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Nonlinear interaction of resonator photons with parametric magnon pairs*

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We consider the process of parametric resonance of magnons in magneto-ordered crystals in a microwave resonator. The nonlinear interaction of the resonator photons with the parametric magnon pairs is analysed theoretically, and the result expressed by the deviation frequencies relative to the pump frequency further demonstrates that the dynamics of resonator modes plays an important role once the pump power exceeds the threshold of parallel pumping.

Keywords: nonlinear interaction, magnon, parametric resonance
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1. Introduction

Theory of magnetic materials and its application have been attracted much attention in recent years.[1−3] The research on the theory of spin-wave (magnon) is important for improving the performance of microwave ferrite device. Parametric resonance under a microwave pump $h\cos\omega_{\rm p}t$ is a convenient tool to study the nonlinear spin-wave system of magnetic crystals.[4−7] When the microwave field $h\cos\omega_{\rm p}t$ is parallel to the static magnetic field H and above a threshold amplitude $h_{\rm thr}$, the parametric resonance is defined by the process of annihilation of microwave photon which excites a pair of magnons with equal but oppositely oriented wavevectors: $\omega_{\rm p} = \varpi_k + \varpi_{-k}$. Experimental measurements for the process of parametric resonance are usually performed using a microwave resonant cavity into which the magnetic sample is mounted. The resonant spectrum is observed by a nonlinear microwave absorption by the sample at $P > P_{\rm thr} \propto h_{\rm thr}^2$. The resonant cavity is assumed to be a part of experimental installation which does not influence the excited state of the sample, but only increases the amplitude of the microwave magnetic field and makes the experimental set-up more sensitive to the absorption of microwave power. Such a point of view is traditional in the analysis of the experimental results on the basis of so-called S-theory.[8−9] Thus, the sample was considered as if it were placed in an alternating magnetic field $h\cos\omega_{\rm p}t$, and the dynamics of the forced electromagnetic oscillations (photons in the cavity) was ignored. With different consideration of previous researcher, Bryant *et al*[10] first mentioned that the dynamics of photon mode in the cavity should be taken into account. Their equations were used by Yamazaki *et al*[11] to simulate a non-equilibrium behaviour of spin-waves. Safonov[12] has shown that the nonlinear coupling of photons and parametric magnons leads to an effective nonlinear damping in a parametric system. The effect of the cavity quality factor Q on the threshold of auto-oscillations of parametrically excited magnons in YIG was observed experimentally by Yamazaki and Mino[13] Andrienko and Safonov[14,15] studied parallel pumping of magnetoelastic waves in antiferromagnet $FeBO_3$ placed in a microwave resonator, in which they obtained the spectrum of the coupled photon-phonon oscillations and demonstrated the fact that the parametric pair of magnetoelastic waves can behave as a single-mode oscillator above the threshold of pumping.

However, although the cavity resonant mode has been considered to play an important role in a combined system of a sample and a resonator and there is some relative experimental evidence, no one has systematically analysed the interaction between the pho-

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ton of cavity and the magnon of sample. Based on the recent theory that the magnon pair can be considered as a nonlinear oscillator,[16] here we shall analyse the nonlinear interaction of sample and resonant cavity mode by the process of the parametric excitation of spin-waves.

2. Theoretical analyses for the interaction of photons and magnon pairs

In the case of parametric resonance of spin-waves in magnetically ordered crystals, the pair of waves usually consists of the following elements:

$$K_{-}(k) = b_k b_{-k}, \qquad K_{+}(k) = b_k^* b_{-k}^*,$$

$$K_0(k) = \frac{1}{2}\left(|b_k|^2 + |b_{-k}|^2\right). \tag{1}$$

Here b_k and b_{-k}^* are the classical complex amplitudes of wave with wavevector k and frequency ϖ_k, in which a shift of spectrum due to nonlinear interactions is taken into account. The resonant condition for the wave pair in the external microwave field of the frequency $\omega_{\rm p}$ is

$$\omega_{\rm p} = \varpi_k + \varpi_{-k}. \tag{2}$$

The effective Hamiltonian of a spin-wave system can be written as follows:

$$\begin{aligned} H/h =& \omega_R R^* R + \sum_k{}' 2\omega_k K_0(k) \\ &+ \sum_k{}' iG_k\left(RK_{+}(k) - R^* K_{-}(k)\right) \\ &+ F\left\{R^* \exp(-i\omega_{\rm p} t) + R\exp(i\omega_{\rm p} t)\right\} + H_{\rm int}/\hbar, \end{aligned} \tag{3}$$

where R^* and R are the complex variable of the resonant mode (with frequency $\omega_{\rm R}$); ω_k is the "bare" spin-wave frequency; G_k is the nonlinear coupling parameter between photons and magnons; and F is the parameter of pumping field.

