Vol 12 No 3, March 2003
1009-1963/2003/12(03)/0287-07

Chinese Physics

© 2003 Chin. Phys. Soc.
and IOP Publishing Ltd

Quantum entanglement in a two-dimensional ion trap*

Wang Cheng-Zhi(王成志) and Fang Mao-Fa(方卯发)†

Department of Physics, Hunan Normal University, Changsha 410081, *China*

(Received 21 May 2002; revised manuscript received 9 December 2002)

In this paper, we investigate the quantum entanglement in a two-dimensional ion trap system. We discuss the quantum entanglement between the ion and phonons by using reduced entropy, and that between two degrees of freedom of the vibrational motion along x and y directions by using quantum relative entropy. We discuss also the influence of initial state of the system on the quantum entanglement and the relation between two entanglements in the trapped ion system.

Keywords: trapped ion, quantum entanglement, reduced entropy, relative entropy
PACC: 4250

1. Introduction

When the quantum states of two or more particles cannot be factored into a direct product of the quantum states of each of the particles, the states are said to be entangled. Quantum entanglement (QE) is a very important notion in quantum mechanics. Aside from being of importance for testing Bell's theorem,[1] QE plays a central role in quantum information (QI). Therefore, it is significant for both revealing the quantum property of a system and QI processing to quantify the entanglement when there are interactions among the particles. Recently, much attention has been paid to the cooling and trapping of ions due to the fundamental physics involved and potential applications in QI[2] and the generation of quantum states.[3−5] Some authors investigated QE in the dynamics of a trapped ion, and discussed the QE between the external degrees of freedom(phonons) and the internal degrees of freedom(ion) of the trapped ion system.[6] However, these results are obtained only under the condition that the trapped ion system is one-dimensional, and no discussion of the entanglement is made in the two-dimensional (2D) trapped ion system. In fact, when the vibrational motion of the ionic centre of mass is 2D, the two degrees of freedom of the phonons, namely 2D harmonic motions, are entangled, although there is no direct interaction between the two degrees of freedom of the phonons. In this paper, we discuss not only the QE between the ion and phonons by using reduced entropy but also the QE between the two degrees of freedom of the vibrational motions along x and y directions by using quantum relative entropy.

2. Model and effective Hamiltonian

An ion confined in an electromagnetic trap is described as a particle moving in a harmonic potential where the vibrational motion of the centre of the trapped ion mass is quantized as a harmonic oscillation. When the trapped ion is driven by a classical laser, its quantized internal degrees of freedom (called an ion characterized by the electronic states $|\pm\rangle$) and its quantized external degrees of freedom (named as a phonon characterized by the vibrational motion of the trapped ion) are coupled and interacted. We consider a two-level trapped ion of mass M in a 2D harmonic potential in which vibrational motion has two components along x and y directions, respectively. At time t=0, two π-out of phase laser beams having the same frequency ω_l, intensity E_0 and wavevector modulus k_l illuminate the trapped ion along the two orthogonal directions x' and y' with an angle of θ relative to the x and y directions, respectively. At any time $t > 0$, in the dipole and the rotating wave approximations (RWA),[7] the Hamiltonian of the system can be writ-

*Projects supported by the National Natural Science Foundation of China (Grant No 19874020), the Natural Science Foundation of Hunan province (Grant No 01JJY3030) and the Ministry of Education, China.
†Corresponding author.

ten as

$$H = \frac{p_x^2 + p_y^2}{2M} + \frac{1}{2}M\gamma^2(x^2+y^2) + \frac{\hbar\omega_0}{2}\sigma_z + \{\varepsilon E_0[\mathrm{e}^{\mathrm{i}(\omega_l t - k_l x')} - \mathrm{e}^{\mathrm{i}(\omega_l t - k_l y')}]\sigma_- + \mathrm{h.c.}\}, \quad (1)$$

where $p_{x(y)}$ is the x-component (y-component) of the momentum along $x(y)$ direction, γ is the trap frequency, ε (assumed to be real) is the dipole element, $\sigma_z = |+\rangle\langle+| - |-\rangle\langle-|$ and $\sigma_\pm = |\pm\rangle\langle\mp|$ are the inversion operator and two transition operators of the trapped ion, $\hbar\omega_0$ is the energy between the ionic excited and ground states. Defining

$$a = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{M\gamma}{\hbar}}x + \mathrm{i}\frac{1}{\sqrt{M\gamma\hbar}}p_x\right),$$

$$b = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{M\gamma}{\hbar}}y + \mathrm{i}\frac{1}{\sqrt{M\gamma\hbar}}p_y\right) \quad (2)$$

in a frame rotating at the laser frequency ω_l, we write H as $H = H_0 + H_1$ with

