Vol 12 No 4, April 2003
1009-1963/2003/12(04)/0377-04

Chinese Physics

© 2003 Chin. Phys. Soc.
and IOP Publishing Ltd

Stochastic resonance in parabolic-bistable potential system with an additive colour noise*

Liang Gui-Yun(梁贵云)[a)b)]

[a)] *National Astronomical Observatories, Chinese Academy of Science, Beijing* 100012, *China*

[b)] *Laboratory of Optical Physics, Institute of Physics, Chinese Academy of Science, Beijing* 100080, *China*

(Received 18 October 2002; revised manuscript received 25 November 2002)

We propose a new model (parabolic bistable system) with a colour noise source. In the presence of periodic input signal, we investigate the output signal to-noise ratio and the power spectral density of output signal, in which the self-correlation time may be an arbitrary value with no restriction. We find that the curves of signal-to-noise ratio versus the noise intensity D exhibits a bell-shape behaviour. The peak height increases with the increasing value of amplitude A_0 of the input signal. However, the position of peak does not shift. Moreover, the signal-to-noise ratio is insensitive to the modulation frequency Ω, which completely differs from result of Ref.[6]. The power spectral density also has similar behaviour.

Keywords: stochastic resonance, parabolic-bistable potential, signal-to-noise ratio
PACC: 0540, 0250

1. Introduction

Stochastic resonance (SR) is a nonlinear phenomenon that has attracted considerable attention.[1−3] In the broad sense, SR means that performances of the system, such as the response to weak input of periodic signals, can be enhanced and optimized at certain nonzero noise level. This phenomenon is rather counter-intuitive, but the mechanism requires three basic ingredients:[4] (i) an energetic activation barrier or, more generally, a form of threshold; (ii) a weak coherent input (such as a periodic signal); (iii) a source of noise that is inherent in the system, or is added to the coherent input. Since the concept of SR was originally put forward in the seminal papers by Benzi and collaborators, much more research has been done both in theory and in experiments. In theory, adiabatic limit and linear-response approximation have been put forward successively. The first experimental verification of the SR in study noise dependence of the spectral line of an ac-driven Schmitt trigger presently created a buzz of activity in other fields,[4−9] optical system (bistable ring laser),[10] electronic and magnetic systems (tunnel diode and superconducting quantum interference devices),[11] neuronal system, and so forth. By far, more works are focused on systems with white noise source. In the physical world, however, such an idealization is never exactly realized. Though a few works explore system with colour noise source, the auto-correlation time τ was be restricted to regions of small τ or large τ. When the value of correlation-time τ equals to a moderate value, the SR is never observed in the system driven by external force in theory or in experiment. In this paper, we observed the SR in a special parabolic bistable potential system, in which the τ may have arbitrary value and the analytical expression of escape rate can be obtained exactly, when τ equals to the moderate value.

In this paper, we propose a new model, namely the parabolic bistable system with a colour noise source. An analytical expression of mean first passage time $T_k(D)$ (escape rate $r_k = 1/T_k(D)$), in which the self-correlation time τ may have an arbitrary value with no restriction, has been obtained in the absence of periodic signal in our other paper.[12] This result is in good agreement with the numerical calculation for arbitrary τ, even for the moderate τ. It should be noted that all values of the parameter are in the regime of adiabatic limit and linear-response approximation is assumed in the paper. In the presence of periodic input signal, we investigate the signal-to-noise ratio

*Project supported by the National Natural Science Foundation of China (Grant No 19975020).

(SNR) and the power spectral density (PSD) $S(\omega)$ of the output signal. We find that the SR will appear in curves of SNR versus the noise intensity D in the presence of periodic input signal ($A_0 \neq 0$). The peak height will be higher with increasing value of amplitude A_0 of input signal, and the position of the peak will not shift for different A_0. Moreover, the SNR is insensitive to the angular modulation frequency Ω, which differs from that in Ref.[6]. Furthermore, we explore the curves of PSD $S(\omega)$ for different values of noise intensity D, modulation amplitude A_0 and modulation frequency Ω.

