Vol 12 No 4, April 2003
1009-1963/2003/12(04)/0357-04
Chinese Physics
© 2003 Chin. Phys. Soc.
and IOP Publishing Ltd

A set of the Lie symmetrical conservation laws for the rotational relativistic Birkhoffian system*

Luo Shao-Kai(罗绍凯)[a)b)] and Cai Jian-Le(蔡建乐)[b)]

[a)] *Institute of Mathematical Mechanics and Mathematical Physics, Changsha University, Changsha* 410003, *China*

[b)] *Department of Physics, Hunan University, Changsha* 410082, *China*

(Received 5 December 2002; revised manuscript received 25 December 2002)

For a rotational relativistic Birkhoffian system, a set of the Lie symmetries and conservation laws is given under infinitesimal transformations. On the basis of the invariance of rotational relativistic Birkhoffian equations under infinitesimal transformations, Lie symmetrical transformations of the system are constructed, which only depend on the Birkhoffian variables. The determining equations of Lie symmetries are given, and a new type of non-noether conserved quantities are directly obtained from Lie symmetries of the system. An example is given to illustrate the application of the results.

Keywords: rotational relativistic, Birkhoffian system, Lie symmetry, non-noether conserved quantity
PACC: 0320, 0330, 0220

1. Introduction

With the developments in science and technology, more and more experimental facts have been related to the problem of high speed spinning.[1] In 1985, Carmeli[2] constructed the theory of rotational relativistic mechanics. Since 1996, we have obtained the theory of analytical mechanics for the rotational relativistic system, and given its basic theoretical frame.[3−12] The dynamics of the Birkhoffian system is more general than the Hamiltonian mechanics, and much attention is paid to it in the field of mathematics, mechanics and physics.[13−26] The Hamiltonian mechanics has been extensively applied to the fields of modern physics. Thus the dynamics of Birkhoffian systems will also be applied to the fields of modern physics. Recently, Refs.[27–32] have given the basic theoretical frame of the rotational relativistic Birkhoffian system.

The conservation laws for a dynamical system play an important role in the fields of mathematics, mechanics and physics. There are two main methods used to seek the conserved quantity. One is based on the theory of invariance of Hamilton's action under infinitesimal transformations, namely, the Noether theory.[33] The other was given by Lutzky[34] in 1979 who introduced Lie's method of expansive group of invariance for ordinary differential equations under infinitesimal transformations to the mechanical system, and presented the Lie symmetries of differential equations of motion. In recent 20 years, in Refs.[35–49] has been developed the method of Lie symmetries, and many important results have been obtained. In general, the Lie symmetry of a dynamical system does not always imply the conserved quantity, it can lead to the conserved quantity only under certain conditions. There are two main methods used to seek the Lie symmetrical conserved quantities. One[7,24,25,35−44] is based on the Noether symmetries, and the other[45−49] is directly based on the Lie symmetries of the system.

This paper presents a method for obtaining conserved quantities from the Lie symmetries of rotational relativistic Birkhoffian system. In terms of the invariance of rotational relativistic Birkhoffian equations under infinitesimal transformations, the Lie symmetrical transformations of the system are given, which only depend on the Birkhoffian variables. A set of non-noether conserved quantities are directly obtained from the Lie symmetries of the system. Also, an example is given to illustrate the application of the results.

*Project supported by the National Natural Science Foundation of China (Grant No 19972010), the Natural Science Foundation of Henan Province, China (Grant Nos 998040080 and 984053100), and the Scientific Research Foundation of the Education Bureau of Hunan Province, China (Grant No 02C033).
†Tel:0731-4250396, 13054188159; E-mail:mmmplsk@163.com

2. A set of the Lie symmetries of rotational relativistic Birkhoffian systems

Here we study the fixed axis rotation of a mechanical system composed of N particles. At time t, the ith particle has an angular velocity $\dot{\theta}_i$, where as the limiting angular velocity is Γ_i, the classical rotational moment of inertia is I_{0i}, and its relativistic rotational moment of inertia is [2,3]

$$I_i = \frac{I_{0i}}{\sqrt{1-\dot{\theta}_i^2/\Gamma_i^2}}. \tag{1}$$

We can construct the Birkhoffian B^* and the Birkhoff functions $R_\nu^*(\nu=1,\cdots,2n)$ of the rotational relativistic system as

$$B^* = B^*(I_i(t,a^\mu),t,a^\mu),$$

$$R_\nu^* = R_\nu^*(I_i(t,a^\mu),t,a^\mu), \quad (\nu,\mu=1,\cdots,2n); \tag{2}$$

and we let

$$\tilde{B}^* = \tilde{B}^*(t,a^\mu) = B^*(I_i(t,a^\mu),t,a^\mu),$$

$$\tilde{R}_\nu^**{} = \tilde{R}_\nu^*(t,a^\mu) = R_\nu^*(I_i(t,a^\mu),t,a^\mu). \tag{3}$$