$$\begin{aligned} H_{\rm int}/\hbar =& 2\sum_{k,q}{}' \left(2T_{kq} K_0(k) K_0(q) + S_{kq} K_{+}(k) K_{-}(q)\right) \\ &+ \sum_k{}' T_k^{(R)} R^* R K_0(k). \end{aligned} \tag{4}$$

Here T_{kq} and S_{kq} are the amplitudes of four-magnon interactions; $T_k^{(R)}$ is the amplitude of photon-magnon scattering. We consider the wave system with the inversion symmetry ($k \to -k$), $\sum_k{}'$ denotes the summation over k-semispace. Planck's constant $\hbar$ is used as a dimensional constant in order to have dimensionless complex variables as a classical analogue of Bose operators of creation and annihilation. It should be pointed out that in the earlier parametric spin-wave theories the dynamics of photon mode in the cavity has not been taken into account.[8–9]

The Hamilton equations of motion with phenomenological damping parameters (Γ and γ_k are the relaxation rates of photons and magnons, respectively) can be written in the form

$$i\left(\frac{\mathrm{d}}{\mathrm{d}t} + \Gamma\right) R = \{R, H/\hbar\},$$

$$i\left(\frac{\mathrm{d}}{\mathrm{d}t} + \gamma_k\right) b_k = \{b_k, H/\hbar\}. \tag{5}$$

Here

$$\begin{aligned} \{A, B\} \equiv& \frac{\delta A}{\delta R}\frac{\delta B}{\delta R^*} - \frac{\delta B}{\delta R}\frac{\delta A}{\delta R^*} \\ &+ \sum_k{}' \left(\frac{\delta A}{\delta b_q}\frac{\delta B}{\delta b_q^*} - \frac{\delta B}{\delta b_q}\frac{\delta A}{\delta b_q^*}\right) \end{aligned}$$

is the classical analogue of a commutator.

Introducing the variables in the rotating frame

$$b_k = c_k \exp(-i\omega_p t/2), \quad R = R\exp(-i\omega_p t),$$

$$\sigma_k = N_k \exp\{i(\theta_k - \pi/2)\}, \tag{6}$$

where θ_k is the phase angle between the pumping field and the forced magnetic oscillations. $|\sigma_k| = N_k$ and $|c_k| = |c_{-k}|$ can be derived from $\sigma_k \equiv c_k c_{-k}$. Finally, one can obtain the equations of motion in the following form in a region Q of k-semispace, where $N_k \neq 0$ and $N_k = 0$ if $k \notin Q$:

$$\left\{i\left(\frac{\mathrm{d}}{\mathrm{d}t} + \Gamma\right) + (\omega_{\rm p} - \varpi_R)\right\}\tilde{R} + GN\exp(i\theta) - F = 0, \tag{7}$$

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\theta - G\left(\tilde{R}\exp(-i\theta) + \tilde{R}^*\exp(i\theta)\right) \\ &- (\omega_{\rm p} - 2\varpi_k) + 4SN = 0, \end{aligned} \tag{8}$$

$$\left\{\frac{\mathrm{d}}{\mathrm{d}t} + 2\gamma_k - iG\left(\tilde{R}\exp(-i\theta) - \tilde{R}^*\exp(i\theta)\right)\right\} N = 0. \tag{9}$$

