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Explicit and exact travelling plane wave solutions of the (2+1)-dimensional Boussinesq equation

Huang Wen-Hua(黄文华)[a)b)] and Jin Mei-Zhen(金美贞)[c)]

[a)] *Department of Physics, Jiangxi Yichun University, Yichun 336000, China*
[b)] *Institute of Nonlinear Physics, Zhejiang Normal University, Jinhua 321004, China*
[c)] *College of Educational Science and Technology, Zhejiang Normal University, Jinhua 321004, China*

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The deformation mapping method is applied to solve a system of (2+1)-dimensional Boussinesq equations. Many types of explicit and exact travelling plane wave solutions, which contain solitary wave solutions,periodic wave solutions,Jacobian elliptic function solutions and others exact solutions, are obtained by a simple algebraic transformation relation between the (2+1)-dimensional Boussinesq equation and the cubic nonlinear Klein–Gordon equation.

Keywords: The (2+1)-dimensional Boussinesq equation, nonlinear Klein–Gordon equation, travelling wave solution
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1. Introduction

Nonlinear partial differential equations (NPDEs) are widely used to describe complex phenomena in various sciences, especially in physical science. Finding explicit and exact solutions, in particular, solitary wave solutions, of nonlinear evolution equations in mathematical physics play an important role in nonlinear science. It is well known that there are infinite solutions for every NPDE, and it is a difficult task to find an exact solution. A vast variety of methods including inverse scattering theory,[1] Bäcklund transformation,[2] Hirota's bilinear method,[3] Darboux transformation,[4] the hyperbola function method,[5] the mixed exponential method,[6] and homogeneous balance method,[7−10], etc., have been developed to obtain travelling solutions to nonlinear partial equations. Besides these methods, we can also obtain solutions of a NPDE by the technical of establishing and take full advantage of an algebraic mapping relation between the given NPDE and another NPDE with known solutions sometimes.

Recently, using the deformation mapping method first presented by Lou and Ni,[11] Li[12−15] obtained many soliton solutions and periodic solutions of (3+1)-dimensional nonlinear models through finding some relations between the travelling wave solutions of the given models and those of the cubic nonlinear Klein–Gordon (NKG) system which has been studied widely in the literature.[11] This deformation mapping method can be extended to solve other $(n+1)(n \geq 1)$-dimensional nonlinear equation systems actually. For the (2+1)-dimensional nonlinear model equation, the main idea of the algorithm is as follows.

We give a (2+1)-dimensional nonlinear equation

$$P(u, u_x, u_y, u_t, u_{xx}, u_{yy}, u_{tt}, u_{xy}, u_{xt}, u_{yt}, \cdots) = 0, \tag{1}$$

where P is in general a polynomial function of its argument, and the subscripts denote the partial derivatives. By using the travelling wave transformation, Eq.(1) has the following form of solution

$$u = u(\xi), \xi = k_1 x + k_2 y + k_3 t, \tag{2}$$

where k_1, k_2 and k_3 are all arbitrary constants. Substituting Eq.(2) into Eq.(1) yields an ordinary differential equation (ODE)

$$P(u, u', u'', \cdots) = 0, \tag{3}$$

where $'$denotes d/dξ.In order to seek for the travelling wave solutions of system (1),we take the following transformations:

$$u(\xi) = U(\phi(\xi)), \tag{4}$$

where the new variable $\phi(\xi)$ is a solution of the cubic NKG equation

$$\phi_\xi^2 = c + \lambda\phi^2 + \frac{\mu}{2}\phi^4,$$

$$\phi_{\xi\xi} = \lambda\phi + \mu\phi^3, \tag{5}$$

where c, λ and μ are all arbitrary constants.

Using the mapping relation (4) and the solutions of Eq.(5),we can obtain many explicit and exact travelling wave solutions of system (1).

In this paper, we still use the deformation mapping method to find some new travelling wave solutions of the (2+1)-dimensional Boussinesq equation[15]

$$u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxxx} - u_{yy} = 0. \tag{6}$$

Eq.(6) describes the propagation of gravity waves on the surface of water, in particular the head-on collision of oblique waves.This equation combines the two-way propagation of the classical Boussinesq equation with the (weak) dependence on a second spatial variable, as occurs in the two (2+1)-dimensional Korteweg–de Vries (KdV) equation. Exact and general solitary-wave, two soliton and resonant solutions were obtained by Jonhson from the Hirota bilinear form of the equation in literature.[15] We find a simple mapping relation between the system of the (2+1)-dimensional Boussinesq equation and the cubic NKG equation. With the aid of this relation and the solutions of the cubic NKG equation,many new explicit and exact travelling plane wave solutions of the system of the (2+1)-dimensional Boussinesq equation, which contain solitary wave solutions, periodic wave solutions, Jacobian elliptic function solutions and other exact solutions, are obtained.