Then the Birkhoffian equations of the rotational relativistic system can be written in the form[27,28]

$$\left(\frac{\partial\tilde{R}_\nu^*}{\partial a^\mu}-\frac{\partial\tilde{R}_\mu^*}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial\tilde{B}^*}{\partial a^\mu}-\frac{\partial\tilde{R}_\mu^*}{\partial t}=0,$$

$$(\nu,\mu=1,\cdots,2n), \tag{4}$$

where the repeated subscripts represent the summation. In general, it is supposed that the system (4) is non-singular, i.e.,

$$\det(\omega_{\mu\nu}^*)\neq 0, \quad \omega_{\mu\nu}^* = \left(\frac{\partial\tilde{R}_\nu^*}{\partial a^\mu}-\frac{\partial\tilde{R}_\mu^*}{\partial a^\nu}\right). \tag{5}$$

If $\tilde{R}_\mu^*$ and $\tilde{B}^*$ do not include time t explicitly, the system (4) is autonomous. If $\tilde{R}_\mu^*$ do not include time t explicitly, the system (4) is semi-autonomous.

From Eqs.(4), we can obtain

$$\dot{a}^\mu = \omega^{\mu\nu*}\left(\frac{\partial\tilde{B}^*}{\partial a^\nu}+\frac{\partial\tilde{R}_\nu^*}{\partial t}\right), \quad (\mu,\nu=1,\cdots,2n), \tag{6}$$

where

$$\omega^{\mu\nu*}\omega_{\nu\tau}^* = \delta_{\mu\tau}. \tag{7}$$

Eqs. (6) can be expressed as follows

$$\dot{a}^\mu = h_\mu(t,a^\nu), \quad (\mu,\nu=1,\cdots,2n). \tag{8}$$

We introduce infinitesimal transformations for a^μ

$$t^*=t, \quad a^{\mu*}(t^*)=a^\mu(t)+\Delta a^\mu, \quad (\mu=1,\cdots,2n), \tag{9}$$

and their expanding forms are

$$t^*=t, \quad a^{\mu*}=a^\mu+\varepsilon\xi_\mu(t,a^\nu), \tag{10}$$

where ε is an infinitesimal parameter, and ξ_μ are called the infinitesimal generators. Under the infinitesimal transformations (10), we introduce a vector of generator

$$X^{(0)} = \xi_\mu\frac{\partial}{\partial a^\mu}, \tag{11}$$

and its first extension

$$X^{(1)} = \xi_\mu\frac{\partial}{\partial a^\mu}+\dot{\xi}_\mu\frac{\partial}{\partial\dot{a}^\mu}. \tag{12}$$

From the theory of the invariance of differential equations under infinitesimal transformations, we know that the invariance of Eqs.(4) under the infinitesimal transformations(10) is

$$X^{(1)}=\left[\dot{a}^\nu-\omega^{\mu\nu*}\left(\frac{\partial\tilde{B}^*}{\partial a^\nu}+\frac{\partial\tilde{R}_\nu^*}{\partial t}\right)\right]=0. \tag{13}$$

Noticing the inverse Eqs. (6) of Eq.(4), we can obtain the determining equations of the rotational relativistic Birkhoffian system

$$\dot{\xi}_\mu = X^{(0)}\left[\omega^{\mu\nu*}\left(\frac{\partial\tilde{B}^*}{\partial a^\nu}+\frac{\partial\tilde{R}_\nu^*}{\partial t}\right)\right],$$

$$(\mu,\nu=1,\cdots,2n). \tag{14}$$

If the generators ξ_μ of the infinitesimal transformations (10) satisfy the determining Eqs. (14), then the transformations (10) are called Lie symmetries. So we have the following Lie symmetrical theorem for the rotational relativistic Birkhoffian system.

Theorem 1. For the rotational relativistic Birkhoffian system (4), if the generators ξ_μ of the infinitesimal transformations (10) satisfy the determining Eqs. (14), then the corresponding symmetries are the Lie symmetries of the system.