$$H_0 = \hbar\gamma(a^+a + b^+b) + \frac{\hbar\Delta}{2}\sigma_z, \quad (3)$$

$$H_1 = \hbar\Omega\{[\mathrm{e}^{-\mathrm{i}\eta(a_\theta + a_\theta^+)} - \mathrm{e}^{-\mathrm{i}\eta(b_\theta + b_\theta^+)}] + \mathrm{h.c.}\}, \quad (4)$$

where $\Delta = \omega_0 - \omega_l = 2\gamma$ is the atom-laser detuning, $\Omega = \dfrac{\varepsilon E_0}{\hbar}$ is the Rabi frequency proportional to the amplitude of the laser, $\eta = k_l\sqrt{\dfrac{\hbar}{2M\gamma}}$ is the Lamb–Dicke parameter, $a_\theta = a\cos\theta + b\sin\theta$ and $b_\theta = -a\sin\theta + b\cos\theta$ are annihilation operators of vibrational quanta in the directions x' and y', respectively. In the interaction picture, the Hamiltonian takes the form

$$\begin{aligned} H_I =& \exp\left(\frac{\mathrm{i}}{\hbar}H_0 t\right)H_1\exp\left(-\frac{\mathrm{i}}{\hbar}H_0 t\right) \\ =& \hbar\Omega\exp[-(2\mathrm{i}\gamma t - \eta^2/2)]\sigma_- \\ & \times\{\exp[-\mathrm{i}\eta a_\theta\exp(-\mathrm{i}\gamma t)]\exp[-\mathrm{i}\eta a_\theta^+\exp(\mathrm{i}\gamma t)] \\ & - \exp[-\mathrm{i}\eta b_\theta\exp(-\mathrm{i}\gamma t)]\exp[-\mathrm{i}\eta b_\theta^+\exp(\mathrm{i}\gamma t)]\} + \mathrm{h.c.} \\ =& \hbar\Omega\exp(-\eta^2)\Bigg\{\sum_{k,l=0}^{\infty}(-1)^{k+l}\frac{(\mathrm{i}\eta)^{k+l}}{k!l!} \\ & \times a_\theta^l(a_\theta^+)^k\exp[\mathrm{i}\gamma t(k-l-2)] \\ & - \sum_{k,l=0}^{\infty}(-1)^{k+l}\frac{(\mathrm{i}\eta)^{k+l}}{k!l!}b_\theta^l(b_\theta^+)^k \\ & \times\exp[\mathrm{i}\gamma t(k-l-2)]\Bigg\}\sigma_- + \mathrm{h.c.} \end{aligned} \quad (5)$$

Under the conditions $\gamma \gg \Omega$, $\eta \ll 1$ (Lamb–Dicke limit), and in the frame of the RWA, we neglect all the oscillating terms of the harmonic frequency γ and make η^{k+l} maximum. For $\theta = -\pi/4$, the effective Hamiltonian can be expressed as[5]

$$H_I = \hbar\lambda(a^+b^+\sigma_- + ab\sigma_+), \quad (6)$$

where $\lambda = \Omega\eta^2\mathrm{e}^{-\eta^2/2}$. In the next section, we discuss some problems using Eq.(6).

3. Entropy dynamics and entanglement

If we assume that at $t = 0$ the ion is in a coherent superposition of the excited state $|+\rangle$ and ground state $|-\rangle$

$$|\Psi_a(0)\rangle = \cos\left(\frac{\alpha}{2}\right)|+\rangle + \exp(-\mathrm{i}\varphi)\sin\left(\frac{\alpha}{2}\right)|-\rangle, \quad (7)$$

and the phonons are in the $SU(1,1)$ related coherent state,[8]

$$|\xi,q\rangle = (1-|\xi|^2)^{\frac{1+q}{2}}\sum_{n=0}^{\infty}\sqrt{\frac{(n+q)!}{n!q!}}\xi^n|n+q,n\rangle$$

$$(0<|\xi|^2<1, q=0,1,2,\ldots), \quad (8)$$

where $0 \le \alpha \le \pi$ denotes the ionic distribution angle, $0 \le \varphi \le 2\pi$ is the phase of the ion dipole, $|n+q,n\rangle = |n+q\rangle_a|n\rangle_b$ is the product of the vibrational Fock states along the directions x and y. $|\xi,q\rangle$, the eigenstate of operator $a^+a - b^+b$, belongs to $SU(1,1)$ related coherent states, if q=0, it reduces to two-mode squeezed vacuum state.