2.The escape rate of particles

Considering overdamped particles in a potential $U(x)$ modulated by an input signal $A_0 x\cos(\Omega t)$, the stochastic dynamics is subject to the following Langevin equation

$$\dot{x} = -U^{'}(x) + A_0\cos(\Omega t) + \xi(t), \tag{1}$$

where

$$U(x)\begin{cases} U_1(x) = \dfrac{a}{2}(x+1)^2 + \dfrac{k}{2}, & x < -\dfrac{1}{\sqrt{3}}, \\ U_2(x) = \dfrac{b}{2}x^2, & |x| \leq \dfrac{1}{\sqrt{3}}, \\ U_3(x) = \dfrac{a}{2}(x-1)^2 + \dfrac{k}{2}, & x > \dfrac{1}{\sqrt{3}}, \end{cases} \tag{2}$$

and $a > 0, b < 0, |b| < 1/\tau$, and $\xi(t)$ represents O-U noise with zero mean and autocorrelation

$$< \xi(t)\xi(s) >= \frac{D}{\tau}\exp\left(-\frac{|t-s|}{\tau}\right). \tag{3}$$

In the absence of periodic signal, the exact analytical expression of stationary parabolic distribution (SPD) is obtained in another paper of ours:

$$P(x) = \begin{cases} NH_1(x)\exp\left[-\dfrac{\alpha(x+1)^2}{2D}\right], & x < -\dfrac{1}{\sqrt{3}}, \\ NH_2(x)\exp\left[-\dfrac{\beta x^2}{2D}\right], & |x| \leq -\dfrac{1}{\sqrt{3}}, \\ NH_3(x)\exp\left[-\dfrac{\alpha(x-1)^2}{2D}\right], & x > \dfrac{1}{\sqrt{3}}, \end{cases} \tag{4}$$

where $\alpha = a(a\tau+1)$, $\beta = b(b\tau+1)$, $H_1(x) = H_3(x) = \sqrt{\dfrac{a\tau+1}{b\tau+1}}\exp(-\dfrac{\beta}{2\sqrt{3}D})$, $H_2(x) = \sqrt{\dfrac{2\pi D}{b\tau^2+\tau}}$, N is the normalization constant. The equations are subject to the continuous condition given by

$$b(b\tau+1) = a(a\tau+1)(1-\sqrt{3}). \tag{5}$$

Then the mean first passage time is determined by when particles surmount a potential starting from $x = -1$ to $x = +1$,

$$\begin{aligned} T_k(D) =& D^{-1}\int_{-1}^{1} H(x)\exp[\phi(x)]\mathrm{d}x \\ & \times \int_{-\infty}^{x} H(y)\exp[-\phi(y)]\mathrm{d}y, \end{aligned} \tag{6}$$

here

$$\phi(x) = \begin{cases} \phi_1(x) = \dfrac{\alpha(x+1)^2}{2}, & x < -\dfrac{1}{\sqrt{3}}, \\ \phi_2(x) = \dfrac{\beta x^2}{2}, & |x| \leq -\dfrac{1}{\sqrt{3}}, \\ \phi_3(x) = \dfrac{\alpha(x-1)^2}{2}, & x > \dfrac{1}{\sqrt{3}}. \end{cases}$$

After some algebraic calculation, the analytical expression which can be in good accordance with the numerical calculation for arbitrary τ of $T_k(D)$, is obtained in the absence of input signal:

$$\begin{aligned} T_k(D) =& D^{-1}\Bigg\{H_1^2\sqrt{\frac{\pi D}{2\alpha}}\int_{-1}^{-1/\sqrt{3}}\exp\left[\frac{\phi_1(x)}{D}\right]\mathrm{d}x \\ & \times\sqrt{\frac{\pi D}{2\alpha}}\left[1+\Phi\left(\sqrt{\frac{\alpha}{2D}}\left(1-\frac{1}{\sqrt{3}}\right)\right)\right. \\ & \left.+\frac{H_2}{H_1}\sqrt{\frac{2\alpha}{\pi D}}\int_0^{1/\sqrt{3}}\exp\left(-\frac{\beta y^2}{2D}\right)\right] \\ & +H_1H_2\sqrt{\frac{2\pi D}{-\beta}}\Phi\left(\sqrt{\frac{-\beta}{6D}}\right)\sqrt{\frac{\pi D}{2\alpha}} \\ & \times\left[1+\Phi\left(\sqrt{\frac{\alpha}{2D}}\left(1-\frac{1}{\sqrt{3}}\right)\right)\right. \\ & \left.+\frac{H_2}{H_1}\sum_{n=0}^{\infty}\left(\frac{-\beta}{2D}\right)^n\frac{1}{2n+1}\left(\frac{1}{\sqrt{3}}\right)^{2n+1}\right]\Bigg\}. \end{aligned} \tag{7}$$