Here

$$\varpi_R \equiv \omega_R + 2T^{(R)}N,$$

$$\varpi_k \equiv \omega_k + 4TN + T^{(R)}\left|\tilde{R}\right|^2 \tag{10}$$

are the renormalized frequencies.[11,12,17] N is the number of magnon pairs. According to Eqs.(7)–(9)

and the resonant condition $2\varpi_k = \omega_p$, the stationary state (with $\tilde{R}_0$, $N^{(0)}$, $\theta^{(0)}$) of the excited resonant mode and spin-wave system can be defined as

$$(i\Gamma + \Delta\varpi_{\rm R})\,\tilde{R}_0 = -GN^{(0)}\exp\left(i\theta^{(0)}\right) + F, \quad (11)$$

$$hV_k\sin(\theta_0 - \theta_{\rm R}) = -\tilde{S}N^{(0)}, \quad (12)$$

$$hV_k\cos(\theta_0 - \theta_{\rm R}) = \gamma_k + \eta_{\rm rd}N^{(0)}, \quad (13)$$

where

$$\sin\theta_{\rm R} \equiv \frac{\Delta\varpi_{\rm R}}{\sqrt{\Gamma^2 + \Delta\varpi_{\rm R}^2}},\ \cos\theta_{\rm R} \equiv \frac{\Gamma}{\sqrt{\Gamma^2 + \Delta\varpi_{\rm R}^2}},$$

$$\tilde{S} \equiv 2S + \frac{\eta_{\rm rd}}{\Gamma}\Delta\varpi_{\rm R},\quad \eta_{\rm rd} \equiv \frac{G^2\Gamma}{\Gamma^2 + \Delta\varpi_{\rm R}^2},$$

$$hV_k \equiv \frac{FG}{\sqrt{\Gamma^2 + \Delta\varpi_{\rm R}^2}},\quad \Delta\varpi_{\rm R} \equiv \omega_{\rm p} - \varpi_{\rm R}.$$

Here V_k is the effective coupling of magnons with the magnetic microwave field

$$h = (\beta_{\rm R}P_iQ/\omega_{\rm p}V_{\rm R})^{1/2} \quad (14)$$

in the cavity.[6,18] P_i is the input microwave power, $Q \equiv \omega_{\rm R}/2\Gamma$ is the quality factor, $V_{\rm R}$ is the volume of the resonator, and $\beta_{\rm R}$ ~10–100 is the numerical factor which depends on the construction of resonator.

$$(hV_k)^2 = \left(\tilde{S}N^{(0)}\right)^2 + \left(\gamma_k + \eta_{\rm rd}N^{(0)}\right)^2. \quad (15)$$

Taking into account that $h_{\rm thr}\cdot V_k = \gamma_k$ and substituting $(h/h_{\rm thr})^2$ by $P/P_{\rm thr}$ yield

$$\left\{\left(\frac{\tilde{S}}{\gamma_k}\right)^2 + \left(\frac{\eta_{\rm rd}}{\gamma_k}\right)^2\right\}N^{(0)2} + 2\left(\frac{\eta_{\rm rd}}{\gamma_k}\right)N^{(0)} + 1 - \frac{P}{P_{\rm thr}} = 0. \quad (16)$$

Linear analysis of small deviations $\delta\tilde{R}$, $\delta\theta_k$ and $\delta N \propto \exp(\lambda t)$ relative to the stationary state (11)–(13) gives the following equations:

$$\begin{vmatrix} \lambda+\Gamma-i\Delta\varpi_{\rm R} & 0 & C & iD^* \\ 0 & \lambda+\Gamma+i\Delta\varpi_{\rm R} & C^* & -iD \\ D & D^* & \lambda+2\gamma_k & 4(T+2S) \\ -iC^* & iC & -4SN^{(0)2} & \lambda \end{vmatrix} = 0, \quad (17)$$

where $C = GN^{(0)}\exp\left(i\theta^{(0)}\right)$, $D = -GN^{(0)}\exp\left(-i\theta^{(0)}\right) + 2T\tilde{R}^{(0)*}$, and $\lambda \equiv \acute{R}(\lambda) + i\acute{I}(\lambda)$. The condition $\acute{R}(\lambda) < 0$ corresponds to a stationary state, $\acute{I}(\lambda)$ describes the frequencies of new normal modes.

