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Critical slowing down of the Gaussian spin system on diamond-type hierarchical lattices*

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Based on the single-spin transition critical dynamics, we have investigated the critical slowing down of the Gaussian spin model situated on the fractal family of diamond-type hierarchical lattices. We calculate the dynamical critical exponent z and the correlation-length critical exponent ν using the dynamical decimation renormalization-group technique. The result, together with some earlier ones, suggests us to conclude that on a wide range of geometries, $z\nu = 1$ is the general relationship, while the two exponents depend on the specific structure. However, we have investigated for various lattices in an earlier paper, the system studied in this paper shows highly universal $z = 1/\nu = 2$ independent of the structure and the dimensionality.

Keywords: nonequilibrium thermodynamics and irreversible processes, dynamical critical phenomena, dynamical real-space renormalization-group

PACC: 0570L, 6460H, 6460A

1. Introduction

In dynamical phase transitions, the critical slowing down is characterized by a divergent relaxation time τ near the critical point. A reasonable explanation is that the long range fluctuation leads to the long time evolution of the order parameter. According to the scaling hypothesis of Halperin and Hohenberg,[1] the divergent relaxation time τ and the divergent correlation length ζ can be related by $\tau \sim \zeta^z \sim |T - T_c|^{-z\nu}$, where z is the dynamic critical exponent and ν is the correlation-length critical exponent. For a long time z was believed to be universal, but depending on the nature of the conserved quantities (the dynamic mechanisms) and of those features,[2] such as dimensionality, which determine their static universality class. Today, the study of this important exponent in some classic model systems is still enhancing our understanding of the critical theory, especially the scaling and universality of the less well understood phenomena.[3] Continuous effort has been made to generalize and perfect some existing hypotheses, and some exceptions have aroused much interest. While it is established that in some very simple systems this exponent is nonuniversal (see, for example, Ref.[4] and references therein), a very odd and opposite case, the kinetic Gaussian model, has also been noticed. Recently, it has been found that on translationally invariant lattices $z = 1/\nu = 2$ is independent of the dimensionality and the assumed dynamic mechanism, whether it be Glauber-type, Kawasaki-type, or both.[5–10]

The conclusion about the kinetic Gaussian model is reached in an indirect way. The first step is to write down the master equation according to, for example, the Glauber-type mechanism. From the master equation the evolving equations of the individual spins can be obtained. When taken average, they will yield a very simple evolving equation of magnetization. On translationally invariant lattices it is exactly solvable, and the relaxation time τ can be calculated rigorously. This equation is found to be divergent near the critical point as $\tau \sim |T - T_c|^{-1}$. With the scaling hypothesis mentioned above, we obtain $z\nu = 1$, because on translationally invariant lattices, $\nu = 1/2$ is known to be universal. So we have $z = 2$.

To obtain more information of this interesting dynamic process and to clarify its nature, we extend the scope to fractal lattices. In Ref.[7] we have studied the kinetic Gaussian model built on several types of Koch curves. Both the above two exponents depend on the

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geometry and the dimensionality, but we always have $z\nu = 1$. In this paper, using the dynamical real-space renormalization, we study the above model built on diamond-type lattices, an important family of fractal structures having wide applications in physics[a)] With the renormalization group (RG) method, we directly calculate the two exponents z and ν without referring to the divergent relaxation time. Different from the Koch curve, highly universal exponents, $z = 1/\nu = 2$, can be obtained, independent of the dimensionality. This paper is organized as follows: section 2 gives a description of the dynamical real-space renormalization group technique governed by the single-spin transition mechanism. In section 3, the critical dynamics of the Gaussian model on diamond-type hierarchical lattices is studied. Section 4 presents the summary with some discussions.

2.Description of the method

The various dynamical processes in the critical phenomena are believed to be governed by two basic mechanisms: the Glauber-type mechanism with order parameters nonconserved and the Kawasaki-type mechanism with order parameters conserved. Recently, a Glauber-type single-spin transition mechanism[5,6] and a Kawasaki-type spin-pair redistribution mechanism[8] have been presented as the natural generalizations of the Glauber's flipping mechanism[13] and the Kawasaki's exchange mechanism,[14] respectively. They are generally applicable and mathematically well organized. In this paper we use only the Glauber-type mechanism. Our work begins with a brief review of it, along with the renormalization group method.[b)]

Glauber's single-spin flipping mechanism allows an Ising system to evolve with its spins flipping to their opposite. In the single-spin transition mechanism,[5] a single spin σ_i may change itself to any possible values, $\hat{\sigma}_i$, and the master equation is expressed as