3. A set of the Lie symmetrical conserved quantities of rotational relativistic Birkhoffian systems

In general, the Lie symmetry of a dynamical system does not always imply the conserved quantity; however, it can only lead to the conserved quantity

under certain conditions. Now, we give the conditions under which a Lie symmetry can lead to a conserved quantity and the form of the quantity for the rotational relativistic Birkhoffian system.

Theorem 2. For the rotational relativistic Birkhoffian system (4), if the generators ξ_μ of the infinitesimal transformations (10) satisfy the Lie symmetrical determining Eqs. (14), and there exists a function $G = G(a^\mu)$ satisfying the following condition

$$\frac{\partial h_\mu}{\partial a^\mu} + \frac{1}{G}\frac{\partial G}{\partial a^\rho}h_\rho = 0, \tag{15}$$

then the system (4) has the Lie symmetrical conserved quantities as follows

$$\Phi = \frac{1}{G}\frac{\partial}{\partial a^\mu}(G\xi_\mu) = \text{constant}. \tag{16}$$

In fact,

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{G}\frac{\partial G}{\partial a^\mu}\right)\xi_\mu + \frac{1}{G}\frac{\partial G}{\partial a^\mu}\dot{\xi}_\mu + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \xi_\mu}{\partial a^\mu}. \tag{17}$$

Using Eqs.(8) and (11), the determining Eqs. (14) can be written in the form

$$\dot{\xi}_\mu - \xi_\rho\frac{\partial h_\mu}{\partial a^\rho} = 0, \quad (\mu = 1, \cdots, 2n), \tag{18}$$

so we have

$$\frac{\partial \dot{\xi}_\mu}{\partial a^\mu} - \frac{\partial \xi_\rho}{\partial a^\mu}\frac{\partial h_\mu}{\partial a^\rho} - \xi_\rho\frac{\partial^2 h_\mu}{\partial a^\rho \partial a^\mu} = 0. \tag{19}$$

Using Eq.(19), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \xi_\mu}{\partial a^\mu} = \frac{\partial^2 h_\mu}{\partial a^\mu \partial a^\rho}\xi_\rho. \tag{20}$$

Substituting Eqs.(18) and (20) into Eq.(17), we have

$$\begin{aligned}\frac{\mathrm{d}\Phi}{\mathrm{d}t} =& \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{G}\frac{\partial G}{\partial a^\mu}\right)\xi_\mu \\ &+ \frac{1}{G}\frac{\partial G}{\partial a^\mu}\frac{\partial h_\mu}{\partial a^\rho}\xi_\rho + \frac{\partial^2 h_\mu}{\partial a^\mu \partial a^\rho}\xi_\rho.\end{aligned} \tag{21}$$

From Eq.(15), we obtain

$$\frac{\partial^2 h_\mu}{\partial a^\mu \partial a^\rho} = -\frac{\partial}{\partial a^\rho}\left(\frac{1}{G}\frac{\partial G}{\partial a^\nu}\right)h_\nu - \frac{1}{G}\frac{\partial G}{\partial a^\nu}\frac{\partial h_\nu}{\partial a^\rho}. \tag{22}$$

Substituting Eq.(22) into Eq.(21) and Noticing

$$\frac{\partial}{\partial a^\rho}\left(\frac{1}{G}\frac{\partial G}{\partial a^\nu}\right) = \frac{\partial}{\partial a^\nu}\left(\frac{1}{G}\frac{\partial G}{\partial a^\rho}\right), \tag{23}$$

we can obtain

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = 0. \tag{24}$$

So Eq.(16) is the Lie symmetrical conserved quantities of the system (4).

From theorem 2, we can obtain an important deduction.

Deduction For the rotational relativistic Birkhoffian system (4), if the generators ξ_μ of the infinitesimal transformations (10) satisfy the Lie symmetrical determining Eqs. (14), and the system satisfy the following condition

$$\frac{\partial h_\mu}{\partial a^\mu} = 0, \tag{25}$$

then the system (4) has the Lie symmetrical conserved quantities as follows

$$\Phi = \frac{\partial \xi_\mu}{\partial a^\mu} = \text{constant}. \tag{26}$$

4. Example

As an example, we study the high-speed spinning of a body around a fixed axis, of which the classical rotational moment of inertia is I_0 and the relativistic equations of motion are

