\xi}^{i\sigma}_{(s)} = \Lambda^\sigma. \quad (32)$$

Multiplying expression (32) by ε^σ, taking the summation for upper and lower indices σ, and integrating the result on $[t_1, t_2]$, one gets expression (16a). Therefore, we obtain the inverse theorem of the generalized canonical Noether theorem of a HOCSGMS: to any r-independent constants of motion $D^\sigma(t; q^i_{(s)}, p^{(s)}_i)$ for a HOCSGMS, there corresponds an infinitesimal transformation (11) generated by the above generators τ^σ, $\xi^{i\sigma}_{(s)}$, $\eta^{(s)\sigma}_i$, as long as inherent constrained conditions $\phi_a = 0$ and subsidiary constrained conditions $G_w = 0$ satisfy expressions (29)–(31), then the canonical action of this system is invariant under this transformation (or the generalized canonical Lagrangian is invariant up to an exact differential term).

5. Conclusions and discussion

The generalized canonical Noether theorem and its inverse theorem for a HOCSGMS are obtained. From above discussion we see that for a HOCSGMS with subsidiary holonomic constraint, the canonical action and subsidiary and inherent constraints are invariant under the global transformation in phase space, the conservation laws in canonical variables are derived. But for a HOCSGMS with subsidiary nonholonomic constraint, the Mth-order simultaneous variations $\delta q^i_{(M)}$ must satisfy Apell–Chetaev conditions, in this case the conservation laws in phase space can also be deduced.

For a system with a singular Lagrangian, the transition to quantum theory would consist in imposing the constraints. To analyse the inherent constraints in phase space of the system is very important both in canonical quantization (operator formalism) and path integral quantization. In some models of field theories the field variables are subject to subsidiary constraints (for example, the nonlinear σ-model, etc). These holonomic constrained systems can be treated as constrained Hamiltonian systems by introducing Lagrangian multipliers. To analyse the constraints and canonical symmetries of singular Lagrangian with higher-order derivative in the quantum theories is thus necessary.

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