Vol 12 No 3, March 2003
1009-1963/2003/12(03)/0259-05

Chinese Physics

© 2003 Chin. Phys. Soc.
and IOP Publishing Ltd

Analysis of global exponential stability for a class of bi-directional associative memory networks*

Wang Hong-Xia(王宏霞)† and He Chen(何 晨)

Department of Electronic Engineering, Shanghai Jiaotong University, Shanghai 200030, *China*

(Received 16 September 2002; revised manuscript received 26 November 2002)

In real-time applications of bi-directional associative memory (BAM) networks, a global exponentially stable equilibrium is highly desired. The existence, uniqueness and global exponential stability for a class of BAM networks are studied in this paper, the signal function of neurons is assumed to be piece-wise linear from the engineering point of view. A very concise condition for the equilibrium of such a network being globally exponentially stable is derived, which makes the practical design of this kind of networks an easy job.

Keywords: BAM network, global exponential stability, equilibrium
PACC: 0545

1. Introduction

Bi-directional associative memory (BAM) is a two-layer nonlinear feedback neural network introduced by Kosko,[1,2] which has served as a vital component in various intelligent systems, e.g., the classification of spatio-temporal signals, the identification of power density, and more importantly, the associative pattern searching. Because BAM is a hetero-associative memory which provides a flexible nonlinear mapping from input space to output space, it possesses an important attribute in its simple architecture which results in an easy implementation via digital processors, analogue circuits, or optical hardware.[1] In designing an analogue BAM circuit, we consider the continuous BAM model described by the following coupled nonlinear equations

$$\begin{cases} \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -x_i(t) + \displaystyle\sum_{j=1}^{n} a_{ij} f(y_j(t)) + I_i, \\ \dfrac{\mathrm{d}y_i(t)}{\mathrm{d}t} = -y_i(t) + \displaystyle\sum_{j=1}^{n} b_{ij} f(x_j(t)) + J_i, \end{cases} \quad (i = 1, 2, \cdots, n) \tag{1}$$

where x_i, y_i are states of neurons embedded in the two circuit layers respectively; $(x_1, \cdots, x_n)^{\mathrm{T}} \in R^n$, $(y_1, \cdots, y_n)^{\mathrm{T}} \in R^n$, $I_i, J_i (i = 1, 2, \cdots, n)$ are constant external inputs; $(a_{ij})_{n\times n}, (b_{ij})_{n\times n}$ are constant matrices representing the unidirectional connection weights between layers. In practical designing, the output signal function $f(\bullet)$ relating to the states and output of neuron i is usually implemented by an analogue operational amplifier, whose input/output relation can be well approximated by a piece-wise linear (PWL) expression as follows:

$$f(s) = \frac{1}{2}(|s+1| - |s-1|), s \in R. \tag{2}$$

In applications, such as prediction[3] and control engineering, the dynamic behaviour,[4–6] in particular, the global stability is of prime importance. In solving many engineering problems it is desired to have a unique equilibrium for a neural network, while in real-time applications, a rapid convergence toward the equilibrium is critically required.[7] The studies on existence, uniqueness of equilibrium and its global exponential stability for Hopfield network and cellular neural network (CNN) were reported in Refs.[8] and [9] respectively. As regard to the qualitative analysis for BAM network, in studying the existence and uniqueness of its equilibrium, the signal function $f(\bullet)$ is mostly assumed to be continuous and differentiable.[10,11] Furthermore, neither the investigation under a PWL signal function as given by Eq.(2) nor the result of global exponential stability for the equilibrium is reported. Hence we study these problems in section 2. Illustrative examples are given in section 3. Conclusions are drawn in section 4.

*Project supported by the National High Technology Development Program of China (Grant No 2002AA144110).
†E-mail: whongxia@sjtu.edu.cn

2.Global exponential stability analysis

In this section, we study the existence, uniqueness of equilibrium for Eq.(1) and its global exponential stability subject to (2). Firstly, we introduce a Lemma, then define the global exponential stability for the equilibrium of BAM network (1).