$$\ddot{q} = \frac{M_0}{I}\left(1 - \frac{\dot{q}^2}{\Gamma^2}\right) = \frac{M_0}{I_0}\left(1 - \frac{\dot{q}^2}{\Gamma^2}\right)^{3/2},$$

$$I = \frac{I_0}{\sqrt{1 - \dot{q}^2/\Gamma^2}}, \tag{27}$$

where M_0 = constant and Γ=constant. Letting $a^1 = q$ and $a^2 = \dot{q}$, we can obtain[32]

$$\tilde{B}^* = \frac{I_0}{M_0}\Gamma^2\left(1 - \frac{(a^2)^2}{\Gamma^2}\right)^{-1/2} - a^1,$$

$$\tilde{R}_1^* = 0, \quad \tilde{R}_2^* = \frac{I_0}{M_0}\left(\frac{\Gamma^2}{M_0} - a^1\right)\left(1 - \frac{(a^2)^2}{\Gamma^2}\right)^{-1}. \tag{28}$$

Eqs. (4) gives

$$\dot{a}^1 = h_1 = a^2, \quad \dot{a}^2 = h_2 = \frac{M_0}{I_0}\left(1 - \frac{(a^2)^2}{\Gamma^2}\right)^{3/2}. \tag{29}$$

The determining Eq. (14) gives

$$\dot{\xi}_1 = \xi_2, \quad \dot{\xi}_2 = -\frac{3M_0 a^2}{I_0\Gamma^2}\left(1 - \frac{(a^2)^2}{\Gamma^2}\right)^{1/2}\xi_2, \tag{30}$$

and these have solutions

$$\begin{aligned}\xi_1 =& \exp\left[-\frac{3M_0 a^2}{I_0\Gamma^2}\left(1 - \frac{(a^2)^2}{\Gamma^2}\right)^{1/2}t\right] + C_1, \\ \xi_2 =& -\frac{I_0\Gamma^2}{3M_0 a^2}\left(1 - \frac{(a^2)^2}{\Gamma^2}\right)^{-1/2} \\ &\times \exp\left[-\frac{3M_0 a^2}{I_0\Gamma^2}\left(1 - \frac{(a^2)^2}{\Gamma^2}\right)^{1/2}t\right] + C_2.\end{aligned} \tag{31}$$

From theorem 1, we know that the generator (31) corresponds to the Lie symmetry of the system (28).

From Eq.(15), we have

$$\frac{\partial}{\partial a^2}\left[\frac{M_0}{I_0}\left(1-\frac{(a^2)^2}{\Gamma^2}\right)^{3/2}\right]+\frac{1}{G}\frac{\partial G}{\partial a^1}\cdot a^2 + \frac{1}{G}\frac{\partial G}{\partial a^2}\left[\frac{M_0}{I_0}\left(1-\frac{(a^2)^2}{\Gamma^2}\right)^{3/2}\right]=0. \tag{32}$$

The above equation has the following solutions

$$G=-\frac{M_0}{I_0}\left(1-\frac{(a^2)^2}{\Gamma^2}\right)^{3/2}. \tag{33}$$

Substituting Eqs.(31) and (33) into Eq.(16), we can obtain

$$\Phi=\left[\frac{I_0}{M_0}a^2\left(1-\frac{(a^2)^2}{\Gamma^2}\right)^{-1}+\left(1-\frac{(a^2)^2}{\Gamma^2}\right)^{1/2}t - \frac{(a^2)^2}{\Gamma^2}\left(1-\frac{(a^2)^2}{\Gamma^2}\right)^{1/2}t\right] \times\exp\left[-\frac{3M_0a^2}{I_0\Gamma^2}\left(1-\frac{(a^2)^2}{\Gamma^2}\right)^{1/2}t\right]=\text{constant} \tag{34}$$

From theorem 2, we know that Eq.(34) is the Lie symmetrical conserved quantity corresponding to the system (28).

