Vol 12 No 4, April 2003
1009-1963/2003/12(04)/0351-06
Chinese Physics
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# Stability for the equilibrium state manifold of relativistic Birkhoffian systems

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(Received 29 September 2002; revised manuscript received 10 January 2003)

In this paper, the stability of equilibrium state manifold for relativistic Birkhoffian systems is studied. The equilibrium state equations, the disturbance equations and their first approximation are presented. The criteria of stability for the equilibrium state manifold are obtained. The relationship between the stability of the equilibrium-state manifold of relativistic Birkhoffian systems and that of classical Birkhoffian systems is discussed. An example is given to illustrate the results.

**Keywords:** relativity, Birkhoffian systems, stability, equilibrium state, manifold
**PACC:** 0316, 0412

## 1. Introduction

The stability of mechanical systems is an important problem both in theory and application, and the stability of the equilibrium position of nonholonomic systems has attracted many researchers for nearly a century. The first contribution on the problem was made by Whittaker.[1] Bottema,[2] Aiserman and Gantmacher,[3] Karapelyan and Rumyantsev,[4] Neimark and Fufaev,[5] Rumyantsev,[4,6,7] and Mikhailov and Parton[8] also made great progress and obtained a series of important results. These results, however, are confined to linear, homogenous and steady systems with nonholonomic constraints. Mei and co-workers[9−14] studied the stability of the equilibrium-state manifold of Birkhoffian systems as well as that of nonlinear nonholonomic systems systematically, and made great achievement. Lately, Luo *et al*[15] studied the stability of the equilibrium-state manifold of nonholonomic systems in a non-inertial reference frame, and obtained some important results.

These previous works were confined to classical mechanical systems. It is very significant to extend the equilibrium stability of mechanical systems to that of relativistic Birkhoffian systems considering the relativistic aspect due to high-speed motion.[16]

In this paper, the equilibrium-state equations, the disturbance equations and their first approximation of autonomous, semi-autonomous and non-autonomous relativistic Birkhoffian systems are obtained. The stability of the equilibrium-state manifold of the systems is studied.

## 2. Equations of motion and equilibrium equation of relativistic Birkhoffian system

### 2.1. Equations of motion of relativistic Birkhoffian system

Consider a mechanical system in a high-speed motion which is composed of $N$ particles. The velocity of the $i$th particle at instant $t$ is $\dot{\boldsymbol{r}}_i (i = 1, \cdots, N)$, and the relativistic mass of the $i$th particle is

$$m_i = m_{0i}/\sqrt{1 - (\dot{\boldsymbol{r}}_i)^2/c^2}, \quad (i = 1, \cdots, N), \qquad (1)$$

where $c$ is the velocity of light.

When the configuration of the system is specified with the generalized coordinates $q_s (s = 1, \cdots, 2n)$, the

equations of motion of a relativistic Birkhoffian system is then expressed in a general form as[18,19]

$$\left(\frac{\partial \widetilde{R}_\nu^\bullet}{\partial a^\mu}-\frac{\partial \widetilde{R}_\mu^\bullet}{\partial a^\nu}\right)\dot{a}^\nu-\left(\frac{\partial \widetilde{B}^\bullet}{\partial a^\mu}+\frac{\partial \widetilde{R}_\mu^\bullet}{\partial t}\right)=0,$$
$$(\mu,\nu=1,\cdots,2n), \tag{2}$$

in which

$$\widetilde{B}^\bullet=\widetilde{B}^\bullet(t,\boldsymbol{a})=B^\bullet(m_i(t,\boldsymbol{a}),t,\boldsymbol{a}),$$
$$\widetilde{R}_\nu^\bullet=\widetilde{R}_\nu^\bullet(t,\boldsymbol{a})=R_\nu^\bullet(m_i(t,\boldsymbol{a}),t,\boldsymbol{a}),$$
$$(\mu,\nu=1,\cdots,2n;i=1,\cdots,N), \tag{3}$$

are called the relativistic Birkhoffian and the relativistic Birkhoff functions respectively. Partially differentiating $\widetilde{B}^\bullet,\widetilde{R}_\nu^\bullet$ yields