Lemma[5] For any x, $y \in R$, the following inequalities are valid:

(1) $|f(x)-f(y)| \le |x-y|$,

(2) $\int_y^x [f(s)-f(y)]\mathrm{d}s \le [x-y][f(x)-f(y)]$,

(3) $\int_y^x [f(s)-f(y)]\mathrm{d}s \ge \frac{1}{2}[f(x)-f(y)]^2$.

These inequalities are important for deriving our qualitative analysis results. The proof is referred to Ref.[5].

Definition Equilibrium $(x^*, y^*) = (x_1^*, \cdots, x_n^*, y_1^*, \cdots, y_n^*)$ of Eq.(1) is said to be globally exponentially stable if there exist constants $\varepsilon > 0$ and $M \ge 1$ such that

$$\begin{cases} \sum_{i=1}^n |x_i(t)-x_i^*| \le M \cdot \Pi \cdot \mathrm{e}^{-\varepsilon t}, \\ \sum_{i=1}^n |y_i(t)-y_i^*| \le M \cdot \Pi \cdot \mathrm{e}^{-\varepsilon t}, \end{cases}$$

for all $t \ge 0$, where

$$\Pi = \max\{||x(0)-x^*||, ||y(0)-y^*||\}, \tag{3}$$

and

$$\begin{cases} ||x(0)-x^*|| = \sum_{i=1}^n |x_i(0)-x_i^*|, \\ ||y(0)-y^*|| = \sum_{i=1}^n |y_i(0)-y_i^*|. \end{cases}$$

Based on this definition we have the following main result.

Theorem We denote the connection weights $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, if

$$A = -B^{\mathrm{T}}, \tag{4}$$

holds, then there exists a unique equilibrium (x^*, y^*) of Eq.(1) satisfying

$$\begin{cases} -x_i^* + \sum_{j=1}^n f(y_j^*) + I_i = 0, \\ -y_i^* + \sum_{j=1}^n f(x_j^*) + J_i = 0, \end{cases} \quad (i = 1, 2, \cdots, n). \tag{5}$$

Besides, the solution $(x_1, \cdots, x_n, y_1, \cdots, y_n)^{\mathrm{T}}$, starting from any point $(x_1(0), \cdots, x_n(0))^{\mathrm{T}} \in R^n$ and $(y_1(0), \cdots, y_n(0))^{\mathrm{T}} \in R^n$, satisfies

$$\begin{cases} |x_i(t)-x_i^*| \le \Pi \cdot \left(\mathrm{e}^{-t} + 4\sqrt{2}\sum_{j=1}^n |a_{ij}|\mathrm{e}^{-\frac{1}{2}t}\right), \\ |y_i(t)-y_i^*| \le \Pi \cdot \left(\mathrm{e}^{-t} + 4\sqrt{2}\sum_{j=1}^n |b_{ij}|\mathrm{e}^{-\frac{1}{2}t}\right), \end{cases} \tag{6}$$

for all $t \ge 0, (i = 1, \cdots, n)$. This means that the network (1) is globally exponentially stable.

[Proof]

(i) Existence of equilibrium (x^*, y^*)

Let us define a bounded close set $D \subset R^{2n}$,

$$D = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \begin{array}{ll} x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}, & |x_i| \le \alpha_i, \\ y = (y_1, y_2, \cdots, y_n)^{\mathrm{T}}, & |y_i| \le \beta_i, \end{array} \right.$$
$$(i = 1, 2, \cdots, n), \tag{7}$$

where

$$\begin{cases} \alpha_i = \sum_{j=1}^n |a_{ij}| + |I_i| \\ \beta_i = \sum_{j=1}^n |b_{ij}| + |J_i| \end{cases} \quad (i = 1, 2, \cdots, n),$$

and a mapping $F : D \to R^{2n}$ as

$$F(x,y) = \begin{cases} F_1(x,y) = \sum_{j=1}^n a_{ij} f(y_j) + I_i \\ F_2(x,y) = \sum_{j=1}^n b_{ij} f(x_j) + J_i \end{cases}$$
$$(i = 1, 2, \cdots, n). \tag{8}$$