References

[1] Bengtsson R and Frauendorf S 1979 *Nucl. Phys.* A **327** 139
[2] Carmeli M 1986 *Int. J. Theor. Phys.* **25** 89
[3] Luo S K 1996 *J. Beijing Inst. Technol.* **16** (S1) 154 (in Chinese)
[4] Luo S K 1998 *Appl. Math. Mech.* **19** 45
[5] Luo S K, Guo Y X and Chen X W 2001 *Acta Phys. Sin.* **50** 2053(in Chinese)
[6] Fu J L, Chen X W and Luo S K 1999 *Appl. Math. Mech.* **20** 1266
[7] Fu J L, Chen X W and Luo S K 2000 *Appl. Math. Mech.* **21** 549
[8] Zhang Y L, Qiao Y F and Ma Y P 1999 *Acta Mech. Solid. Sin.* **20** 356 (in Chinese)
[9] Qiao Y F, Li R J and Men J 2001 *Acta Phys. Sin.* **50** 1639 (in Chinese)
[10] Fang J H 2000 *Acta Phys. Sin.* **49** 1028 (in Chinese)
[11] Fang J H 2001 *Acta Phys. Sin.* **50** 390 (in Chinese)
[12] Luo S K 2002 *Acta Phys. Sin.* **51** 712 (in Chinese)
[13] Birkhoff G D 1927 *Dynamical System* (New York: American Mathematical Society)
[14] Santilli R M 1983 *Foundations of Theoretical Mechanics II* (New York: Springer-Verlag)
[15] Mei F X 1993 *Sci. China* A **36** 1456
[16] Mei F X 1993 *Chin. Sci. Bull.* **38** 816
[17] Wu H B and Mei F X 1995 *Chin. Sci. Bull.* **40** 885 (in Chinese)
[18] Mei F X and Lévesque B 1995 *Trans. CSME* **19** 59
[19] Mei F X 1996 *Chin. Sci. Bull.* **41** 641
[20] Mei F X, Shi R C, Zhang Y F and Wu H B 1996 *Dynamics of Birkhoff Systems* (Beijing: Beijing Institute of Technology Press) (in Chinese)
[21] Mei F X 1997 *Appl. Math. Mech.* **18** 55
[22] Chen X W, Luo S K and Mei F X 2000 *Acta Mech. Sol. Sin.* **21** 251 (in Chinese)
[23] Guo Y X, Luo S K and Mei F X 2001 *Rep. Math. Phys.* **47** 33
[24] Mei F X 1999 *Applications of Lie Groups and Lie Algebras to Constrined Mechanical Systems* (Beijing: Science Press) (in Chinese)
[25] Chen X W 2002 *Global Analysis for Birkhoff Systems* (Kaifeng: Henan University Press) (in Chinese)
[26] Mei F X 2001 *Int. J Non-linear Mech.* **36** 817
[27] Luo S K, Fu J L and Chen X W 2001 *Acta Phys. Sin.* **50** 383 (in Chinese)
[28] Luo S K, Chen X W and Fu J L 2001 *Chin. Phys.* **10** 271
[29] Luo S K, Guo Y X, Chen X W and Fu J L 2001 *Acta. Phys. Sin.* **50** 2049 (in Chinese)
[30] Luo S K, Chen X W and Guo Y X 2002 *Chin. Phys.* **11** 523
[31] Luo S K 2002 *Chin. Phys.* **11** 1197
[32] Luo S K 2002 *Commun. Theor. Phys.* **38** 257
[33] Noether A E 1918 *Nachr. Akad Wiss. Math. Phys.* **2** 235
[34] Lutzky M 1979 *J. Phys. A: Math. Gen.* **12** 973
[35] Prince G E and Elizer C J 1981 *J. Phys. A: Math. Gen.* **14** 587
[36] Thompson G 1986 *J. Phys. A: Math. Gen.* **19** 105
[37] Zhao Y Y 1994 *Acta Mech. Sin.* **26** 380
[38] Mei F X, Wu R H and Zhang Y F 1998 *Acta. Mech. Sin.* **30** 468
[39] Mei F X, Zhang Y F and Shang M 1999 *Mech. Res. Commun.* **26** 7
[40] Fu J L and Wang X M 2000 *Acta Phys. Sin.* **49** 1023 (in Chinese)
[41] Zhang Y and Mei F X 2000 *Chin. Sci. Bull.* **45** 1354
[42] Luo S K 2002 *Chin. Phys. Lett.* **19** 449
[43] Luo S K 2002 *Chin. Phys.* **11** 429
[44] Luo S K 2000 *J. Shangqiu Tech. Coll.* **16** (2) 5 (in Chinese)
[45] Hojman S A 1992 *J. Phys. A: Math. Gen.* **25** L291
[46] Lutzky M 1998 *Int. J. Non-linear Mech.* **33** 393
[47] Lutzky M 1999 *Int. J. Non-linear Mech.* **34** 387
[48] Zhang Y 2002 *Chin. Quart. Mech.* **23** 392 (in Chinese)
[49] Zhang Y 2002 *Acta. Phys. Sin.* **51** 461 (in Chinese)