$$\begin{aligned}\frac{\partial \widetilde{B}^\bullet}{\partial a^\mu}=&\frac{\partial B^\bullet}{\partial m_i}\frac{\partial m_i}{\partial a^\mu}+\frac{\partial B^\bullet}{\partial a^\mu},\\ \frac{\partial \widetilde{R}_\nu^\bullet}{\partial a^\mu}=&\frac{\partial R_\nu^\bullet}{\partial m_i}\frac{\partial m_i}{\partial a^\mu}+\frac{\partial R_\nu^\bullet}{\partial a^\mu},\\ \frac{\partial \widetilde{R}_\nu^\bullet}{\partial t}=&\frac{\partial R_\nu^\bullet}{\partial m_i}\frac{\partial m_i}{\partial t}+\frac{\partial R_\nu^\bullet}{\partial t}.\end{aligned} \tag{4}$$

The dynamical systems or the physical systems represented by Eqs.(2) are called the relativistic Birkhoffian system.

Relativistic Birkhoff Eqs. (2) can be written in an equivalent form

$$(\widetilde{\omega}_{\mu\nu}^\bullet)\dot{a}^\nu-\left(\frac{\partial \widetilde{B}^\bullet}{\partial a^\mu}+\frac{\partial \widetilde{R}_\mu^\bullet}{\partial t}\right)=0,(\mu,\nu=1,\cdots,2n), \tag{5}$$

where

$$\begin{aligned}\widetilde{\omega}_{\mu\nu}^\bullet=&\left(\frac{\partial \widetilde{R}_\nu^\bullet}{\partial a^\mu}-\frac{\partial \widetilde{R}_\mu^\bullet}{\partial a^\nu}\right)\\ =&\frac{\partial R_\nu^\bullet}{\partial m_i}\frac{\partial m_i}{\partial a^\mu}+\frac{\partial R_\nu^\bullet}{\partial a^\mu}-\frac{\partial R_\mu^\bullet}{\partial m_i}\frac{\partial m_i}{\partial a^\nu}-\frac{\partial R_\mu^\bullet}{\partial a^\nu}\end{aligned} \tag{6}$$

is called the relativistic Birkhoff tensor. Relativistic Birkhoff Eqs. (2) are called autonomous when the functions $\widetilde{R}_\nu^\bullet$ and $\widetilde{B}^\bullet$ do not depend explicitly on time, in which case Eqs. (2) can be expressed in the simplified form

$$(\widetilde{\omega}_{\mu\nu}^\bullet(\boldsymbol{a}))\dot{a}^\nu-\frac{\partial \widetilde{B}^\bullet(\boldsymbol{a})}{\partial a^\mu}=0,\ (\mu,\nu=1,\cdots,2n). \tag{7}$$

Relativistic Birkhoff Eqs. (2) are called semi-autonomous if only functions $\widetilde{R}_\nu^\bullet$ do not depend explicitly on time, in which case they have the more general form

$$(\widetilde{\omega}_{\mu\nu}^\bullet)\dot{a}^\nu-\frac{\partial \widetilde{B}^\bullet(t,\boldsymbol{a})}{\partial a^\mu}=0,\ (\mu,\nu=1,\cdots,2n). \tag{8}$$

Relativistic Birkhoff Eqs.(2) are called non-autonomous when both $\widetilde{R}_\nu^\bullet$ and $\widetilde{B}^\bullet$ are dependent explicitly on time, in which case they are of the form of Eqs.(2), which we rewrite as

$$\widetilde{\omega}_{\mu\nu}^\bullet(t,\boldsymbol{a})\dot{a}^\nu-\left(\frac{\partial \widetilde{B}^\bullet(t,\boldsymbol{a})}{\partial a^\mu}+\frac{\partial \widetilde{R}_\mu^\bullet(t,\boldsymbol{a})}{\partial t}\right)=0,$$
$$(\mu,\nu=1,\cdots,2n). \tag{9}$$