It follows from Eq.(2) that

$$|f(x_i)| \le 1, \quad |f(y_i)| \le 1 \quad (i = 1, 2, \cdots, n),$$

and for any $\begin{cases} x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \\ y = (y_1, y_2, \cdots, y_n)^{\mathrm{T}} \end{cases} \in D$, we have

$$|F(x,y)| = \begin{cases} |F_1(x,y)| \le \sum_{j=1}^n |a_{ij}| + |I_i| \\ |F_2(x,y)| \le \sum_{j=1}^n |b_{ij}| + |J_i| \end{cases}$$
$$(i = 1, 2, \cdots, n), \tag{9}$$

which implies $F(x, y) \in D$. According to the Brouwer fixed point theorem, there exists a fixed point of the mapping F. Obviously the fixed point is the equilibrium of Eq.(1).

(ii) Uniqueness of the equilibrium (x^*, y^*)

Assume there exist two equilibria of Eq.(1), they are $(x_1^*, \cdots, x_n^*, y_1^*, \cdots, y_n^*)$ and $(x_1^\perp, \cdots, x_n^\perp, y_1^\perp, \cdots, y_n^\perp)$. Then from Eq.(4), we have

$$\begin{cases} x_i^* - x_i^\perp = \sum_{j=1}^n a_{ij}[f(y_j^*) - f(y_j^\perp)], \\ y_i^* - y_i^\perp = \sum_{j=1}^n b_{ij}[f(x_j^*) - f(x_j^\perp)], \end{cases} \tag{10}$$

which yield

$$\begin{aligned} &\sum_{i=1}^n (x_i^* - x_i^\perp)[f(x_i^*) - f(x_i^\perp)] \\ =&\sum_{i=1}^n \sum_{j=1}^n a_{ij}[f(y_j^*) - f(y_j^\perp)][f(x_i^*) - f(x_i^\perp)] \end{aligned} \tag{11}$$

and

$$\begin{aligned} &\sum_{i=1}^n (y_i^* - y_i^\perp)[f(y_i^*) - f(y_i^\perp)] \\ =&\sum_{i=1}^n \sum_{j=1}^n b_{ji}[f(x_i^*) - f(x_i^\perp)][f(y_j^*) - f(y_j^\perp)]. \end{aligned} \tag{12}$$

By using condition (4), we have

$$\begin{aligned} &\sum_{i=1}^n (x_i^* - x_i^\perp)[f(x_i^*) - f(x_i^\perp)] \\ &+ \sum_{i=1}^n (y_i^* - y_i^\perp)[f(y_i^*) - f(y_i^\perp)] = 0. \end{aligned}$$

On the other hand, from Eq.(2) we know $f(\bullet)$ is a monotonically increasing function, this means

$$(x_i^* - x_i^\perp)[f(x_i^*) - f(x_i^\perp)] + (y_i^* - y_i^\perp)[f(y_i^*) - f(y_i^\perp)] = 0,$$

$$(i = 1, 2, \cdots, n).$$

Without loss of generality, we assume

$$\begin{aligned} &x_1^* - x_1^\perp = 0, \cdots, x_k^* - x_k^\perp = 0, \\ &f(y_{k+1}^*) - f(y_{k+1}^\perp) = 0, \cdots, f(y_n^*) - f(y_n^\perp) = 0; \\ &y_1^* - y_1^\perp = 0, \cdots, y_k^* - y_k^\perp = 0, \\ &f(x_{k+1}^*) - f(x_{k+1}^\perp) = 0, \cdots, f(x_n^*) - f(x_n^\perp) = 0. \end{aligned}$$