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}P(\{\sigma\},t)\\&=-\sum_i\sum_{\hat{\sigma}_i}(1-\hat{p}_i)W_i(\sigma_i\to\hat{\sigma}_i)P(\{\sigma\},t),\end{aligned}\tag{1}$$

where $\hat{p}_i$ is the transition operation defined by

$$\begin{aligned}&\hat{p}_i f(\sigma_1,\sigma_2,\ldots,\sigma_i,\ldots,\sigma_N,t)\\&=f(\sigma_1,\sigma_2,\ldots,\hat{\sigma}_i,\ldots,\sigma_N,t),\end{aligned}$$

and $W_i(\sigma_i\to\hat{\sigma}_i)$ is the single-spin transition probability

$$W_i(\sigma_i\to\hat{\sigma}_i)=\frac{1}{Q_i}\exp\left[-\beta\mathcal{H}_i\left(\hat{\sigma}_i,\sum_{\langle i,j\rangle}\sigma_j\right)\right],$$

$$Q_i=\sum_{\hat{\sigma}_i}\exp\left[-\beta\mathcal{H}_i\left(\hat{\sigma}_i,\sum_{\langle i,j\rangle}\sigma_j\right)\right].\tag{2}$$

Except for a few cases,[c)] the dynamical evolution equation may be difficult to solve exactly. While numerous approximation methods were being developed, the success of the RG methods in static problems[16−18] motivated attempts to apply them to critical dynamics.[2,7,19−21] The dynamical real-space RG technique, proposed by Achiam and Kosterlitz[19] and perfected by Kandel,[21] will also be the focus of this paper. In order to study the critical slowing down, we shall limit ourselves to the relaxation of an infinitely small perturbation from equilibrium. Following Achiam's idea,[22] we consider the magnetic-like perturbation,

$$P(\{\sigma\},t)=\left[1+\sum_i h_{q_i}(t)\sigma_i\right]P_{\mathrm{eq}}(\{\sigma\}),\tag{3}$$

where q_i distinguishes between points having different R (the order of ramification), and h_{q_i} is the reduced external field.

With Eq.(3), the master Eq. (1) takes the following form

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_i h_{q_i}(t)\sigma_i P_{\mathrm{eq}}(\{\sigma\})\\&=-\sum_i\sum_{\hat{\sigma}_i}h_{q_i}(t)(\sigma_i-\hat{\sigma}_i)W_i(\sigma_i\to\hat{\sigma}_i)P_{\mathrm{eq}}(\{\sigma\}),\end{aligned}\tag{4}$$

or,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{h}(t)\cdot\boldsymbol{\Lambda}(\sigma)P_{\mathrm{eq}}(k,\{\sigma\})\\&=-\boldsymbol{h}(t)\cdot\boldsymbol{\Omega}(k,\sigma)P_{\mathrm{eq}}(k,\{\sigma\}),\end{aligned}\tag{5}$$

where $\boldsymbol{h}(t)$ is a row matrix, $\boldsymbol{\Lambda}(\sigma)$ and $\boldsymbol{\Omega}(k,\sigma)$ are column matrices.

[a)] The advantage is that one can find the static renormalization-group transformation exactly.[11,12]

[b)] The systematic formulation can be found in Ref.[7]. This brief review is for better understanding of the calculations below.

[c)] For example, the one-dimensional kinetic Ising model,[13] the infinite-range Ising model,[15] and the kinetic Gaussian model.[5−9]

The critical dynamical behaviour of the system described by Eq.(5) can be studied using the dynamical real-space RG technique, which consists of two stages: first, performing a RG transformation, we rescale the space by $x \to x' = Lx$, where L is the length-rescaling factor. For example, in the case of the decimation transformation, the spins $\{\sigma\}$ "aggregate" into a new set $\{\mu\}$ under the control of a decimation operator $T(\mu,\sigma)$, then a trace over the $\{\sigma\}$ is performed. The process of the decimation of a spin function $f(\{\sigma\})$ can be demonstrated as

$$R\{f(\{\sigma\})\} = \sum_{\{\sigma\}} T(\mu,\sigma) f(\{\sigma\}) = f(\{\mu\}). \qquad (6)$$

In order to keep an invariant form of the probability distribution, $P'_{\rm eq}(k',\{\mu'\})$, we need to rescale the interaction parameter $k = J/k_\beta T$ and the spin μ, i.e.

$$k \to k' = R(k)k, \quad \mu \to \mu' = \xi(k)\mu, \qquad (7)$$

where $\xi(k)$ is the spin-rescaling factor.