Eqs. (7)–(9) are called regular if

$$\det(\widetilde{\omega}_{\mu\nu}^\bullet)\neq 0, \tag{10}$$

They are said to be covariant when they are of type (7), (8), or (9), in which case the corresponding tensor $\widetilde{\omega}_{\mu\nu}^\bullet$ is called the covariant relativistic Birkhoff tensor; and they are said to be contravariant when Eqs. (7), 8) and (9) are written in the equivalent form

$$\dot{a}^\mu-\widetilde{\omega}^{\bullet\mu\nu}(\boldsymbol{a})\frac{\partial \widetilde{B}^\bullet(\boldsymbol{a})}{\partial a^\nu}=0,\ (\mu,\nu=1,\cdots,2n); \tag{11}$$

$$\dot{a}^\mu-\widetilde{\omega}^{\bullet\mu\nu}(\boldsymbol{a})\frac{\partial \widetilde{B}^\bullet(t,\boldsymbol{a})}{\partial a^\nu}=0,\ (\mu,\nu=1,\cdots,2n); \tag{12}$$

$$\dot{a}^\mu-\widetilde{\omega}^{\bullet\mu\nu}(t,\boldsymbol{a})\left(\frac{\partial \widetilde{B}^\bullet(t,\boldsymbol{a})}{\partial a^\nu}+\frac{\partial \widetilde{R}_\nu^\bullet(t,\boldsymbol{a})}{\partial t}\right)=0,$$
$$(\mu,\nu=1,\cdots,2n), \tag{13}$$

where the tensor

$$\widetilde{\omega}^{\bullet\mu\nu}=\left(\left|\widetilde{\varpi}_{\alpha\beta}^\bullet\right|^{-1}\right)^{\mu\nu}=\left(\left|\frac{\partial \widetilde{R}_\beta^\bullet}{\partial a^\alpha}-\frac{\partial \widetilde{R}_\alpha^\bullet}{\partial a^\beta}\right|^{-1}\right)^{\mu\nu},$$
$$\det(\widetilde{\omega}^{\bullet\mu\nu})\neq 0,\quad(\alpha,\beta,\mu,\nu=1,\cdots,2n), \tag{14}$$

is called the contravariant relativistic Birkhoff tensor.

### 2.2. Equilibrium equations of relativistic Birkhoffian system

Suppose the equilibrium positions of relativistic Birkhoffian system is

$$a^\nu=a_0^\nu=\text{constant},\ \dot{a}_0^\nu=0,\ (\nu=1,\cdots,2n). \tag{15}$$

Substituting Eqs.(15) into Eqs.(11) leads to the equilibrium equations of the autonomous relativistic Birkhoffian system

$$\left(\frac{\partial \widetilde{B}^\bullet}{\partial a^\mu}\right)_0=\left(\frac{\partial B^\bullet}{\partial m_i}\frac{\partial m_i}{\partial a^\mu}+\frac{\partial B^\bullet}{\partial a^\mu}\right)_0=0,$$
$$\widetilde{B}^\bullet=B^\bullet(m_i(\boldsymbol{a}),\boldsymbol{a}),\ (\mu=1,\cdots,2n). \tag{16}$$

Substituting Eqs.(15) into Eqs.(12) leads to the equilibrium equations of the semi-autonomous relativistic Birkhoffian system

$$\left(\frac{\partial \widetilde{B}^{\bullet}}{\partial a^{\mu}}\right)_0 = \left(\frac{\partial B^{\bullet}}{\partial m_i}\frac{\partial m_i}{\partial a^{\mu}} + \frac{\partial B^{\bullet}}{\partial a^{\mu}}\right)_0 = 0,$$

$$\widetilde{B}^{\bullet} = B^{\bullet}(m_i(t,\boldsymbol{a}),t,\boldsymbol{a}),\ \ (\mu = 1,\cdots,2n). \tag{17}$$

and substituting Eqs.(15) into Eqs.(13) yields the equilibrium equations of the non-autonomous relativistic Birkhoffian system

$$\left(\frac{\partial \widetilde{B}^{\bullet}}{\partial a^{\mu}}\right)_0 + \left(\frac{\partial \widetilde{R}^{\bullet}_{\mu}}{\partial t}\right)_0$$
$$= \left(\frac{\partial B^{\bullet}}{\partial m_i}\frac{\partial m_i}{\partial a^{\mu}} + \frac{\partial B^{\bullet}}{\partial a^{\mu}}\right)_0 + \left(\frac{\partial R^{\bullet}_{\mu}}{\partial m_i}\frac{\partial m_i}{\partial t} + \frac{\partial R^{\bullet}_{\mu}}{\partial t}\right)_0 = 0,$$

$$(\mu = 1,\cdots,2n), \tag{18}$$

In Eqs.(16), (17) and (18), $(\ )_0$ denotes the results obtained from Eqs.(15). The systems have equilibrium positions when Eqs.(16), (17) and (18) have solutions. These equilibrium positions form manifolds if the equilibrium equations depend on each other.