Since

$$\begin{aligned} &x_1^* - x_1^\perp = 0, \cdots, x_k^* - x_k^\perp = 0, \\ &y_1^* - y_1^\perp = 0, \cdots, y_k^* - y_k^\perp = 0. \end{aligned}$$

We have

$$\begin{aligned} &f(x_1^*) - f(x_1^\perp) = 0, \cdots, f(x_k^*) - f(x_k^\perp) = 0, \\ &f(y_1^*) - f(y_1^\perp) = 0, \cdots, f(y_k^*) - f(y_k^\perp) = 0; \end{aligned}$$

therefore

$$f(x_j^*) = f(x_j^\perp), \quad f(y_j^*) = f(y_j^\perp), \quad (j = 1, 2, \cdots, n).$$

From Eq.(10), we finally have

$$x_i^* = x_i^\perp, y_i^* = y_i^\perp \quad (i = 1, 2, \cdots, n).$$

The proof of the uniqueness of equilibrium point (x^*, y^*) is completed.

(iii) Global exponential stability of Eq.(1)

Define a function

$$V(t) = \begin{cases} V_1(t) = \sum_{i=1}^n \int_{x_i^*}^{x_i} [f(s) - f(x_i^*)]\mathrm{d}s, \\ V_2(t) = \sum_{i=1}^n \int_{y_i^*}^{y_i} [f(s) - f(y_i^*)]\mathrm{d}s, \end{cases}$$

for all $t \geq 0$. Obviously, $V_1(t) \geq 0, V_2(t) \geq 0$. Calculating their time derivatives yields

$$\begin{aligned} \frac{\mathrm{d}V_1(t)}{\mathrm{d}t} =& -\sum_{i=1}^n [f(x_i) - f(x_i^*)][x_i - x_i^*] \\ &+ \sum_{i=1}^n \sum_{j=1}^n a_{ij}[f(x_i) - f(x_i^*)][f(y_j) - f(y_j^*)], \\ \frac{\mathrm{d}V_2(t)}{\mathrm{d}t} =& -\sum_{i=1}^n [f(y_i) - f(y_i^*)][y_i - y_i^*] \\ &+ \sum_{i=1}^n \sum_{j=1}^n b_{ji}[f(x_i) - f(x_i^*)][f(y_j) - f(y_j^*)]. \end{aligned}$$

Taking the summation of the above two equations and using Eq.(4), then using Lemma (2), we have

$$\begin{aligned} &\frac{\mathrm{d}[V_1(t) + V_2(t)]}{\mathrm{d}t} \\ &\leq -\sum_{i=1}^n \int_{x_i^*}^{x_i} [f(s) - f(x_i^*)]\mathrm{d}s \\ &\quad -\sum_{i=1}^n \int_{y_i^*}^{y_i} [f(s) - f(y_i^*)]\mathrm{d}s. \end{aligned}$$

It implies

$$V_1(t) + V_2(t) \leq [V_1(0) + V_2(0)]\mathrm{e}^{-t}$$

for all $t \geq 0$.

According to Lemmas (1) and (2), we have

$$\begin{aligned} V_1(0) =& \sum_{i=1}^n \int_{x_i^*}^{x_i(0)} [f(s) - f(x_i^*)]\mathrm{d}s \\ &\leq \sum_{i=1}^n |f(x_i(0)) - f(x_i^*)| \\ &\quad \cdot |x_i(0) - x_i^*| \leq ||x(0) - x^*||^2. \end{aligned}$$

From Lemma (3), we have

$$V_1(t) = \sum_{i=1}^{n} \int_{x_i^*}^{x_i} [f(s) - f(x_i^*)]\mathrm{d}s$$
$$\geq \int_{x_i^*}^{x_i} [f(s) - f(x_i^*)]\mathrm{d}s \geq \frac{1}{2}[f(x_i) - f(x_i^*)]^2.$$

Similarly

$$V_2(0) \leq ||y(0) - y^*||^2, \quad V_2(t) \geq \frac{1}{2}[f(y_i) - f(y_i^*)]^2.$$

Note $V_1(t) \geq 0$, therefore
$|f(x_i) - f(x_i^*)| \leq \sqrt{2}[||x(0) - x^*|| + ||y(0) - y_i^*||]\mathrm{e}^{-\frac{1}{2}t}$,

$$(i = 1, 2, \cdots, n)$$

for all $t \geq 0$.