The initial state of the system is given by

$$\begin{aligned} |\Psi(0)\rangle =& |\xi,q\rangle(\cos(\alpha/2)|+\rangle \\ & + \exp(-\mathrm{i}\varphi)\sin(\alpha/2)|-\rangle). \end{aligned} \quad (9)$$

At time $t = 0$, the two laser pulses are applied to the ion. By using Eq.(6), the time evolution operator of the system is given by

$$\begin{aligned} U_I(t) =& \cos[\sqrt{(aa^+bb^+}\lambda t]|+\rangle\langle+| \\ & + \cos[\sqrt{a^+ab^+b}\lambda t]|-\rangle\langle-| \\ & - \mathrm{i}\frac{\sin[\sqrt{aa^+bb^+}\lambda t]}{\sqrt{aa^+bb^+}}|+\rangle\langle-| \\ & - \mathrm{i}\frac{\sin[\sqrt{a^+ab^+b}\lambda t]}{\sqrt{a^+ab^+b}}|-\rangle\langle+|. \end{aligned} \quad (10)$$

At any time, the state of the system can be written as

$$\begin{aligned} |\Psi(t)\rangle =& U_I(t)|\Psi(0)\rangle \\ =& |\Phi_+(t)\rangle|+\rangle + |\Phi_-(t)\rangle|-\rangle, \end{aligned} \quad (11)$$

where

$$
\begin{aligned}
&|\Phi_+(t)\rangle \\
&=\sum_n\{\cos(\alpha/2)f_n\cos[\sqrt{(n+1)(n+1+q)}\lambda t] \\
&\quad -\mathrm{i}\sin(\alpha/2)\exp(-\mathrm{i}\varphi)f_{n+1} \\
&\quad \times\sin[\sqrt{(n+1)(n+1+q)}\lambda t]\}|n+q,n\rangle, \\
&|\Phi_-(t)\rangle \\
&=\sum_n\{\sin(\alpha/2)\exp(-\mathrm{i}\varphi)f_n\cos[\sqrt{n(n+q)}\lambda t] \\
&\quad -\mathrm{i}\cos(\alpha/2)f_{n-1}\sin[\sqrt{n(n+q)}\lambda t]\}|n+q,n\rangle, \quad (12)
\end{aligned}
$$

with

$$
f_n=(1-|\xi|^2)^{\frac{1+q}{2}}\sqrt{\frac{(n+q)!}{n!q!}}\xi^n. \quad (13)
$$

The density operator of the system is given by

$$
\rho(t)=|\Psi(t)\rangle\langle\Psi(t)|. \quad (14)
$$

Tracing over the degrees of the ion, we obtain the reduced phonon density operator

$$
\rho_p(t)=|\Phi_+(t)\rangle\langle\Phi_+(t)|+|\Phi_-(t)\rangle\langle\Phi_-(t)|, \quad (15)
$$

its eigenvalues and eigenvectors after a straightforward calculation can be expressed as, respectively,

$$
\begin{aligned}
\Pi^{\pm}(t)=&\langle\Phi_+(t)|\Phi_+(t)\rangle \\
&\pm\exp(\mp\beta)|\langle\Phi_+(t)|\Phi_-(t)\rangle|, \quad (16)
\end{aligned}
$$

$$
\begin{aligned}
|\psi_p^{\pm}(t)\rangle=&\frac{1}{\sqrt{\Pi^{\pm}(t)+\Pi^{\mp}(t)\exp(\mp2\beta)}} \\
&\times(|\Phi_+(t)\pm\exp(-\mathrm{i}\delta\mp\beta)|\Phi_-(t)\rangle), \quad (17)
\end{aligned}
$$

where

$$
\beta=\sinh^{-1}\left[\frac{\langle\Phi_+(t)|\Phi_+(t)\rangle-\langle\Phi_-(t)|\Phi_-(t)\rangle}{2|\langle\Phi_+(t)|\Phi_-(t)\rangle|}\right], \quad (18)
$$

$$
\delta=\tan^{-1}\left[\frac{\mathrm{Rm}(\langle\Phi_+(t)|\Phi_-(t)\rangle)}{\mathrm{Re}(\langle\Phi_+(t)|\Phi_-(t)\rangle)}\right]. \quad (19)
$$

Similarly, we obtain the reduced ionic density operator

$$
\begin{aligned}
\rho_i(t)=&\langle\Phi_+(t)|\Phi_+(t)\rangle|+\rangle\langle+| \\
&+\langle\Phi_-(t)|\Phi_+(t)\rangle|+\rangle\langle-| \\
&+\langle\Phi_+(t)|\Phi_-(t)\rangle|-\rangle\langle+| \\
&+\langle\Phi_-(t)|\Phi_-(t)\rangle|-\rangle\langle-|. \quad (20)
\end{aligned}
$$