# 3. Disturbance equations and first approximate equations of a relativistic Birkhoffian system

## 3.1. Disturbance equations and first approximate equations for a non-autonomous relativistic Birkhoffian system

Suppose that $a^{\nu} = a^{\nu}_0 (\nu = 1,\cdots,2n)$ are isolated equilibrium positions. Let

$$a^{\nu} = a^{\nu}_0 + \xi^{\nu},\ \ (\nu = 1,\cdots,2n). \tag{19}$$

Substituting Eqs.(19) into Eqs.(13) yields

$$\dot{\xi}^{\mu} - (\widetilde{\omega}^{\bullet\mu\nu})_1\left[\left(\frac{\partial \widetilde{B}^{\bullet}}{\partial a^{\nu}}\right)_1 + \left(\frac{\partial \widetilde{R}^{\bullet}_{\nu}}{\partial t}\right)_1\right] = 0$$

$$(\mu,\nu = 1,\cdots,2n), \tag{20}$$

where the subscript 1 denotes the result obtained by replacing $a^{\nu}$ with $a^{\nu}_0 + \xi^{\nu}$. Expanding $(\ )_1$, etc into Taylor series near equilibrium positions, we obtain

$$(\widetilde{\omega}^{\bullet}_{\mu\nu})_1 = (\omega^{\bullet}_{\mu\nu})_0 + \left(\frac{\partial \omega^{\bullet}_{\mu\nu}}{\partial m_i}\frac{\partial m_i}{\partial a^{\rho}} + \frac{\partial \omega^{\bullet}_{\mu\nu}}{\partial a^{\rho}}\right)_0 \xi^{\rho} + \cdots,$$

$$(\mu,\nu,\rho = 1,\cdots,2n);$$

$$\left(\frac{\partial \widetilde{B}^{\bullet}}{\partial a^{\mu}}\right)_1 = \left(\frac{\partial B^{\bullet}}{\partial m_i}\frac{\partial m_i}{\partial a^{\mu}} + \frac{\partial B^{\bullet}}{\partial a^{\mu}}\right)_0$$
$$+ \frac{\partial}{\partial a^{\mu}}\left(\frac{\partial B^{\bullet}}{\partial m_i}\frac{\partial m_i}{\partial a^{\nu}} + \frac{\partial B^{\bullet}}{\partial a^{\nu}}\right)_0 \xi^{\nu} + \cdots,$$

$$(\mu,\nu,\rho = 1,\cdots,2n);$$

$$\left(\frac{\partial \widetilde{R}^{\bullet}_{\mu}}{\partial t}\right)_1 = \left(\frac{\partial R^{\bullet}_{\mu}}{\partial m_i}\frac{\partial m_i}{\partial t} + \frac{\partial R^{\bullet}_{\mu}}{\partial t}\right)_0$$
$$+ \frac{\partial}{\partial t}\left(\frac{\partial R^{\bullet}_{\mu}}{\partial m_i}\frac{\partial m_i}{\partial a^{\nu}} + \frac{\partial R^{\bullet}_{\mu}}{\partial a^{\nu}}\right)_0 \xi^{\nu} + \cdots,$$

$$(\mu,\nu,\rho = 1,\cdots,2n);$$

$$(\widetilde{\omega}^{\bullet\mu\nu})_1 = (\omega^{\bullet\mu\nu})_0 + \left(\frac{\partial \omega^{\bullet\mu\nu}}{\partial m_i}\frac{\partial m_i}{\partial a^{\rho}} + \frac{\partial \omega^{\bullet\mu\nu}}{\partial a^{\rho}}\right)_0 \xi^{\rho} + \cdots,$$

$$(\mu,\nu,\rho = 1,\cdots,2n), \tag{21}$$

where the second and higher-order terms have been omitted. Substituting Eqs.(21) into Eqs.(20) and using Eqs.(18) lead to

$$\dot{\xi}^{\mu} - (\omega^{\bullet\mu\nu})_0\left[\frac{\partial}{\partial a^{\nu}}\left(\frac{\partial B^{\bullet}}{\partial m_i}\frac{\partial m_i}{\partial a^{\rho}} + \frac{\partial B}{\partial a^{\rho}}\right)_0\right.$$
$$\left. + \frac{\partial}{\partial t}\left(\frac{\partial R^{\bullet}_{\mu}}{\partial m_i}\frac{\partial m_i}{\partial a^{\rho}} + \frac{\partial R^{\bullet}_{\nu}}{\partial a^{\rho}}\right)_0\right]\xi^{\rho} = \widetilde{\Phi}_{\mu}(\xi^{\nu},t), \tag{22}$$