Equation (17) can be expanded in the following form:

$$a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0 \quad (18)$$

where

$$\begin{aligned}
a_0 =& 1,\\
a_1 =& 2\gamma_k + 2\Gamma,\\
a_2 =& 2G^2N^{(0)} - 2H + 16S^2N^{(0)2} + 4\Gamma\gamma_k + \Gamma^2\\
&+ 32(\Delta\varpi_{\rm R})^2SN^{(0)2}T - 4CT\tilde{R}^{(0)*},\\
a_3 =& -2\gamma_kH - 2\Gamma H + 64\Gamma SN^{(0)2}T + 2\Gamma G^2N^{(0)}\\
&+ 2G^2N^{(0)}\gamma_k + 2\Gamma^2\gamma_k + 32\Gamma S^2N^{(0)2}\\
&+ 2(\Delta\varpi_{\rm R})^2\gamma_k - 4CT\tilde{R}^{(0)*},\\
a_4 =& -2\Gamma\gamma_kH - 2G^2N^{(0)}H - 2i\gamma_k(\Delta\varpi_{\rm R})H\\
&+ 2\Gamma G^2N^{(0)}\gamma_k + 16\Gamma^2S^2N^{(0)2} + 32\Gamma^2SN^{(0)2}T\\
&+ 16(\Delta\varpi_{\rm R})^2S^2N^{(0)2} + 4i(\Delta\varpi_{\rm R})CT\gamma_k\tilde{R}^{(0)*}\\
&- 4\Gamma CT\gamma_k\tilde{R}^{(0)*} + 32(\Delta\varpi_{\rm R})^2SN^{(0)2}T\\
&+ 16(\Delta\varpi_{\rm R})G^2N^{(0)2}T + 8(\Delta\varpi_{\rm R})G^2N^{(0)2}S\\
&+ G^4N^{(0)4} - 4G^3\exp\left(-i\theta^{(0)}\right)N^{(0)2}T\tilde{R}^{(0)*}\\
&+ 4T^2\tilde{R}^{(0)*2}C^2 - 2i(\Delta\varpi_{\rm R})G^2N^{(0)}\gamma_k\\
&- 8(\Delta\varpi_{\rm R})SN^{(0)2}G\exp\left(-i\theta^{(0)}\right)Q^*\\
&+ 16(\Delta\varpi_{\rm R})SN^{(0)2}TQ^*\tilde{R}^{(0)*} + \left(C^2Q^2\right)^*\\
&+ 4TG^2N^{(0)2}\tilde{R}^{(0)*}Q^*,\\
Q =& -G\exp\left(-i\theta^{(0)}\right)2T\tilde{R}^{(0)},\\
H =& Q^*GN^{(0)}\exp\left(i\theta^{(0)}\right).
\end{aligned}$$

Let

$$b_i = a_i/a_0 (i = 1, 2, 3, 4),\quad \lambda = u - b_1/4, \quad (19)$$

Eq.(18) becomes

$$u^4 + au^2 + bu + c = 0, \tag{20}$$

where

$$a = -3b_1^2/8 + b_2,$$

$$b = b_1^3/8 - b_1 b_2/2 + b_3,$$

$$c = -3b_1^4/256 + b_1^2 b_2/16 - b_1 b_3/4 + b_4.$$

So the solution of Eq.(20) is

$$u_{1,2} = \frac{\sqrt{6\sigma_2}}{12} \pm \frac{1}{12}\sqrt{-48a - 6(\sigma_1)^{\frac{1}{3}} - \frac{288c + 24a^2}{(\sigma_1)^{\frac{1}{3}}} - \frac{72\sqrt{6}b}{\sqrt{\sigma_2}}},$$

$$u_{3,4} = \frac{\sqrt{6\sigma_2}}{12} \pm \frac{1}{12}\sqrt{-48a - 6(\sigma_1)^{\frac{1}{3}} - \frac{288c + 24a^2}{(\sigma_1)^{\frac{1}{3}}} + \frac{72\sqrt{6}b}{\sqrt{\sigma_2}}},$$

where

$$\begin{aligned}\sigma_1 =& -288ac + 108b^2 + 8a^3 \\ &+ 12\sqrt{-768c^3 + 384c^2a^2 - 48ca^4 - 432acb^2 + 81b^4 + 12b^2a^3}, \\ \sigma_2 =& -4a + \sigma_1^{\frac{1}{3}} + 4(12c + a^2)/\sigma_1^{\frac{1}{3}}.\end{aligned}$$