With the decimation transformation (6), Eq.(5) takes the following form

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{h}(t)\cdot\boldsymbol{R}_{\Lambda}(\boldsymbol{k})\cdot\boldsymbol{\Lambda}'(\mu',k')P'_{\rm eq}(k',\{\mu'\}) \\ &= -\boldsymbol{h}(t)\cdot\boldsymbol{R}_{\Omega}(\boldsymbol{k})\cdot\boldsymbol{\Omega}'(\mu',k')P'_{\rm eq}(k',\{\mu'\}), \end{aligned} \qquad (8)$$

where $\boldsymbol{\Lambda}'(\mu',k')$ and $\boldsymbol{\Omega}'(\mu',k')$ retain the original form. Taking the monomark

$$\boldsymbol{h}'(t,k') = \boldsymbol{h}(t)\cdot\boldsymbol{R}_{\Lambda}(\boldsymbol{k}),$$

which can be regarded as a RG transformation of the dynamic parameter $\boldsymbol{h}(t)$, Eq.(8) can be rewritten as

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{h}'(t,k')\cdot\boldsymbol{\Lambda}'(\mu',k')P'_{\rm eq}(k',\{\mu'\}) \\ &= -\boldsymbol{h}'(t,k')\cdot[\boldsymbol{R}_{\Lambda}^{-1}(\boldsymbol{k})\cdot\boldsymbol{R}_{\Omega}(\boldsymbol{k})] \\ &\quad\cdot\boldsymbol{\Omega}'(\mu',k')P'_{\rm eq}(k',\{\mu'\}). \end{aligned} \qquad (9)$$

The second stage is to rescale the time by $t \to t' = L^{-z}t$. As a result, Eq.(9) is restored back to the invariant form of the master Eq. (5):

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t'}\boldsymbol{h}'(t',k')\cdot\boldsymbol{\Lambda}'(\mu',k')P'_{\rm eq}(k',\{\mu'\}) \\ &= -\boldsymbol{h}'(t',k')\cdot\boldsymbol{\Omega}'(\mu',k')P'_{\rm eq}(k',\{\mu'\}). \end{aligned} \qquad (10)$$

From the relationship of t and t' we can determine z, while the critical point $k_{\rm c}$ and the correlation-length critical exponent ν can be determined with the relationship of k and k' (see below).

3. Kinetic Gaussian model on diamond-type hierarchical lattices

The hierarchical lattice that we are going to study is constructed in an iterative manner. We start from a two-vertex bond. In every stage, each bond is repeatedly replaced by p branches with q bonds on each one of them. Figure 1 shows some examples of them. The dimensionality of such self-similar structures is defined as $D_f = \log_q(pq)$.

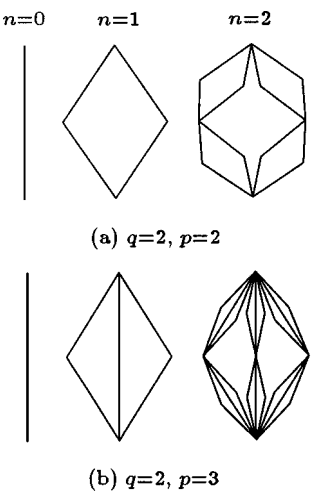


Fig.1. A family of the diamond-type hierarchical lattices. n, q and p denote the order of construction, the number of bonds per branch, and the number of branches, respectively.

In a Gaussian spin system located on these lattices, the Hamiltonian is defined as

$$-\beta\mathcal{H} = k\sum_{\langle i,j\rangle}\sigma_i\sigma_j, \qquad (11)$$

where $\beta = 1/k_\beta T$, $k = J/k_\beta T$ and the summation $\sum_{\langle i,j\rangle}$ is taken over the nearest neighbours. Unlike the Ising system, the spins of the Gaussian model can take any real value. To prevent the continuous spin value from tending to infinity, the probability that we can find a given spin between σ_i and $\sigma_i + \mathrm{d}\sigma_i$ is of the Gaussian type,

$$f(\sigma_i)\mathrm{d}\sigma_i \sim \exp\left(-\frac{b_{q_i}}{2}\sigma_i^2\right)\mathrm{d}\sigma_i, \qquad (12)$$

where b_{q_i} is a distribution constant only depending on the coordination number (just like h). Actually, the summation to be taken below is an integration, $\sum_\sigma(\cdots) \to \int(\cdots)f(\sigma)\mathrm{d}\sigma$.