The eigenvalues $\Pi^{\pm}(t)$ of this operator are identical to the phonon eigenvalues, Eq.(16), and its eigenstates are

$$
\begin{aligned}
|\psi_i^{\pm}(t)\rangle=&\frac{1}{\sqrt{\Pi^{\pm}(t)+\Pi^{\mp}(t)\exp(\mp2\beta)}} \\
&\times(|+\rangle\pm\exp(\mathrm{i}\delta\mp\beta)|-\rangle). \quad (21)
\end{aligned}
$$

According to the Schmidt theorem,[9] the state vector of the system can be rewritten as

$$
\begin{aligned}
|\Psi(t)\rangle=&\sqrt{\Pi^+(t)}|\psi_p^+(t)\rangle|\psi_i^+(t)\rangle \\
&+\sqrt{\Pi^-(t)}|\psi_p^-(t)\rangle|\psi_i^-(t)\rangle. \quad (22)
\end{aligned}
$$

In what follows, we will utilize the above results to investigate QE in the trapped ion system.

3.1. Quantum entanglement between the ion and phonons

The quantum entanglement between the ion and phonons can be measured by the reduced entropy of the ion or the phonons,

$$
\begin{aligned}
S(t)=&-\mathrm{Tr}[\rho_{p(i)}(t)\log_2\rho_{p(i)}(t)] \\
=&-\Pi^+(t)\log_2\Pi^+(t)-\Pi^-(t)\log_2\Pi^-(t). \quad (23)
\end{aligned}
$$

It reflects the degree of entanglement between the ion and the phonons: if it takes the minimal value zero, the ion and phonons are disentangled; if it takes the maximal value 1, the ion and phonons are maximally entangled.

Numerical results for the time evolution of the reduced entropy of the entanglement are displayed in Fig.1(assuming $\varphi=0$ for simplicity in all figures in this paper) for different q and α but same ξ. It is found that q and α have strong influence on the reduced entropy of the entanglement.

Firstly, we discuss the influence of q on the reduced entropy of the entanglement. In the case of $q=0$, as in Fig.1(a), the reduced entropy of the entanglement is periodic. At the time $\lambda t=n\pi$ and $\frac{2n-1}{2}\pi(n\in N)$, the reduced entropy of the entanglement is zero, this means that the trapped ion and the phonons are disentangled. This can be interpreted from Eqs.(16) and (22). At $\lambda t=n\pi$ and $\frac{2n-1}{2}\pi$, $\Pi^+=1,\Pi^-=0$, the system state $|\Psi(t)\rangle$ described by Eq.(22) is written as

$$
\left|\Psi\left(\lambda t=n\pi,\frac{2n-1}{2}\pi\right)\right\rangle=|\Phi_p^+\rangle\otimes|\Phi_i^+\rangle, \quad (24)
$$

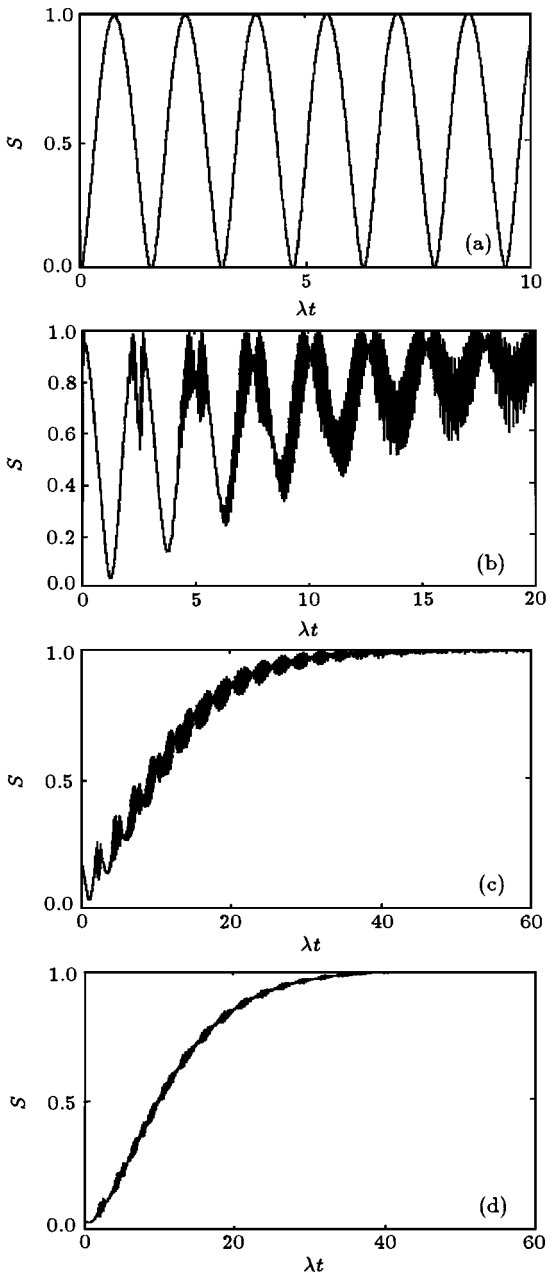


Fig.1. The time evolution of the reduced entropy of the entanglement. (a) $q = 0, \alpha = 0$; (b) $q = 40, \alpha = 0$; (c) $q = 40, \alpha = \pi/4$; (d) $q = 40, \alpha = \pi/2$.