Therefore, the escape rate under the Kramers meaning with no input signal can be given by

$$r_k = \frac{1}{T_k(D)}. \tag{8}$$

In the presence of input signal, the transition probability densities $W_{\pm}(t)$ proposed by McNamara and Wiesefeld is[4]

$$W_{\mp}(t) = r_k\exp\left(\mp\frac{A_0x_m}{D}\cos(\Omega t)\right). \tag{9}$$

On assuming that the modulation amplitude is small, i.e., $A_0x_m \ll D$, we can use the following expansions in the small parameter A_0x_m/D,

$$\begin{aligned} W_{\mp}(t) =& r_k\left[1\mp\frac{A_0x_m}{D}\cos(\Omega t)\right. \\ & \left.+\frac{1}{2}\left(\frac{A_0x_m}{D}\right)^2\cos^2(\Omega t)\mp\cdots\right], \end{aligned}$$

$$W_+(t)+W_-(t) = 2r_k\left[1+\frac{1}{2}\left(\frac{A_0x_m}{D}\right)^2\cos^2(\Omega t)\mp\cdots\right],$$

where $x_m = \pm 1$.

3. The power spectral dentistry and SNR

In order to investigate the SR, it is necessary to calculate the power spectral density and SNR.

We know that the power spectral density is the Fourier transform of autocorrelation function. Firstly, we write down the expression of autocorrelation. The general definition of the autocorrelation function $< x(t+s)x(t) \mid x_0, t_0 >$ is

$$\begin{aligned}
&< x(t+s)x(t) \mid x_0, t_0 > \\
&= \int\int xyP(x, t+s \mid y, t)P(y, t \mid x_0, t_0)\mathrm{d}x\mathrm{d}y \\
&= x_m^2\exp(-2r_k|s|)\left[1-\frac{1}{2}\left(\frac{A_0x_m}{D}\right)^2\frac{4r_k^2}{4r_k^2+\Omega^2}\right] \\
&\quad + \frac{x_m^2}{2}\left(\frac{A_0x_m}{D}\right)^2\frac{4r_k^2}{4r_k^2+\Omega^2}\cos(\Omega s). \qquad (10)
\end{aligned}$$

The power spectral density $S(\omega)$ commonly reported in the literature is the Fourier transform of Eq.(10):

$$\begin{aligned}
S(\omega) &= \left[1-\frac{1}{2}\left(\frac{A_0x_m}{D}\right)^2\frac{4r_k^2}{4r_k^2+\Omega^2}\right]\frac{4r_kx_m^2}{4r_k^2+\omega^2} \\
&\quad + \frac{\pi}{2}\left(\frac{A_0x_m}{D}\right)^2\frac{4r_k^2x_m^2}{4r_k^2+\Omega^2} \\
&\quad \times [\delta(\omega-\Omega)+\delta(\omega+\Omega)]. \qquad (11)
\end{aligned}$$

SNR is the important measure for the SR phenomenon. Researchers usually investigate the SR through observing the SNR in theory or in experiment. The SNR adopted here has the following definition:[4]

$$SNR = 2\left[\lim_{\Delta\omega\to 0}\int_{\Omega-\Delta\omega}^{\Omega+\Delta\omega}S(\omega)\mathrm{d}\omega\right]/S_N(\Omega). \qquad (12)$$

Here $S_N(\Omega) = \left[1-\frac{1}{2}\left(\frac{A_0x_m}{D}\right)^2\frac{4r_k^2}{4r_k^2+\Omega^2}\right]\frac{4r_kx_m^2}{4r_k^2+\omega^2}$ is the product of the Lorentzian curve obtained with no input signal $A_0 = 0$. The total output power, signal plus noise, for the two-state model discussed here, is $2\pi x_m^2$, independent of the input-signal amplitude A_0 and frequency Ω. Finally, according to Eq.(11) and Eq.(12), the SNR follows as

$$SNR = \pi\left(\frac{A_0x_m}{D}\right)^2 r_k + O(A_0^4) \qquad (13)$$

4. Conclusion and discussion

(i) Fig.1 shows the relation between the SNR and stochastic parameters D for different amplitudes A_0 of input signal when the self-correlation time of noise source has a fixed value ($\tau = 10$). Due to the continuous condition Eq.(5), the values of a, b and k equal to 0.0260263, -0.0402702 and -0.018, respectively, when $\tau = 10$. When the threshold of signal amplitude is exceeded, the curve of SNR versus noise strength D will exhibit a bell-shape behaviour, that is, the SR is observed. Moreover, the peak height will become higher with increasing amplitude A_0 of the input signal, and the position of peak will not shift for different A_0, as shown in Fig.1.