With Eq.(2), in which the normalization factor

$$Q_i = \int_{-\infty}^{\infty} \exp\left[k\hat{\sigma}_i \sum_{\langle i,j\rangle} \sigma_j\right] f(\hat{\sigma}_i)\mathrm{d}\hat{\sigma}_i$$
$$= \exp\left[-\frac{k^2}{2b_{q_i}}(\sum_{\langle i,j\rangle} \sigma_j)^2\right],$$

we can obtain

$$\sum_{\hat{\sigma}_i}(\sigma_i - \hat{\sigma}_i)W_i(\sigma_i \to \hat{\sigma}_i)$$
$$= \int_{-\infty}^{\infty}(\sigma_i - \hat{\sigma}_i)W_i(\sigma_i \to \hat{\sigma}_i)f(\hat{\sigma}_i)\mathrm{d}\hat{\sigma}_i$$
$$= \sigma_i - \frac{k}{b_{q_i}}\sum_{\langle i,j\rangle}\sigma_j. \qquad (13)$$

Thus, with magnetic-like perturbation, the master equation of the modified Gaussian model on fractal lattices can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_i h_{q_i}(t)\sigma_i P_{\mathrm{eq}}(k,\{\sigma\})$$
$$= -\sum_i h_{q_i}(t)\left(\sigma_i - \frac{k}{b_{q_i}}\sum_w \sigma_{i+w}\right)P_{\mathrm{eq}}(k,\{\sigma\}). \qquad (14)$$

In order to describe the inhomogeneity of the branching lattice, we have assumed that the reduced external field parameter $h_{q_i}(t)$ and the distribution constant b_{q_i} both depend on the coordination number q_i of the lattice. A step further, we assume

$$b_{q_i}/b_{q_j} = h_{q_i}/h_{q_j} = q_i/q_j. \qquad (15)$$

This choice is based on considerations in two aspects. In physics: the RG transformation is also known as the "aggregation" or "miniaturization". A part of the spins may survive in higher stages while the others may not; in every stage, a certain number of spins aggregate into and enrich the one in the centre. In these diamond-type structures, spins of larger coordination number q_i have longer lives, and thus, this relationship is acceptable, though coincidental to some extent. In mathematics, as we shall see below, this choice is necessary in order to put the RG technique into practice in such an inhomogeneous fractal lattice.

Now we begin to treat the modified Gaussian model on diamond lattices with an arbitrary p, but fixed $q = 2$. As shown in Fig.2, we separate the whole system into unit diamonds. At first glance, one may find it difficult to go on since, for different diamonds, the vertices have different coordination number. It seems difficult to find a universal term that can describe them all. Fortunately, with Eq.(15), the individually obtained mathematical expressions actually turn to be universal. For example, for a unit diamond at construction stage $n = 2$, we can write the effective Hamiltonian as

$$-\frac{1}{k_B T}\mathcal{H}^{\alpha}_{\mathrm{eff}}(\sigma^{\alpha},k)$$
$$= k(\sigma_1^{\alpha}+\sigma_2^{\alpha})\sum_{i=3}^{p+2}\sigma_i^{\alpha} - \frac{b_{p^2}}{2}\frac{1}{p}(\sigma_1^{\alpha})^2$$
$$-\frac{b_{2p}}{2}\frac{1}{2}(\sigma_2^{\alpha})^2 - \frac{b_2}{2}\sum_{i=3}^{p+2}(\sigma_i^{\alpha})^2$$
$$= k(\sigma_1^{\alpha}+\sigma_2^{\alpha})\sum_{i=3}^{p+2}\sigma_i^{\alpha} - \frac{b_p}{2}[(\sigma_1^{\alpha})^2+(\sigma_2^{\alpha})^2]$$
$$-\frac{b_2}{2}\sum_{i=3}^{p+2}(\sigma_i^{\alpha})^2. \qquad (16)$$

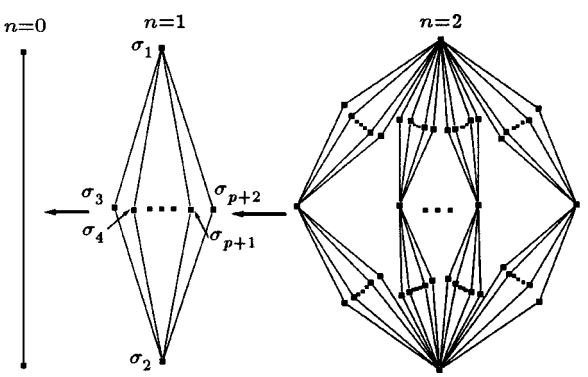


Fig.2. A diagrammatic sketch of the decimation RG procedure about the modified Gaussian model on diamond lattices with an arbitrary p, but fixing $q = 2$.