The solution of Eq.(18) is

$$\lambda_{1,2,3,4} = u_{1,2,3,4} - b_1/4. \tag{21}$$

This is a general expression about λ, which includes complicated real and imaginary parts. If we omit Γ and γ_k in Eq.(21) (i.e. a relation of new mixed modes is neglected), then we can obtain a simplified solution for λ in the form of $\acute{R}(\lambda) = 0$ and

$$\acute{I}(\lambda)_{\pm\pm} = \pm\frac{S}{\sqrt{2}}\left[A \pm B^{1/2}\right]^{1/2}, \tag{22}$$

where

$$\begin{aligned}A =& 16\left(1 + 2\frac{T}{S}\right)N^{(0)2} + 4\left(\frac{G}{S}\right)^2 N^{(0)} + \left(\frac{\Delta\varpi_R}{S}\right)^2 \\ B =& 256\left(1 + 2\frac{T}{S}\right)^2 N^{(0)4} + 128\left(\frac{G}{S}\right)^2\left(1 + 2\frac{T}{S}\right)N^{(0)3} \\ &- 32\frac{\Delta\varpi_R}{S}\left(\frac{\Delta\varpi_R}{S}\left(1 + 2\frac{T}{S}\right)\right. \\ &\left. + 2\left(\frac{G}{S}\right)^2\left(1 + \frac{T}{S}\right)\right)N^{(0)2} \\ &+ 8\left(\frac{\Delta\varpi_R}{S}\right)\left(\frac{G}{S}\right)^2 N^{(0)} + \left(\frac{\Delta\varpi_R}{S}\right)^4.\end{aligned}$$

For simplicity, we consider the case $T_k^{(R)} = 0$ when $\Delta\varpi_{\mathrm{R}} = \omega_{\mathrm{p}} - \omega_{\mathrm{R}}$. We can calculate the number N of magnon pairs from Eq.(16) and obtain the deviation frequencies relative to the pump frequency from Eq.(22):

$$\Delta\omega_+ = \omega_+ - \omega_{\mathrm{p}} = \left\{ \begin{array}{ll} \acute{I}(\lambda)_{++} & (\omega_{\mathrm{p}} < \omega_{\mathrm{R}}) \\ \acute{I}(\lambda)_{-+} & (\omega_{\mathrm{p}} \geq \omega_{\mathrm{R}}) \end{array} \right\}, \tag{23}$$

$$\Delta\omega_- = \omega_- - \omega_{\mathrm{p}} = \left\{ \begin{array}{ll} \acute{I}(\lambda)_{--} & (\omega_{\mathrm{p}} < \omega_{\mathrm{R}}) \\ \acute{I}(\lambda)_{+-} & (\omega_{\mathrm{p}} \geq \omega_{\mathrm{R}}) \end{array} \right\}. \tag{24}$$

3. Discussion

The above analyses demonstrate that there is a nonlinear interaction between the parametric spin-wave pairs and resonant cavity photons. When the driving microwave power exceeds the threshold, $P > P_{\mathrm{thr}}$, the nonlinear interaction between the parametric spin-wave pairs and the photons will exhibit a physical picture of spectrum in which two resonant peaks correspond to two new normal modes of the system. We can find that the frequency of one mixed mode is lower, and that of another mode is higher than ω_{p} from Eqs.(23) and (24). The distance between the two resonant peaks increases with increasing pump power, indicating an increase of interaction between photons and magnon pairs. It is further demonstrated that the dynamics of resonant cavity modes plays an important role in the process of the parametric resonance of magnons once the pump power exceeds the threshold of parallel pumping. Also, parametric pairs shift

the resonator frequency and decrease the amplitude of the pump field. These lead to a restriction of absorbed microwave power even when $T_{kq} = S_{kq} = 0$. The existence of nonlinear interactions between photons and magnon pairs stimulates us to reconsider some physical results on parallel pumping of spin-waves in the overthreshold region. It is valuable to perform experiments to quantitatively parallel and validate our predicted result.

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