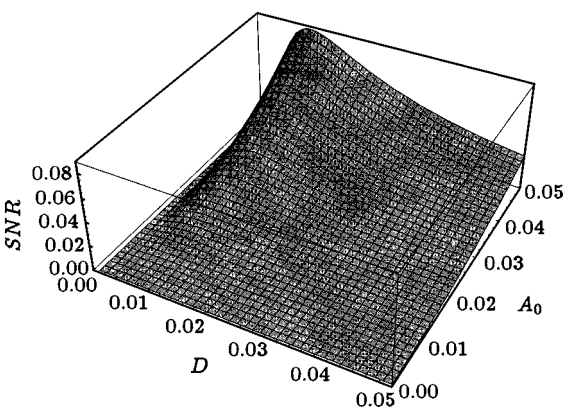


Fig.1. The curve plane of SNR versus noise intensity D and signal amplitude A_0. The values of the parameters: $\Omega = 0.001$, $\tau = 10$.

(ii) In the case of $\tau = 10$, the curve surface of SNR versus strength D and signal frequency Ω demonstrates that the SR is insensitive to the modulated frequency Ω (see Fig.2). Such point differs from that observed in other double-well potential system.

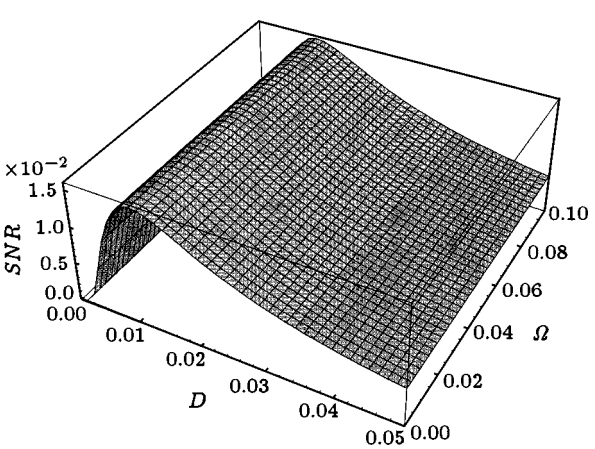


Fig.2. The curve plane of SNR versus noise intensity D and modulation frequency Ω. The values of the parameters: $A_0 = 0.03$, $\tau = 10$.

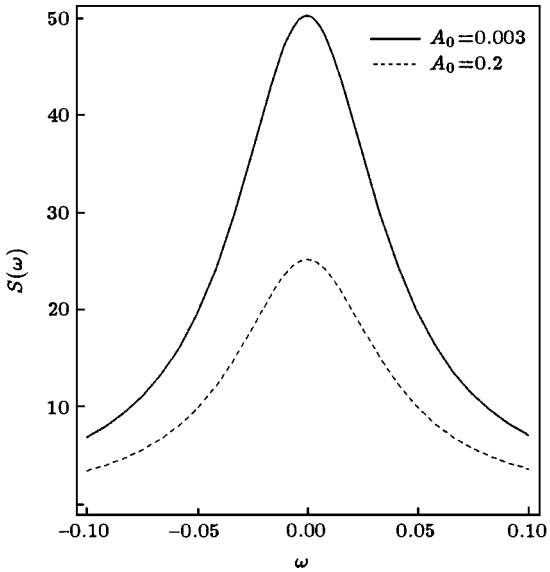


Fig.3. The curve of PSD $S(\omega)$ versus frequency ω for different input-signal amplitude A_0. The values of the parameters: $\Omega = 0.001$, $D = 0.2$, $\tau = 10$, solid line versus $A_0 = 0.003$, dotted line versus $A_0 = 0.2$.

(iii) When the autocorrelation time has a moderate value ($\tau = 10$), the power spectral density $S(\omega)$ as a function of frequency ω for different amplitudes A_0 of input signal is depicted in Fig.3. From the figure, we can see that the $S(\omega)$ is symmetric to the zero frequency ($\omega = 0$) and its peak will be higher with the increasing of A_0. And all values of $S(\omega)$ for large A_0 is larger than that for small A_0 because the input energy is larger. So there is no junction between the curves for different A_0. However, the curves of $S(\omega)$-ω for different noise intensities D indicate that the output signal will be submerged in the large noise background, that is, the peak height will be lower when increasing the D, as shown in Fig.4. Due to the input-energy is held a constant, there are sure junctions between the curves for different noise intensities D.