In an arbitrary diamond, the spins located on the branches have coordination number 2; the spins at each end are shared by, say, m branches, and thus have coordination number pm. At the same time, because the total Hamiltonian is the summation of all the unit ones, the terms with regard to these shared spins should be divided by m. Thus, generally, the effective Hamiltonian for the αth diamond can be obtained as

$$
\begin{aligned}
&-\frac{1}{k_BT}\mathcal{H}^{\alpha}_{\rm eff}(\sigma^{\alpha},k)\\
=&k(\sigma^{\alpha}_1+\sigma^{\alpha}_2)\sum_{i=3}^{p+2}\sigma^{\alpha}_i-\frac{b_{pm_1}}{2}\frac{1}{m_1}(\sigma^{\alpha}_1)^2\\
&-\frac{b_{pm_2}}{2}\frac{1}{m_2}(\sigma^{\alpha}_2)^2-\frac{b_2}{2}\sum_{i=3}^{p+2}(\sigma^{\alpha}_i)^2\\
=&k(\sigma^{\alpha}_1+\sigma^{\alpha}_2)\sum_{i=3}^{p+2}\sigma^{\alpha}_i-\frac{b_p}{2}[(\sigma^{\alpha}_1)^2+(\sigma^{\alpha}_2)^2]\\
&-\frac{b_2}{2}\sum_{i=3}^{p+2}(\sigma^{\alpha}_i)^2. \qquad (17)
\end{aligned}
$$

With this universal form, we need not differentiate among the diamonds about their positions or the construction stages they belong to. The space-rescaling procedure is performed through the decimation renormalization transformation. Multiplied by the transformation operator

$$T^{\alpha}(\mu^{\alpha},\sigma^{\alpha})=\delta(\mu^{\alpha}_1-\sigma^{\alpha}_1)\delta(\mu^{\alpha}_2-\sigma^{\alpha}_2), \qquad (18)$$

and integrated over $\{\sigma\}^{\alpha}$, the effective Hamiltonian becomes

$$
\begin{aligned}
&R\left\{\exp\left[-\frac{1}{k_BT}\mathcal{H}^{\alpha}_{\rm eff}(\sigma^{\alpha},k)\right]\right\}\\
=&\int_{-\infty}^{\infty}{\rm d}\sigma^{\alpha}_1{\rm d}\sigma^{\alpha}_2\prod_{i=1}^{p}{\rm d}\sigma^{\alpha}_i\cdot T^{\alpha}(\mu^{\alpha},\sigma^{\alpha})\\
&\times\exp\left[-\frac{1}{k_BT}\mathcal{H}^{\alpha}_{\rm eff}(\sigma^{\alpha},k)\right]\\
=&C\exp\left\{k_0\mu^{\alpha}_1\mu^{\alpha}_2-\frac{b_2}{2}\xi\left[\frac{1}{2}(\mu^{\alpha}_1)^2+\frac{1}{2}(\mu^{\alpha}_2)^2\right]\right\}, \qquad (19)
\end{aligned}
$$

where

$$k_0=pk^2/b_2,\xi=p(b_2^2-2k^2)/b_2^2,C=(2\pi/b_2)^{p/2}.$$

In order to keep an invariant form of the effective Hamiltonian, we take

$$\mu'=\sqrt{\xi}\mu, \qquad (20)$$

and

$$k'=\frac{k_0}{\xi}=\frac{b_2k^2}{b_2^2-2k^2}, \qquad (21)$$

and then

$$
\begin{aligned}
&R\left\{\exp\left[-\frac{1}{k_BT}\mathcal{H}^{\alpha}_{\rm eff}(\sigma,k)\right]\right\}\\
=&C\exp\left\{k'\mu'^{\alpha}_1\mu'\alpha_2-\frac{b_2}{2}\left[\frac{1}{2}(\mu'^{\alpha}_1)^2+\frac{1}{2}(\mu'^{\alpha}_2)^2\right]\right\}. \qquad (22)
\end{aligned}
$$

If taking summation over the next stage, we bind $2p$ diamonds into a new larger diamond, and clearly it leads to an invariant form of the effective Hamiltonian, as we have expected before. Obviously,

$$
\begin{aligned}
&R\{P_{\rm eq}(k,\{\sigma\})\}\\
=&\frac{1}{Z'}\exp\left[k'\sum_{\langle i,j\rangle}\mu'_i\mu'_j-\frac{b_{q_i}}{2}\sum_i\mu'^2_i\right]\\
=&P'_{\rm eq}(k',\{\mu'\}). \qquad (23)
\end{aligned}
$$