2. Transformation relation between the travelling plane wave solutions of the (2+1)-dimensional Boussinesq equation and the cubic nonlinear Klein–Gordon equation

For Eq.(6), we first make the following travelling plane wave transformation:

$$u = u(\xi), \xi = k_1 x + k_2 y + k_3 t, \tag{7}$$

where k_1, k_2 and k_3 are all arbitrary constants. Substituting Eq.(7) into Eq.(6) and integrating it twice with respect to ξ, we obtain:

$$(k_3^2 - k_2^2 - k_1^2)u + 3k_1^2u^2 - k_1^4u_{\xi\xi} + c_0 = 0, \tag{8}$$

where c_0 is an integral constant. Now it is straightforward to establish the transformation relation between the solutions of (8) and the cubic nonlinear Klein–Gordon equation, the result reads

$$u = a_0 + a_1\phi^2, \tag{9}$$

where a_0 and a_1 are undetermined constants. From Eq.(9), we have

$$u_\xi = 2a_1\phi\phi_\xi, \tag{10}$$

$$u_{\xi\xi} = 2a_1(\phi_\xi^2 + \phi\phi_{\xi\xi}). \tag{11}$$

In Eq.(9), $\phi(\xi)$ is a solution of the cubic NKG equation (5). Substituting Eqs.(9)–(11) and Eqs.(5) into Eq.(8), we obtain:

$$\begin{aligned}&(3a_1^2k_1^2 - 3a_1k_1^4\mu)\phi^4 + [6a_0a_1k_1^2 - a_1k_1^2\\ &- a_1k_2^2 + a_1k_3^2 - 4a_1k_1^4\lambda]\phi^2 + 3a_0k_1^2\\ &+ a_0k_3^2 - a_0k_1^2 - a_0k_2^2 - 2a_1ck_1^4 + c_0 = 0.\end{aligned} \tag{12}$$

Owing to ϕ being arbitrary solution of the cubic NKG equation, all the coefficients of different powers of ϕ in Eq.(12) should vanish. That means

$$3a_1^2k_1^2 - 3a_1k_1^4\mu = 0, \tag{13}$$

$$6a_0a_1k_1^2 - a_1k_1^2 - a_1k_2^2 + a_1k_3^2 - 4a_1k_1^4\lambda = 0, \tag{14}$$

$$3a_0k_1^2 + a_0k_3^2 - a_0k_1^2 - a_0k_2^2 - 2a_1ck_1^4 + c_0 = 0. \tag{15}$$

Solving Eqs.(13)–(15) yields:

$$a_1 = \mu k_1^2, \tag{16}$$

$$a_0 = \frac{4\lambda k_1^4 + k_1^2 + k_2^2 - k_3^2}{6k_1^2}, \tag{17}$$

$$\begin{aligned}c_0 =&\frac{k_1^2 + 2k_2^2 - 2k_3^2}{12} + \frac{k_2^4 + k_3^4}{12k_1^2}\\ &- \frac{k_2^2k_3^2}{6k_1^2} + \frac{k_3^4}{12k_1^2} - \frac{4k_1^6\lambda^2}{3} + 2ck_1^6\mu.\end{aligned} \tag{18}$$

From the above derivations, we can see that if $\phi(\xi)$ is a solution of Eqs. (5), then $u(\xi) = a_0 + a_1\phi^2$ with Eqs.(16)–(18) must be a solution of Eq.(8).

3. Exact travelling plane wave solutions of the (2+1)-dimensional Boussinesq equations

From Ref.[11], many exact solutions of the cubic NKG equation have been given. Using of the mapping relations (9),we can obtained many explicit and exact travelling plane wave solutions of the (2+1)-dimensional Boussinesq equation,which contain solitary solutions,periodic wave solutions,Jacobian elliptic function solutions and others travelling plane wave solutions.

3.1. Solitary wave solution

From Ref.[11],we know that when $\lambda=-2,\mu=2$ and $c=1$,the solitary wave solution of (5) reads as $\phi=\tanh\xi(or\coth\xi)$; when $\lambda=1,\mu=-2$ and $c=0,\phi=\mathrm{sech}\xi$; for $\lambda=1,\mu=2$ and $c=0,\phi=\mathrm{csch}\xi$. The corresponding plane solitary wave solution of the (2+1)-dimensional Boussinesq equation Eq.(6) reads as

$$\begin{aligned}u_1 &= \frac{{k_1}^2-8\,{k_1}^4+{k_2}^2-{k_3}^2}{6\,{k_1}^2}+2\,{k_1}^2\tanh^2\xi\\ &= \frac{{k_1}^2+4\,{k_1}^4+{k_2}^2-{k_3}^2}{6\,{k_1}^2}-2\,{k_1}^2\,\mathrm{sech}^2\xi\\ &= \frac{{k_1}^2+4\,{k_1}^4+{k_2}^2-{k_3}^2}{6\,{k_1}^2}+2\,{k_1}^2\,\mathrm{csch}^2\xi. \quad (19)\end{aligned}$$