(iv) The curve surface of $S(\omega)$ versus frequency ω and signal angular-frequency Ω demonstrates that the $S(\omega)$ is also insensitive to the modulated frequency Ω (see Fig.5). In other words, the input signal frequency Ω does not influence the output signal configuration of frequency spectra.

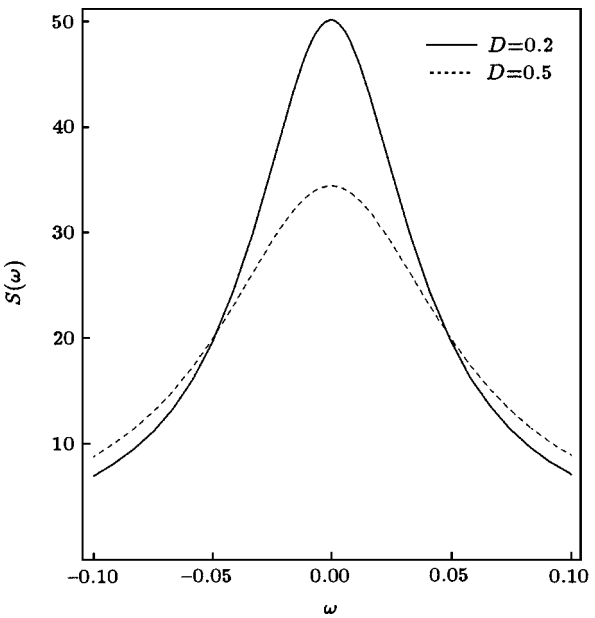


Fig.4. The curve of PSD $S(\omega)$ versus frequency ω for noise intensity D. The values of the parameters: $A_0 = 0.003$, $\Omega = 0.001$, $\tau = 10$, solid line versus $D = 0.2$, dotted line versus $D = 0.5$.

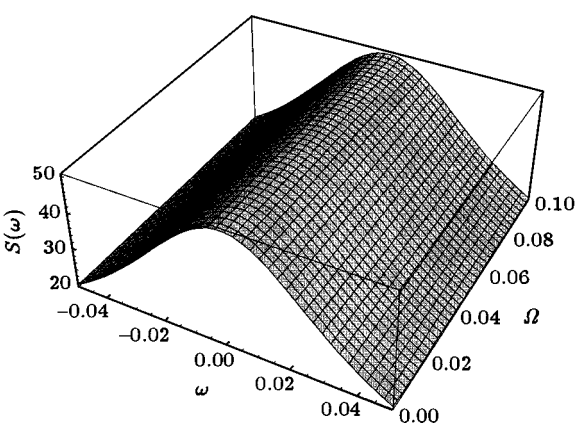


Fig.5. The curve plane of PSD $S(\omega)$ versus frequency ω and modulation frequency Ω. The values of the parameters: $D = 0.2$, $\tau = 10$, $A_0 = 0.003$.

References

[1] Liu Z H, Lai Y C and Nachman A 2002 *Phys. Lett.* A **297** 75

[2] Zaikin A A, Kurths J and Geier L S 2000 *Phys. Rev. Lett.* **85** 227

[3] Mao X M, Sun K and Ou Y X 2002 *Chin. Phys.* **11** 1

[4] Gammaitoni L *et al* 1998 *Rev. Mod. Phys.* **70** 223

[5] Zhou T and Moss F 1990 *Phys. Rev.* A **42** 3161

[6] Jia Y, Yu S N and Li J R 2000 *Phys. Rev.* E **62** 1869

[7] Luo X T and Zhu S Q 2002 *Acta. Phys. Sin.* **51** 977 (in Chinese)

[8] Mei D C *et al* 1999 *Chin. Phys.* **8** 808

[9] Luo X T, Zhu S Q and Gao W J 2001 *Chin. Phys.* **10** 1011

[10] Liu Y W, Zhao H and Wang Y H 1999 *Acta. Phys. Sin.* **48** 198 (in Chinese)

[11] Inchiosa M E *et al* 1999 *Phys. Lett.* A **252** 20

[12] Liang G Y, Cao L, Ke S Z and Wu D J 2002 *Commun. Theor. Phys.* **37** 155