This means that the Gibbs distribution function is invariant under the RG transformation. The critical point $k_{\rm c}$ can also be obtained from the recursion relation (21),

$$k^*=k'=k\Rightarrow k_{\rm c}=\frac{b_2}{2}. \qquad (24)$$

Before treating the master equation, we derive some useful terms:

$$
\begin{aligned}
&R\{\sigma^{\alpha}_jP_{\rm eq}(k,\{\sigma\})\}\\
=&\sum_{\{\sigma\}}\{T(\mu,\sigma)\sigma^{\alpha}_jP_{\rm eq}(k,\{\sigma\})\}\\
=&\int_{-\infty}^{\infty}\prod_{\beta}\left[{\rm d}\sigma^{\beta}_1{\rm d}\sigma^{\beta}_2\prod_{i=3}^{p+2}{\rm d}\sigma^{\beta}_iT_{\beta}(\mu,\sigma)\right]\sigma^{\alpha}_jP_{\rm eq}(k,\{\sigma\})\\
=&P'_{\rm eq}(k',\{\mu'\})\cdot\frac{W_{\sigma^{\alpha}_j}}{W},\ (j=3,4,\ldots,p+2)
\end{aligned}
$$

where

$$
\begin{aligned}
W=&\int_{-\infty}^{\infty}{\rm d}\sigma^{\alpha}_1{\rm d}\sigma^{\alpha}_2\prod_{i=3}^{p+2}{\rm d}\sigma^{\alpha}_iT_{\alpha}(\mu,\sigma)\\
&\times\exp\left[-\frac{1}{k_BT}\mathcal{H}^{\alpha}_{\rm eff}(\sigma,k)\right]\\
=&\left\{\sqrt{\frac{2\pi}{b_2}}\exp\left[\frac{k^2(\mu^{\alpha}_1+\mu^{\alpha}_2)^2}{2b_2}\right]\right\}^p,
\end{aligned}
$$

$$
\begin{aligned}
W_{\sigma^{\alpha}_j}=&\int_{-\infty}^{\infty}{\rm d}\sigma^{\alpha}_1{\rm d}\sigma^{\alpha}_2\prod_{i=3}^{p+2}{\rm d}\sigma^{\alpha}_iT_{\alpha}(\mu,\sigma){\rm d}\sigma^{\alpha}_j\\
&\times\exp\left[-\frac{1}{k_BT}\mathcal{H}^{\alpha}_{\rm eff}(\sigma,k)\right]\\
=&\frac{k}{b_2}(\mu^{\alpha}_1+\mu^{\alpha}_2)W.
\end{aligned}
$$

Then

$$
\begin{aligned}
&R\{\sigma^{\alpha}_jP_{\rm eq}(k,\{\sigma\})\}\\
=&\frac{k}{b_2}\frac{1}{\sqrt{\xi}}(\mu'^{\alpha}_1+\mu'^{\alpha}_2)P'_{\rm eq}(k',\{\mu'\}),\\
&(j=3,4,\ldots,p,p+1,p+2) \qquad (25)
\end{aligned}
$$

similarly, we can obtain

$$R\{\sigma^{\alpha}_{1,2}P_{\rm eq}(k,\{\sigma\})\}=\frac{1}{\sqrt{\xi}}\mu'^{\alpha}_{1,2}P'_{\rm eq}(k',\{\mu'\}). \qquad (26)$$

Our purpose is the renormalization of the master equation Eq.(14), or,

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{h}(t)\cdot\boldsymbol{O}(\sigma)P_{\mathrm{eq}}(k,\{\sigma\}) = -\boldsymbol{h}(t)\cdot\boldsymbol{\Omega}(k,\sigma)P_{\mathrm{eq}}(k,\{\sigma\}). \tag{27}$$