which can be written in a direct product form. Therefore, S equals zero, and the ion and phonons are disentangled. At the rest of the time, the system state cannot be written in the direct product form, so the external and internal degrees of freedom are entangled. In particular, at $\lambda t = \dfrac{2n-1}{4}\pi$, the reduced entropy is approximately equal to 1. This means that the ion and the phonons are maximally entangled, and the system state is an EPR (Einstein–Podolsky–Rosen) state [10] which is widely used in QI processing,

$$\left|\Psi\left(\lambda t = \frac{2n-1}{4}\pi\right)\right\rangle \approx \frac{1}{\sqrt{2}}\left[|\Phi_p^+\rangle|\Phi_i^+\rangle + |\Phi_p^-\rangle|\Phi_i^-\rangle\right]. \quad (25)$$

When $q \neq 0$, three cases for different initial ion states, namely $\alpha = 0, \dfrac{\pi}{4}$, and $\dfrac{\pi}{2}$, are considered in Figs.1(b), 1(c) and 1(d). Comparing with the case of $q = 0$ in Fig.1(a), the reduced entropy of the entanglement has no regular period and does not achieve the minimum value zero, indicating that the trapped ion and the phonons are entangled all the time during the interaction. Given a longer time, the reduced entropy of the entanglement for $q \neq 0$ tends to the maximum value 1, indicating that the ion and the phonons are maximally entangled after a long evolution time. Further study shows that, when q increases, the reduced entropy of the entanglement oscillates rapidly. This stems from the fact that the increase in q leads to a more strong interaction between the ion and the phonons.

It is apparent from Fig.1 that the initial ionic state has also a considerable influence on the reduced entropy of the entanglement. When $q \neq 0$, the case for $\alpha = 0$(the excited state), $\alpha = \pi/4$ and $\pi/2$ (the ion initial state is in coherent state) are considered in Figs.1(b)–(d). Comparing Fig.1(c) with Fig.1(b), we find that in Fig.1(c), the peak of the maximum reduced entropy of the entanglement becomes small and the oscillation of the reduced entropy of the entanglement becomes slow. Especially, in Fig.1(d) corresponding to the case of $\alpha = \pi/2$, the peak and oscillation almost disappear, and as a result, the reduced entropy of the entanglement increases gradually as time goes on. Finally, it reaches the maximum value 1, showing that the ion and phonons are maximally entangled. This corresponds to the fact that, when $\alpha = \pi/2$, the ion is initially in the trapped state, the ionic state is a dressed-state(stationary state), not showing any dynamic evolution. The small vibrations of the reduced entropy in these cases, however, come from the contributions of the interactions between the ion and phonons.

3.2. Quantum entanglement between the two degrees of freedom of the phonons

The above discussion shows that the Hamiltonian given by Eq.(6) leads to the entanglement between the ion and phonons, so the state of the phonons con-

sisting of two degrees of freedom of the phonons may evolve into a mixed state. In this case, the QE between the two degrees of freedom of the phonons cannot be measured by the reduced entropy of the entanglement. However, the relative entropy of the entanglement is a good measure for the degree of entanglement of two-particle mixed states, which is defined as,[11]

$$E_R(\rho) = \min_{\sigma \in D} S(\rho||\sigma), \tag{26}$$

where $S(\rho||\sigma) = \mathrm{Tr}[\rho(\log_2 \rho - \log_2 \sigma)]$. This is the quantum relative entropy, its minimum is taken over D, the set of all disentangled states. The relative entropy of the entanglement is viewed as the minimal "distance" between the state ρ and the disentangled state σ. For a pure state, the relative entropy of the entanglement reduces to the reduced entropy of the entanglement.

For the two-particle mixed states, it is difficult to calculate numerically the relative entropy of the entanglement. Before investigating the QE, we introduce a theorem given by Ref.[12] which is very useful to numerically compute the relative entropy of the entanglement.