where $\widetilde{\Phi}_{\mu}$ denotes for the second and higher-order terms of $\xi^{\mu}$ and $t$. From Eqs.(22), one obtains the first approximate equations of the disturbance equations for non-autonomous relativistic Birkhoffian system as

$$\dot{\xi}^{\mu} - (\widetilde{\omega}^{\bullet\mu\nu})_0(\widetilde{\Omega}^{\bullet}_{\nu\rho})_0\xi^{\rho} = 0, \tag{23}$$

in which

$$\widetilde{\Omega}^{\bullet}_{\mu\nu} = \frac{\partial}{\partial a^{\mu}}\left(\frac{\partial B^{\bullet}}{\partial m_i}\frac{\partial m_i}{\partial a^{\nu}} + \frac{\partial B^{\bullet}}{\partial a^{\nu}}\right)$$
$$+ \frac{\partial}{\partial t}\left(\frac{\partial R^{\bullet}_{\mu}}{\partial m_i}\frac{\partial m_i}{\partial a^{\nu}} + \frac{\partial R^{\bullet}_{\mu}}{\partial a^{\nu}}\right). \tag{24}$$

## 3.2. Disturbance equations and first approximate equations of an autonomous relativistic Birkhoffian system

Substituting Eqs.(19) into Eqs.(11) leads to

$$\dot{\xi}^{\mu} - (\widetilde{\omega}^{\bullet\mu\nu})_1\left(\frac{\partial B^{\bullet}}{\partial m_i}\frac{\partial m_i}{\partial a^{\nu}} + \frac{\partial B^{\bullet}}{\partial a^{\nu}}\right)_1 = 0,$$

$$(\mu,\nu = 1,\cdots,2n). \tag{25}$$

Expanding $(\ )_1$, etc near the equilibrium positions, and then substituting these results and Eqs.(16)

into Eqs.(25), the disturbance equations of the autonomous relativistic Birkhoffian system are obtained as

$$\dot{\xi}^{\mu} - (\widetilde{\omega}^{\bullet\mu\nu})_0(\widetilde{\Omega}^{\bullet}_{\mu\rho})_0\xi^{\rho} = \Phi_{\mu}(m_i(\xi^{\nu}),\xi^{\nu}),$$
$$(\mu,\nu,\rho = 1,\cdots,2n), \tag{26}$$

where

$$\widetilde{\Omega}^{\bullet}_{\mu\nu}(m_i(\xi^{\nu}),\xi^{\nu}) = \frac{\partial}{\partial a^{\mu}}\left(\frac{\partial B^{\bullet}}{\partial m_i}\frac{\partial m_i}{\partial a^{\nu}} + \frac{\partial B^{\bullet}}{\partial a^{\nu}}\right), \tag{27}$$

and $\Phi_{\mu}$ are the second and higher-order terms of $\xi^{\mu}$.

The first approximations of the disturbance equations for the autonomous relativistic Birkhoffian system are obtained from Eqs.(22) as

$$\dot{\xi}^{\mu} - (\widetilde{\omega}^{\bullet\mu\nu})_0(\widetilde{\Omega}^{\bullet}_{\mu\nu})_0\xi^{\rho} = 0. \tag{28}$$

### 3.3. Disturbance equations and first approximate equations for semi-autonomous relativistic Birkhoffian systems

Substituting Eqs.(19) into (12) yields the disturbance equations

$$\dot{\xi}^{\mu} - (\widetilde{\omega}^{\bullet\mu\nu})_0(\widetilde{\Omega}^{\bullet}_{\mu\rho})_0\xi^{\rho} = \Theta_{\mu}(m_i(t,\xi^{\nu}),t,\xi^{\nu}),$$
$$(\mu,\nu,\rho = 1,\cdots,2n), \tag{29}$$

and the first approximate equations

$$\dot{\xi}^{\mu} - (\widetilde{\omega}^{\bullet\mu\nu})_0(\widetilde{\Omega}^{\bullet}_{\mu\rho})_0\xi^{\rho} = 0, \tag{30}$$

where

$$\widetilde{\Omega}^{\bullet}_{\mu\nu}(m_i(t,\xi^{\nu}),t,\xi^{\nu}) = \frac{\partial}{\partial a^{\mu}}\left(\frac{\partial B^{\bullet}}{\partial m_i}\frac{\partial m_i}{\partial a^{\nu}} + \frac{\partial B^{\bullet}}{\partial a^{\nu}}\right). \tag{31}$$