We divide the whole system at a given construction stage into unit diamonds, and the whole term is the summation of the individual unit terms. For example, at construction stage $n=2$, for the αth diamond, we can write the left and right-hand sides of Eq.(27) as, respectively,

$$\begin{aligned}&\boldsymbol{h}(t)\cdot\boldsymbol{O}(\sigma)P_{\mathrm{eq}}(k,\{\sigma\})\\ =&\left[h_{p^2}\frac{1}{p}\sigma_1^\alpha + h_{2p}\frac{1}{2}\sigma_2^\alpha + h_2\sum_{i=3}^{p+2}\sigma_i^\alpha\right]P_{\mathrm{eq}}(k,\{\sigma\})\\ =&h_2\left[\frac{p}{2}(\sigma_1^\alpha+\sigma_2^\alpha)+\sum_{i=3}^{p+2}\sigma_i^\alpha\right]P_{\mathrm{eq}}(k,\{\sigma\}),\end{aligned}$$

and

$$\begin{aligned}&\boldsymbol{h}(t)\cdot\boldsymbol{\Omega}(k,\sigma)P_{\mathrm{eq}}(k,\{\sigma\})\\ =&\Bigg\{h_{p^2}\left(\frac{1}{p}\sigma_1^\alpha-\frac{k}{b_{p^2}}\sum_{i=3}^{p+2}\sigma_i^\alpha\right)\\ &+h_{2p}\left(\frac{1}{2}\sigma_2^\alpha-\frac{k}{b_{2p}}\sum_{i=3}^{p+2}\sigma_i^\alpha\right)\\ &+h_2\sum_{i=3}^{p+2}\left[\sigma_i^\alpha-\frac{k}{b_2}(\sigma_1^\alpha+\sigma_2^\alpha)\right]\Bigg\}P_{\mathrm{eq}}(k,\{\sigma\})\\ =&h_2\left(1-\frac{2k}{b_2}\right)\left[\frac{p}{2}(\sigma_1^\alpha+\sigma_2^\alpha)+\sum_{i=3}^{p+2}\sigma_i^\alpha\right]P_{\mathrm{eq}}(k,\{\sigma\}).\end{aligned}$$

With the same considerations and the relationship (15), these forms are also universal. Generally, for the αth diamond, spins σ_3^α, σ_4^α through σ_{p+2}^α have coordination number 2. σ_1^α and σ_2^α are each shared by m_1 and m_2 branches, and they have a larger reduced field h and distribution constant b, while at the same time the terms regarding them will be divided by m_1 and m_2 respectively. Thus, we have

$$\begin{aligned}&\boldsymbol{h}(t)\cdot\boldsymbol{O}(\sigma)P_{\mathrm{eq}}(k,\{\sigma\})\\ =&\left[h_{m_1p}\frac{1}{m_1}\sigma_1^\alpha + h_{m_2p}\frac{1}{m_2}\sigma_2^\alpha + h_2\sum_{i=3}^{p+2}\sigma_i^\alpha\right]P_{\mathrm{eq}}(k,\{\sigma\})\\ =&h_2\left[\frac{p}{2}(\sigma_1^\alpha+\sigma_2^\alpha)+\sum_{i=3}^{p+2}\sigma_i^\alpha\right]P_{\mathrm{eq}}(k,\{\sigma\}),\end{aligned} \tag{28}$$

and

$$\begin{aligned}&\boldsymbol{h}(t)\cdot\boldsymbol{\Omega}(k,\sigma)P_{\mathrm{eq}}(k,\{\sigma\})\\ =&\Bigg\{h_{m_1p}\left(\frac{1}{m_1}\sigma_1^\alpha-\frac{k}{b_{m_2p}}\sum_{i=3}^{p+2}\sigma_i^\alpha\right)\\ &+h_{m_2p}\left(\frac{1}{m_2}\sigma_2^\alpha-\frac{k}{b_{m_2p}}\sum_{i=3}^{p+2}\sigma_i^\alpha\right)\\ &+h_2\sum_{i=3}^{p+2}\left[\sigma_i^\alpha-\frac{k}{b_2}(\sigma_1^\alpha+\sigma_2^\alpha)\right]\Bigg\}P_{\mathrm{eq}}(k,\{\sigma\})\\ =&h_2\left(1-\frac{2k}{b_2}\right)\left[\frac{p}{2}(\sigma_1^\alpha+\sigma_2^\alpha)+\sum_{i=3}^{p+2}\sigma_i^\alpha\right]P_{\mathrm{eq}}(k,\{\sigma\}).\end{aligned} \tag{29}$$

This procedure, as well as that taken in the calculation of the effective Hamiltonian, is the key in the whole paper, providing the foundation for later calculations.