If a bipartite quantum state can be expressed as

$$\rho = \sum_{n_1,n_2} a_{n_1,n_2} |\Phi_{n_1}\phi_{n_1}\rangle\langle\Phi_{n_2}\phi_{n_2}|, \tag{27}$$

then the relative entropy of the entanglement is given by

$$E_R(\rho) = -\sum_n a_{n,n} \log_2 a_{n,n} + \mathrm{Tr}(\rho \log_2 \rho), \tag{28}$$

and the disentangled state σ that minimizes the quantum relative entropy is given by

$$\sigma = \sum_n a_{n,n} |\Phi_n\phi_n\rangle\langle\Phi_n\phi_n|.$$

It is clear that $\rho_p(t)$ given by Eq.(15) can be written in the same form as that in Eq.(27)

$$\rho_p(t) = \sum_{m,n}^{\infty} a_{m,n} |m+q, m\rangle\langle n+q, n|, \tag{29}$$

with

$$a_{m,n} = \langle m+p, m|\rho_p|n+p, n\rangle. \tag{30}$$

Consequently, the relative entropy of the entanglement of the state Eq.(15) is written as

$$\begin{aligned} E_R(\rho_p) &= -\sum_n a_{nn} \log_2 a_{nn} - S(\rho_p) \\ &= -\sum_n a_{nn} \log_2 a_{nn} + \Pi^+(t) \log_2 \Pi^+(t) \\ &\quad + \Pi^-(t) \log_2 \Pi^-(t). \end{aligned} \tag{31}$$

The numerical results of Eq.(31) are shown in Fig.2 for the parameter values being the same as in Fig.1.

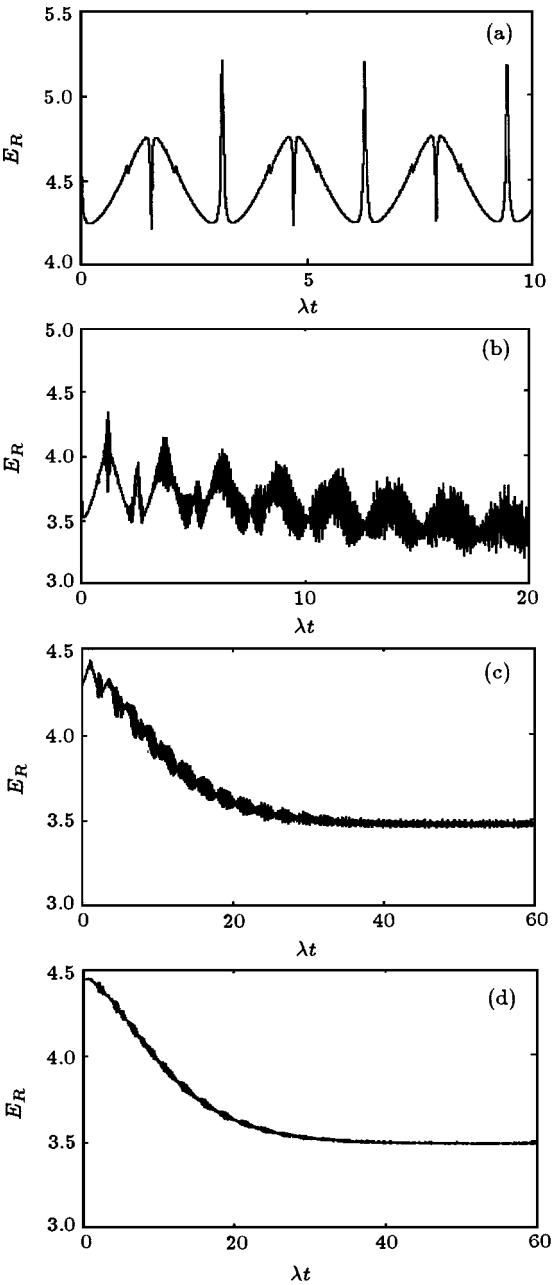


Fig.2. The time evolution of the relative entropy of the entanglement. (a) $q = 0, \alpha = 0$; (b) $q = 40, \alpha = 0$; (c) $q = 40, \alpha = \pi/4$; (d) $q = 40, \alpha = \pi/2$.

Fig. 2(a) for $q = 0$ shows the QE between the two degrees of freedom of the phonons. It is obvious that QE exhibits the evolution with a π periodicity. At time $\lambda t = \dfrac{2n-1}{2}\pi$ when the phonons are disentangled from the ion (Fig.1(a)), the relative entropy of

entanglement achieves minimal values. While at time $\lambda t = n\pi$ when the ion and phonons are also disentangled, the relative entropy of the entanglement achieves maximal values. The difference of the two cases can be explained from Eqs.(11) and (12)as follows. At $\lambda t = \frac{2n-1}{2}\pi$, the phonon state is given by

$$\begin{aligned}&\left|\psi_p\left(\lambda t = \frac{2n-1}{2}\pi\right)\right\rangle \\ =&[1+|\xi|^2]^{1/2}\sum_{n=0}^{\infty}(-1)^n f_{2n} \\ &\times[\sin(\alpha/2)\exp(-\mathrm{i}\varphi)|2n,2n\rangle \\ &-\mathrm{i}\cos(\alpha/2)|2n+1,2n+1\rangle, \end{aligned} \tag{32}$$

the corresponding relative entropy of the entanglement takes the minimal value, given by

$$\begin{aligned}&E_R\left(\lambda t = \frac{2n-1}{2}\pi\right) \\ =&-\log_2(1-|\xi|^4) - 2\cos^2(\alpha/2)\log_2\cos(\alpha/2) \\ &-2\sin^2(\alpha/2)\log_2\sin(\alpha/2) \\ &+\frac{4|\xi|^4}{1-|\xi|^4}\log_2|\xi|. \end{aligned} \tag{33}$$