## 4. Criteria of the stability of equilibrium state manifold for a relativistic Birkhoffian system

We have proved from the above discussion that the manifold of the equilibrium positions of a relativistic Birkhoffian system is formed when the equilibrium positions are not isolated. The stability of equilibrium-state manifold of the system is discussed in this section.

Suppose an autonomous relativistic Birkhoffian system (11) has the integral

$$I(m_i(a^{\nu}),a^{\nu}) = C_0, \tag{32}$$

which forms manifold $\pounds$ as follows:

$$\pounds = \{a^{\nu}|I(m_i(a^{\nu}),a^{\nu}) = C_0 = \text{constant}\}. \tag{33}$$

The manifold $\pounds$ is an invariant set of Eqs.(7). Suppose that the equilibrium positions of Eqs.(16) form a manifold $\varepsilon$ as

$$\varepsilon = \left\{a^{\nu}\left|\left(\frac{\partial B^{\bullet}}{\partial m_i}\frac{\partial m_i}{\partial a^{\mu}} + \frac{\partial B^{\bullet}}{\partial a^{\mu}}\right)_0 = 0\right.\right\}. \tag{34}$$

Based on Lyapnuov's stability theories,[20] some criteria of the stability for the equilibrium-state manifold of the system are given as follows

**Proposition 1** For the disturbance Eqs. (25) of the autonomous relativistic Birkhoffian system (11), if there is a positively (negatively) function $V(a^{\nu})$ for $\varepsilon$ in the manifold $\pounds$, and its orbital derivative $\dot{V}$ in the direction of the vector-field defined by Eqs. (25) is negatively (positively) semi-definite of $\varepsilon$ in the manifold $\pounds$, the equilibrium-state manifold $\varepsilon$ is stable in the manifold $\pounds$.

**Proposition 2** For the disturbance Eqs. (25) of the autonomous relativistic Birkhoffian system (11), if there is a positively (negatively) function $V(a^{\nu})$ for $\varepsilon$ in the manifold $\pounds$, and its orbital derivative $\dot{V}$ in the direction of the vector-field defined by Eqs.(25) is not a negatively (positively) semi-definite function of $\varepsilon$ in the manifold $\pounds$, the equilibrium-state manifold $\varepsilon$ is unstable in the manifold $\pounds$.

Propositions 1 and 2 are general results. We can study the stability of the equilibrium-state manifold in the semi-autonomous relativistic Birkhoffian system (12) and non-autonomous relativistic Birkhoffian system (13) in the same way. Then one can selected relativistic Birkhoffian $\widetilde{B}^{\bullet}$ as function $V$ to study the stability of the equilibrium-state manifold for the autonomous relativistic Birkhoffian systems, since the following equality is always held

$$\frac{\mathrm{d}\widetilde{B}^{\bullet}}{\mathrm{d}t} = \frac{\partial B^{\bullet}}{\partial m_i}\frac{\partial m_i}{\partial a^{\nu}}\dot{a}^{\nu} + \frac{\partial B^{\bullet}}{\partial a^{\nu}}\dot{a}^{\nu} = 0. \tag{35}$$

**Proposition 3** For the disturbance Eqs. (25) of the autonomous relativistic Birkhoffian system (11), if the relativistic Birkhoffian $\widetilde{B}^{\bullet}$ is a positively (negatively) function of $\varepsilon$ in the manifold $\pounds$, the equilibrium-state manifold $\varepsilon$ of the autonomous relativistic Birkhoffian system is stable in the manifold $\pounds$.