Now let us calculate the time derivative

$$D^+|x_i - x_i^*| \leq -|x_i - x_i^*| + \sum_{j=1}^{n} |a_{ij}| \cdot |f(y_j) - f(y_j^*)|$$
$$\leq -|x_i - x_i^*| + \sqrt{2}\sum_{j=1}^{n} |a_{ij}|$$
$$\times [||x(0) + x^*|| + ||y(0) - y^*||]\mathrm{e}^{-\frac{1}{2}t}. \quad (13)$$

Consequently, we have

$$|x_i - x_i^*| \leq \Pi \cdot \left(\mathrm{e}^{-t} + 4\sqrt{2}\sum_{j=1}^{n} |a_{ij}|\mathrm{e}^{-\frac{1}{2}t}\right),$$
$$(i = 1, 2, \cdots, n)$$

for all $t \geq 0$.

Similarly, we have

$$|y_i - y_i^*| \leq \Pi \cdot \left(\mathrm{e}^{-t} + 4\sqrt{2}\sum_{j=1}^{n} |b_{ij}|\mathrm{e}^{-\frac{1}{2}t}\right),$$
$$(i = 1, 2, \cdots, n)$$

valid for all $t \geq 0$.

Thus Eq.(6) in the Theorem is derived, and according to the definition, network (1) is globally exponentially stable.

3. Examples

Example 1 Consider a two-dimensional BAM network described by:

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = -x_1 + \dfrac{1}{4}f(y_1) - \dfrac{1}{7}f(y_2) + 0.1, \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = -x_2 - \dfrac{1}{3}f(y_1) + \dfrac{1}{6}f(y_2) - 0.1, \\ \dfrac{\mathrm{d}y_1}{\mathrm{d}t} = -y_1 - \dfrac{1}{4}f(x_1) + \dfrac{1}{3}f(x_2) + 0.2, \\ \dfrac{\mathrm{d}y_2}{\mathrm{d}t} = -y_2 + \dfrac{1}{7}f(x_1) - \dfrac{1}{6}f(x_2) - 0.2, \end{cases} \quad (14)$$

where $f(s) = \dfrac{1}{2}(|s+1| - |s-1|)$, $s \in R$. Obviously, the connection matrices are antisymmetric. It follows from Theorem that (14) has a unique equilibrium $(x_1^*, x_2^*, y_1^*, y_2^*)$ and for any point $(x_1(0), x_2(0)) \in R^2$, $(y_1(0), y_2(0)) \in R^2$, the solution $(x_1, x_2), (y_1, y_2)$, of Eq.(14) starting from this point satisfies

$$\begin{cases} |x_1 - x_1^*| \leq (11\sqrt{2}/7 + 1)\Pi\mathrm{e}^{-t/2}, \\ |x_2 - x_2^*| \leq (2\sqrt{2} + 1)\Pi\mathrm{e}^{-t/2}, \\ |y_1 - y_1^*| \leq (7\sqrt{2}/3 + 1)\Pi\mathrm{e}^{-t/2}, \\ |y_2 - y_2^*| \leq (26\sqrt{2}/21 + 1)\Pi\mathrm{e}^{-t/2}, \end{cases}$$

for all $t \geq 0$. This means that the unique equilibrium is globally exponentially stable, and the above expression can be used to evaluate the convergence rate of a motion trajectory of the network. Figure 1 shows the output waveforms of system (14). The equilibrium is located at x^*=(0.2026, –0.2679), y^*=(0.2807, –0.2243).