With Eqs.(25) and (26) we can obtain the RG transformation of Eqs.(28) and (29) as

$$\begin{aligned}&R\{\boldsymbol{h}(t)\cdot\boldsymbol{O}(\sigma)P_{\mathrm{eq}}(k,\{\sigma\})\}\\ =&\frac{1}{\sqrt{\xi}}ph_2\left(1+\frac{2k}{b_2}\right)\left(\frac{1}{2}\mu_1'^\alpha+\frac{1}{2}\mu_2'^\alpha\right)P'_{\mathrm{eq}}(k',\{\mu'\}),\end{aligned} \tag{30}$$

$$\begin{aligned}&R\{\boldsymbol{h}(t)\cdot\boldsymbol{\Omega}(k,\sigma)P_{\mathrm{eq}}(k,\{\sigma\})\}\\ =&\frac{1}{\sqrt{\xi}}ph_2\left(1+\frac{2k}{b_2}\right)\left(1-\frac{2k}{b_2}\right)\left(1-\frac{2k'}{b_2}\right)^{-1}\\ &\times\left(1-\frac{2k'}{b_2}\right)\left(\frac{1}{2}\mu_1'^\alpha+\frac{1}{2}\mu_2'^\alpha\right)P'_{\mathrm{eq}}(k',\{\mu'\}).\end{aligned} \tag{31}$$

Let

$$h_2' = ph_2\left(1+\frac{2k}{b_2}\right),$$

then the master equation will retain the same form only if we take

$$t' = \left(1-\frac{2k}{b_2}\right)\left(1-\frac{2k'}{b_2}\right)^{-1}t = \frac{b_2^2-2k^2}{(b_2+2k)b_2}t.$$

When $k\to k_{\mathrm{c}}$,

$$t'\to\frac{1}{4}t.$$

With $L=2$,

$$z=\frac{\ln 4}{\ln 2}=2.$$

At the same time,

$$\frac{1}{\nu}=\frac{1}{\ln L}\ln\left(\frac{\partial k'}{\partial k}\right)_{k\to k_{\mathrm{c}}=b_2/2}=2.$$

Thus, the final result,

$$z=1/\nu=2, \tag{32}$$

is obtained once more.

Structures with q larger than 2 can be treated in the same way. The effective Hamiltonian includes interaction between spins both on the same branch,

and thus the RG transformation is more complex in mathematics.

4. Results and Discussion

In this paper we apply the RG method based on the single-spin transition mechanism to the Gaussian model built on the diamond-type hierarchical lattices. Two of the results are most interesting. First, we obtain the critical point $K_c = b_2/2$. Note that both on the one-dimensional lattice[5] and the nonbranching Koch curve,[7] $K_c = b/2$, Thus, if we take $b_2 = b$, the three structures do not differ at all from each other as far as the critical point is concerned. Another highly universal result is that $z\nu = 1$, which is true for all the systems under study by far.[5–10] About these two exponents, the earlier results obtained on the Koch curves seem to show that $z\nu = 1$ is the more fundamental relationship, and the above universality stops when we refer to z or ν separately. While the study reported in this paper provides support for such a hidden common mechanism, we are surprised to see that, the result obtained on the translationally invariant lattices, $z = 1/\nu = 2$, is also valid for the diamond-type structure. Thus, as far as the Gaussian model is concerned, the diamond-type lattice, interestingly, is just like a twisty one-dimensional loop.

Based on the investigations in this paper and some previous publications, we are now able to roughly offer the explanation for the above-mentioned "coincidence" of the critical point and the exponents. We shall refer to Eq.(14) and the similar equation in Ref.[7]. With the help of Eq.(15), this equation becomes similar to the evolving equation of the Gaussian model on an one-dimensional lattice,[5]

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_i \langle\sigma_i\rangle = -\left(1 - \frac{Ck}{b}\right)\sum_i \langle\sigma_i\rangle, \qquad (33)$$

where $C = 2$ is deemed as the effective coordination number. The critical point is solely determined by this number, which becomes $2d$ on d-dimensional translationally invariant lattices. Equation (33) can be easily solved

$$M(t) = M(0)\exp(-t/\tau),$$

where $M \equiv \sum_i \langle\sigma_i\rangle/N$, and the relaxation time $\tau = 1/(1 - Ck/b)$. τ becomes divergent[a)] near the critical point $k_c = b/C$ and $\tau \sim |T - T_c|^{-1}$, thus $z\nu = 1$. Therefore, the universality of z can be deemed as a result of the universality of ν.

In the studies of the critical dynamics, analytical results are always valuable, due to the often formidable mathematical task. The Gaussian model is well-known for its mathematical simplicity. As more studies have been carried out, its simplicity and universality in the dynamic aspect is now becoming more and more remarkable. Idealized to some extent, it often serves as a starting point for the further studies on other types of systems. We hope further studies about it will continue to yield more interesting features, with the mathematical treatment being very convenient.

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[a)] With the RG method, we do not calculate the relaxation time on the diamond-type lattices, but we expect it to be similar to that on the one-dimensional lattice.