While at $\lambda t = n\pi$, the phonon state is

$$\begin{aligned}&|\psi_p(\lambda t = n\pi)\rangle \\ =&\begin{cases}\sum\limits_n f_n(-1)^n|n,n\rangle, & \text{for } n = \text{odd}, \\ \sum\limits_n f_n|n,n\rangle, & \text{for } n = \text{even}, \end{cases}\end{aligned} \tag{34}$$

the relative entropy of entanglement of the above two states is equal to the relative entropy of the entanglement of initial phonon state for $q = 0$

$$\begin{aligned}E_R(\lambda t = n\pi) =& -\frac{1}{1-|\xi|^2}\log_2(1-|\xi|^2) \\ &-\frac{|\xi|^2}{1-|\xi|^2}\log_2\frac{|\xi|^2}{1-|\xi|^2}. \end{aligned} \tag{35}$$

It is easy to find that the latter is larger than the former.

Figures 2(b), 2(c) and 2(d) display the time evolution of the relative entropy of the entanglement for $q \neq 0$. We can find that the relative entropy of the entanglement becomes smaller and smaller, and the oscillation and the peaks of maximum values of the relative entropy of entanglement decrease as time goes on. This is opposite to the case for the reduced entropy of entanglement shown in Figs.1(b), 1(c) and 1(d). This comes from the fact that the entanglement between the ion and the phonons can impair the entanglement between the two degrees of freedom of the phonons. This also holds for the case $q = 0$. In addition, When the ion initial state is in the coherent superposition state of the excited and ground states, the oscillation and the peaks decrease, which is the same as the case of the reduced entropy of entanglement. The explanation is similar to that in section 2.

4. Conclusion

In this paper we have investigated the QE in a 2D trapped ion system. We can conclude as follows:

(1) When $q = 0$, both the QE between the ion and phonons and the QE between two degrees of freedom of the phonons evolve periodically. At $\lambda t = \frac{2n-1}{2}\pi$, the phonons are disentangled from the ion, but there is still a small entanglement for the two degrees of freedom of the vibrations of the phonons. At $\lambda t = n\pi$, the ion and phonons are also disentangled, but the two degrees of freedom of the phonons are maximally entangled. For $q \neq 0$, the evolution of QE is not periodic, the ion and phonons tend to be maximally entangled, but the two degrees of freedom of the phonons tend to be minimally entangled.

(2) When the ion initial state is in the ion coherent state $1/\sqrt{2}(|+\rangle + |-\rangle$, the oscillation of the reduced entropy and the relative entropy of the entanglement become small, and the entanglement between the ion and phonons(two degrees of the phonons) increases(decreases) gradually.

(3) The entanglement between the ion and phonons can impair the entanglement between the two degrees of freedom of the phonons, which is an interesting result in our paper.

Appendix A: the derivation of Eq.(1)–(4)

The Hamiltonian of the system, obviously, is

$$H = \frac{p_x^2+p_y^2}{2M} + \frac{1}{2}M(x^2+y^2)\gamma^2 + \frac{\hbar\omega_0}{2}\sigma_z + \boldsymbol{D}\cdot\boldsymbol{E}, \tag{A1}$$

with

$$\begin{aligned}\boldsymbol{D}\cdot\boldsymbol{E} =&(\boldsymbol{D}_{+-}|+\rangle\langle-| + \boldsymbol{D}_{-+}|-\rangle\langle+|) \\ &\cdot\boldsymbol{E}_0\{[\mathrm{e}^{\mathrm{i}(\omega_l t - k_l x')} + \mathrm{e}^{-\mathrm{i}(\omega_l t - k_l x')}] \\ &-[\mathrm{e}^{\mathrm{i}(\omega_l t - k_l y')} + \mathrm{e}^{-\mathrm{i}(\omega_l t - k_l y')}]\}, \end{aligned} \tag{A2}$$

where $\boldsymbol{D}$ is dipole moment,

$$\boldsymbol{D}_{+-} = \langle+|D|-\rangle, \boldsymbol{D}_{-+} = \langle-|D|+\rangle.$$