Example 2 Consider another two-dimensional BAM network described by:

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = -x_1 + \dfrac{1}{4}f(y_1) - \dfrac{1}{7}f(y_2) + 0.1, \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = -x_2 - 3f(y_1) + 6f(y_2) - 0.1, \\ \dfrac{\mathrm{d}y_1}{\mathrm{d}t} = -y_1 - 4f(x_1) + \dfrac{1}{3}f(x_2) + 0.2, \\ \dfrac{\mathrm{d}y_2}{\mathrm{d}t} = -y_2 + 7f(x_1) - \dfrac{1}{6}f(x_2) - 0.2. \end{cases} \quad (15)$$

Note that the condition of the Theorem is not satisfied. Next, we will show that the equilibrium is not global exponential stability. Figure 2 is the output waveforms of system (15). Suppose we have an initial point (0,0.4), (0,0.4), the solution $(x_1(t), x_2(t))$, $(y_1(t), y_2(t))$, of system (15) starting from this point eventually converges to the stable equilibrium located at x^*= (0.0000, 17.2951), y^*= (1.9672, 3.8525). However, it can be seen from Fig.2 that there are overshoots for some output waveforms which may not be acceptable in some practical applications.

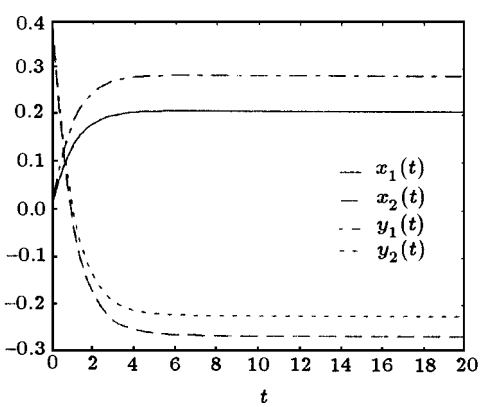


Fig.1. Trajectory of system (14).

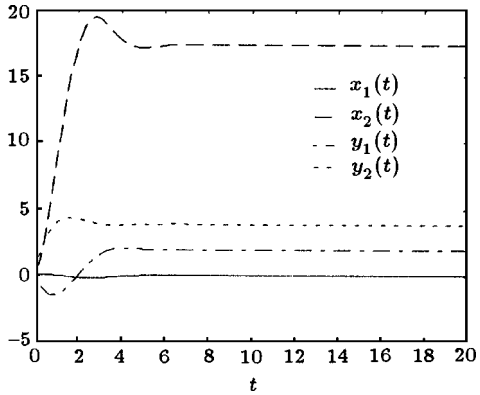


Fig.2. Trajectory of system (15).

4. Conclusions

A sufficient condition for a class of BAM networks to have a globally exponentially stable equilibrium is derived in this paper. The condition (4) is in a fairly concise form. The convergence rate of trajectories for a BAM network satisfying condition (4) can be evaluated by use of Eq.(6). The results given in our Theorem are justified by two simulation examples, and expressions (4) and (6) are quite easy to use in practical network design.

References

[1] Kosko B 1987 *Appl. Opt.* **26** 4947
[2] Kosko B 1988 *IEEE Trans. Syst. Man Cybernet.* **18** 49
[3] Zhang J S and Xiao X C 2000 *Chin. Phys. Lett.* **17** 88
[4] Sun Z C and Xu J X 2001 *Chin. Phys.* **10** 599
[5] Li L X *et al* 2001 *Chin. Phys.* **10** 796
[6] Chen G R *et al* 1999 *Acta Phys. Sin.* (Overseas Edition) **8** 416
[7] Wang H X and Yu J B 2001 *Acta Phys. Sin.* **50** 2303 (in Chinese)
[8] Zhang Y 1996 *Int. J. Syst. Sci.* **27** 227
[9] Zhang Y and Yu J B 2001 *Sci. China, Section* E **44** 1
[10] K Gopalsamy and X Z He 1994 *IEEE Trans. Neural Network* **5** 998
[11] Glorieux C *et al* 1999 *J. Phys. D: Appl. Phys.* **32** 616