## 5. Example

The fourth-order Birkhoffian and Birkhoff functions are given by

$$\begin{aligned}\widetilde{B}^{\bullet} =& m\{b_{11}(a^1+a^2)^2+2b_{12}(a^1+a^2)\\ &\times(a^3+a^4)+b_{22}(a^3+a^4)^2\},\\ \widetilde{R}_1^{\bullet} =& \widetilde{R}_2^{\bullet}=0, \quad \widetilde{R}_3^{\bullet}=m\left[\frac{1}{2}(a^1)^2+a^1\right],\\ \widetilde{R}_4^{\bullet} =& m\left[\frac{1}{2}(a^2)^2+a^2\right],\end{aligned} \tag{36}$$

in which

$$a^1\neq 1, \quad \begin{vmatrix} b_{11} & b_{12}\\ b_{12} & b_{22}\end{vmatrix}\neq 0, \tag{37}$$

and

$$m=m_0/\sqrt{1-[(a^1+a^2)^2+(a^3+a^4)^2]/c^2} \tag{38}$$

is the relativistic mass of the system, and $c$ is the velocity of light.

The relativistic Birkhoffian system is autonomous. Suppose the positions of equilibrium of the system are

$$a^{\nu}=a_0^{\nu}=\text{constant}, \quad \dot{a}_0^{\nu}=0, \tag{39}$$

Substituting functions $\widetilde{B}^{\bullet}$, $\widetilde{R}_{\nu}^{\bullet}$ and Eqs.(d) into Eqs.(11), one has the equilibrium equations

$$\frac{(a_0^1+a_0^2)[b_{11}(a_0^1+a_0^2)^2+2b_{12}(a_0^1+a_0^2)(a_0^3+a_0^4)+b_{22}(a_0^3+a_0^4)^2]}{c^2-(a_0^1+a_0^2)^2-(a_0^3+a_0^4)^2}+2b_{11}(a_0^1+a_0^2)+2b_{12}(a_0^3+a_0^4)=0, \tag{40}$$

$$\frac{(a_0^3+a_0^4)[b_{11}(a_0^1+a_0^2)^2+2b_{12}(a_0^1+a_0^2)(a_0^3+a_0^4)+b_{22}(a_0^3+a_0^4)^2]}{c^2-(a_0^1+a_0^2)^2-(a_0^3+a_0^4)^2}+2b_{12}(a_0^1+a_0^2)+2b_{22}(a_0^3+a_0^4)=0. \tag{41}$$

One of the solutions is as follows

$$(a_0^1+a_0^2)=0, \quad (a_0^3+a_0^4)=0. \tag{42}$$

The equilibrium-state manifold of the system is described as

$$\varepsilon=\{a^{\mu}|a_0^1+a_0^2=0, \quad a_0^3+a_0^4=0\}. \tag{43}$$

The relativistic Birkhoffian $\widetilde{B}^{\bullet}$ is positive definite on the manifold $\varepsilon$ if and only if

$$b_{11}>0, \quad \begin{vmatrix} b_{11} & b_{12}\\ b_{12} & b_{22}\end{vmatrix}>0; \tag{44}$$

and the relativistic Birkhoffian $\widetilde{B}^{\bullet}$ is negative definite if and only if

$$-b_{11}>0, \quad \begin{vmatrix} -b_{11} & -b_{12}\\ -b_{12} & -b_{22}\end{vmatrix}>0. \tag{45}$$

It can be deduced from proposition 3 that the equilibrium-state manifold of the system (36) is stable when $b_{11}$, $b_{12}$ and $b_{22}$ satisfy the conditions (44) or (45).

## 6. The relationship between the stability equilibrium state manifold of the relativistic Birkhoffian systems and that of the classical Birkhoffian systems

In the case of the classical approximation, expanding $\sqrt{1-(\dot{\boldsymbol{r}}_i)^2/c^2}$ into power series on $(\dot{\boldsymbol{r}}_i)^2/c^2$ and extracting the first two terms, we obtain

$$m_i=m_{0i}/\sqrt{1-(\dot{\boldsymbol{r}}_i)^2/c^2}\approx m_{0i},$$

$$\widetilde{B}^{\bullet}(t,\boldsymbol{a})=B^{\bullet}(m_i(t,\boldsymbol{a}),t,\boldsymbol{a})\approx B(t,\boldsymbol{a}),$$

$$\widetilde{R}_{\nu}^{\bullet}(t,\boldsymbol{a})=R_{\nu}^{\bullet}(m_i(t,\boldsymbol{a}),t,\boldsymbol{a})=R_{\nu}(t,\boldsymbol{a}).$$

So the stability of the equilibrium-state manifold of the classical Birkhoffian systems is obtained.[20]

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