Let $\boldsymbol{D}_{+-}\cdot\boldsymbol{E}_0=\boldsymbol{D}_{-+}\cdot\boldsymbol{E}_0=\varepsilon E_0$, where ε, assumed real for simplicity, is dipole moment element. In the rotating wave approximations, Eq.(A2) is

$$\boldsymbol{D}\cdot\boldsymbol{E}=\varepsilon E_0[\mathrm{e}^{\mathrm{i}(\omega_l t-k_l x')}-\mathrm{e}^{\mathrm{i}(\omega_l t-k_l y')}]\sigma_-+\mathrm{h.c.}\quad\text{(A3)}$$

with $\sigma_-=|-\rangle\langle+|$. According Eq.(A3), H is given by

$$\begin{aligned}H=&\frac{p_x^2+p_y^2}{2M}+\frac{1}{2}M(x^2+y^2)\gamma^2+\frac{\hbar\omega_0}{2}\sigma_z\\&+\{\varepsilon E_0[\mathrm{e}^{\mathrm{i}(\omega_l t-k_l x')}-\mathrm{e}^{\mathrm{i}(\omega_l t-k_l y')}]\sigma_-+\mathrm{h.c.}\}.\end{aligned}\quad(1)$$

By defining

$$\begin{aligned}a=&\frac{1}{\sqrt{2}}\left(\sqrt{\frac{M\gamma}{\hbar}}x+\mathrm{i}\frac{1}{\sqrt{M\gamma\hbar}}p_x\right),\\b=&\frac{1}{\sqrt{2}}\left(\sqrt{\frac{M\gamma}{\hbar}}y+\mathrm{i}\frac{1}{\sqrt{M\gamma\hbar}}p_y\right).\end{aligned}\quad(2)$$

The expressions of $x=x(a,a^+)$, $P_x=P_x(a,a^+)$ and $y=y(b,b^+)$, $P_y=P_y(b,b^+)$ can be easily obtained according to Eq.(2), and the relations between x and x', and y and y'. Thus Hamiltonian of the system $H=H_0+H_1$ can be written as

$$H_0=\hbar\gamma(a^+a+b^+b)+\frac{\hbar\Delta}{2}\sigma_z,\quad(3)$$

$$H_1=\hbar\Omega\{[\mathrm{e}^{-\mathrm{i}\eta(a_\theta+a_\theta^+)}-\mathrm{e}^{-\mathrm{i}\eta(b_\theta+b_\theta^+)}]+\mathrm{h.c.}\},\quad(4)$$

where

$$\begin{aligned}a_\theta=&\frac{1}{\sqrt{2}}\left(\sqrt{\frac{M\gamma}{\hbar}}x'+\mathrm{i}\frac{1}{\sqrt{M\gamma\hbar}}p_x'\right)\\=&a\cos\theta+b\sin\theta,\\b_\theta=&\frac{1}{\sqrt{2}}\left(\sqrt{\frac{M\gamma}{\hbar}}y'+\mathrm{i}\frac{1}{\sqrt{M\gamma\hbar}}p_y'\right)\\=&-a\sin\theta+b\cos\theta.\end{aligned}\quad\text{(A4)}$$

References

[1] Bell S 1964 *Ann. Phys., NY* **1** 195

[2] Cirac J I and Zoller P 1995 *Phys. Rev. Lett.* **74** 4091
Wu Y and Yang X X 1997 *Phys. Rev. Lett.* **78** 3086
Liu X and Fang M F 2002 *Chin. Phys.* **11** 0926
Liu X and Fang M F 2002 *Chin. Phys.* **11** 0635
Fang M F and Liu X 2001 *Chin. Phys.* **10** 720

[3] Meekhof D M, Monroe C, King B E, Itano W M and Wineland D J 1996 *Phys. Rev. Lett.* **76** 1796

[4] Monroe C, Meekhof D M, King B E and Wineland D J 1996 *Science* **212** 1131

[5] Gerry C C, Gou S-C and Steinbach J 1997 *Phys. Rev.* A **55** 630

[6] Fang M F, Swain S and Zhou P 2001 *Phys. Rev.* A **63** 013812

[7] Sukumar C V and Buck B 1981 *Phys. Lett.* A **83** 211

[8] Zhou L Z and Zeng G J 2001 *Chin. Phys.* **10** 821

[9] Schmidt E 1907 *Math. Ann.* **63** 433

[10] Einstein A, Podolsky B and Rosen N 1935 *Phys. Rev.* **47** 777

[11] Vedral V, Plenio M B, Rippin M A and Knight P L 1997 *Phys. Rev. Lett.* **78** 2275
Vedral V and Plenio M B 1998 *Phys. Rev.* A **57** 1619

[12] Wu S J and Zhang Y D, LANL e-print quant-ph/0004018
Rains E 1999 *Phys. Rev.* A **60** 179