3.2. Periodic wave solution

When $\lambda=2,\mu=2$ and $c=1,\phi=\tan\xi(\mathrm{or}\cot\xi)$; when $\lambda=-1,\mu=2$ and $c=0,\phi=\sec\xi(\mathrm{or}\csc\xi)$. The corresponding periodic plane wave solution of Eq.(6) reads as

$$\begin{aligned}u_2 &= \frac{{k_1}^2+8\,{k_1}^4+{k_2}^2-{k_3}^2}{6\,{k_1}^2}+2\,{k_1}^2\tan^2\xi\\ &= \frac{{k_1}^2-4\,{k_1}^4+{k_2}^2-{k_3}^2}{6\,{k_1}^2}+2\,{k_1}^2\sec^2\xi. \quad (20)\end{aligned}$$

3.3. Jacobian elliptic function solution

When $\lambda=-(1+k^2),\mu=2k^2$ and $c=1,\phi=\mathrm{sn}\xi$; when $\lambda=2k^2-1,\mu=-2k^2$ and $c=1-k^2,\phi=\mathrm{cn}\xi$; and $\lambda=2-k^2,\mu=-2$ and $c=k^2-1,\phi=\mathrm{d}n\xi$.The corresponding Jacobian elliptic function solution of Eq.(6) reads as

$$\begin{aligned}u_3 &= \frac{{k_1}^2-4\left(1+k^2\right){k_1}^4+{k_2}^2-{k_3}^2}{6\,{k_1}^2}+2k^2k_1^2\mathrm{sn}^2\xi\\ &= \frac{{k_1}^2-4\left(1-2\,k^2\right){k_1}^4+{k_2}^2-{k_3}^2}{6\,{k_1}^2}+2k^2k_1^2\mathrm{cn}^2\xi\\ &= \frac{{k_1}^2+4\left(2-k^2\right){k_1}^4+{k_2}^2-{k_3}^2}{6\,{k_1}^2}-2k_1^2\mathrm{d}n^2\xi, \quad (21)\end{aligned}$$

where $\mathrm{sn}\xi$, $\mathrm{cn}\xi$ and $\mathrm{dn}\xi$ are Jacobian elliptic functions with modulus k.

And we have the following other Jacobian elliptic function solutions of Eq.(6):

When $\lambda=2-k^2,\mu=2(1-k^2)$ and $c=1,\phi=\mathrm{tn}\xi$,

$$\begin{aligned}u_4 =&\frac{{k_1}^2+4\left(2-k^2\right){k_1}^4+{k_2}^2-{k_3}^2}{6\,{k_1}^2}\\ &+2\left(1-k^2\right){k_1}^2\,\mathrm{tn}^2\xi; \quad (22)\end{aligned}$$

When $\lambda=(1+k^2)/2,\mu=(k^2-1)/2$ and $c=(k^2-1)/4,\phi=\mathrm{dn}\xi/(1+k\mathrm{sn}\xi)$,

$$\begin{aligned}u_5 =&\frac{{k_1}^2+2\left(1+k^2\right){k_1}^4+{k_2}^2-{k_3}^2}{6\,{k_1}^2}\\ &-\frac{\left(1-k^2\right){k_1}^2\,\mathrm{dn}^2\xi}{2\left(1+k\,\mathrm{sn}\xi\right)^2}. \quad (23)\end{aligned}$$

When $\lambda=(1+k^2)/2,\mu=(1-k^2)/2$ and $c=(1-k^2)/4,\phi=\mathrm{cn}\xi/(1+\mathrm{sn}\xi)$,

$$\begin{aligned}u_6 =&\frac{{k_1}^2+2\left(1+k^2\right){k_1}^4+{k_2}^2-{k_3}^2}{6\,{k_1}^2}\\ &+\frac{\left(1-k^2\right){k_1}^2\,\mathrm{cn}^2\xi}{2\left(1+k\,\mathrm{sn}\xi\right)^2}. \quad (24)\end{aligned}$$

When $\lambda=(k^2-2)/2,\mu=k^2/2$ and $c=k^2/4,\phi=(k\mathrm{sn}\xi)/(1+\mathrm{dn}\xi)$,

$$\begin{aligned}u_7 =&\frac{{k_1}^2-2\left(2-k^2\right){k_1}^4+{k_2}^2-{k_3}^2}{6\,{k_1}^2}\\ &+\frac{k^4\,{k_1}^2\,\mathrm{sn}^2\xi)}{2\left(1+\mathrm{dn}\xi\right)^2}. \quad (25)\end{aligned}$$

Using relations (9) and the Table 1 of Ref.[11], we can obtain many other Jacobian elliptic function solutions of Eq.(6).

3.4. The other solution

When $\lambda=0,\mu=2$ and $c=0,\phi=1/\xi$,the solution of Eq.(6) reads

$$u_8=\frac{{k_1}^2+{k_2}^2-{k_3}^2}{6\,{k_1}^2}+\frac{2\,{k_1}^2}{\xi^2}. \quad (26)$$

4. Summary

Using the deformation mapping method, we obtain many types of plane travelling wave solutions of the (2+1)-dimensional Boussinesq equation. This method we use here is standard and simple, and can be used to various types of nonlinear equations, no matter they are integrable or not. More about this method

and the properties of the plane travelling wave solutions (especially the soliton solutions)are worthwhile to study further.

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