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New applications of the homogeneous balance principle*

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The homogeneous balance principle has been widely applied to the exploration of nonlinear transformation, exact solutions (especially solitary wave solution), dromion and similarity reduction to the nonlinear partial differential equations in mathematical physics. In this paper, we use the homogeneous balance principle to derive Bäcklund transformations for nonlinear partial differential equations that have more nonlinear terms and more highest-order partial derivative terms. With the aid of the Bäcklund transformations derived here, we could obtain exact solutions to the nonlinear partial differential equations. The Davey–Stewartson equation and the Nizhnik–Novikov–Veselov equation are considered as the examples.

Keywords: homogeneous balance principle, Davey–Stewartson equation, Nizhnik–Novikov–Veselov equation, Bäcklund transformation, exact solutions

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1. Introduction

It is well known that nonlinear partial differential equations in mathematical physics are widely applied to the description of complex phenomena in various sciences, for example, in fluid mechanics, plasma physics, atmospheric science, optical fibre communication, etc. In order to understand the physical mechanism of the phenomena in those fields, one has to study the solutions and their properties of the associated equations. Exploring exact solutions to nonlinear partial differential equations has long been paid much attention by many researchers in mathematical and physical fields. In particular, great progress has been made in finding solitary wave solutions to nonlinear partial differential equations in mathematical physics. A number of powerful methods have been introduced, such as the inverse scattering method,[1] Hirota method,[2] Bäcklund transformation (BT) method,[2] Hopf–Cole transformation method,[2] Darboux transformation method,[2] Lie group analysis,[3–5] Miura transformation method,[6] hyperbolic function method,[7–10] sine–cosine method,[11–13] Jacobi elliptic function method,[14–16] truncated expansion method,[17] direct reduction method,[18,19] formally variable separation approach,[20–22] rank analysis technique,[23], etc. Due to the non-integrability of nonlinear partial differential equations, and many nonlinear equations without passing the Painlevé test, to find the exact solutions to nonlinear equations is one of the most difficult tasks in mathematical physics.

Recently,the homogeneous balance principle[24–37] put forward by one of the authors of this paper has been applied to derive the nonlinear transformations and exact solutions (especially the solitary wave solution),[24–37] auto-Bäcklund transformations (auto-BT)[38–40] and similarity reductions[41] to nonlinear partial differential equations (PDEs) in mathematical physics. The dromion solutions, breathers, instantons and ring soliton solutions of some nonlinear PDEs were obtained by using the improved homogeneous balance principle.[42] The homogeneous balance principle has been applied to explore the exact solutions to nonlinear PDEs which have one highest-order

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partial derivative term. In this paper, we use the homogeneous balance principle to derive BT for nonlinear equations in mathematical physics, which have more nonlinear terms and more highest-order partial derivative terms.

This paper is organized as follows: a (2+1)-dimensional Davey–Stewartson equation is considered as an example in section 2 of this paper, We derive the Bäcklund transformation for Davey–Stewartson equation firstly, and then give the solution of the Davey–Stewartson equation. The Bäcklund transformation and the solitary wave solution of the Nizhnik–Novikov–Veselov (NNV) equation are derived in section 3. Finally, some conclusions are given in section 4.

2. The (2+1)-dimensional Davey–Stewartson equation

Firstly, we explore the exact solution of the Davey–Stewartson equation[43−45]

$$\mathrm{i}q_t = q_{xx} + q_{yy} + \frac{1}{2}q[\partial_y^{-1}(|q|^2)_x + \partial_x^{-1}(|q|^2)_y]. \quad (1)$$

Since q is a complex function, we assume that

$$q = \exp[\mathrm{i}(\alpha x + \beta y + \lambda t)]u(x,y,t) = \exp(\mathrm{i}\eta)u(x,y,t), \quad (2)$$

where $\eta = \alpha x + \beta y + \lambda t$, and α, β, λ are to be determined later.

From Eq.(2), we could derive

$$q_t = \mathrm{i}\lambda \mathrm{e}^{\mathrm{i}\eta}u + \mathrm{e}^{\mathrm{i}\eta}u_t, \quad (3)$$

$$q_{xx} = -\alpha^2\mathrm{e}^{\mathrm{i}\eta} + 2\mathrm{i}\alpha\mathrm{e}^{\mathrm{i}\eta}u_x + \mathrm{e}^{\mathrm{i}\eta}u_{xx}, \quad (4)$$

$$q_{yy} = -\beta^2\mathrm{e}^{\mathrm{i}\eta}u + 2\mathrm{i}\beta\mathrm{e}^{\mathrm{i}\eta}u_y + \mathrm{e}^{\mathrm{i}\eta}u_{yy}. \quad (5)$$

Substituting Eqs.(3)–(5) into Eq.(1) and cancelling $\mathrm{e}^{\mathrm{i}\eta}$ yield

$$u_t - 2\alpha u_x - 2\beta u_y = 0, \quad (6)$$

$$u_{xx} + u_{yy} + \frac{1}{2}u[\partial_y^{-1}(u^2)_x + \partial_x^{-1}(u^2)_y] + (\lambda - \alpha^2 - \beta^2)u = 0. \quad (7)$$

In order to explore the solutions of the Davey–Stewartson equation, we suppose that

$$u = w_{xy}^{\frac{1}{2}}. \quad (8)$$

Substituting Eq.(8) into Eqs.(6) and (7), we obtain

$$w_{xyt} - 2\alpha w_{xxy} - 2\beta w_{xyy} = 0, \quad (9)$$

$$(\lambda - \alpha^2 - \beta^2)w_{xy}^2 - \frac{1}{4}(w_{xxy}^2 + w_{xyy}^2) + \frac{1}{2}w_{xy}^2(w_{xx} + w_{yy}) + \frac{1}{2}w_{xy}(w_{xxxy} + w_{xyyy}) = 0. \quad (10)$$

According to the idea of the homogeneous balance principle, considering the homogeneous balance $w_{xy}w_{xxxy}$ and $-\frac{1}{4}w_{xxy}^2 + \frac{1}{2}w_{xy}^2w_{xx}$, as well as $w_{xy}w_{xyyy}$ and $-\frac{1}{4}w_{xyy}^2 + \frac{1}{2}w_{xy}^2w_{yy}$ in Eq.(10) simultaneously, we assume that the solution of Eqs.(9) and (10) is of the form

$$w = f(\varphi). \quad (11)$$

From Eq.(11), we have

$$w_{xyt} = f'''\varphi_x\varphi_y\varphi_t + f''(\varphi_x\varphi_{yt} + \varphi_{xt}\varphi_y + \varphi_{xy}\varphi_t) + f'\varphi_{xyt}, \quad (12)$$

$$w_{xxy} = f'''\varphi_x^2\varphi_y + f''(\varphi_{xx}\varphi_y + 2\varphi_x\varphi_{xy}) + f'\varphi_{xxy}, \quad (13)$$

$$w_{xyy} = f'''\varphi_y^2\varphi_x + f''(\varphi_x\varphi_{yy} + 2\varphi_{xy}\varphi_y) + f'\varphi_{xyy}, \quad (14)$$

$$w_{xy}^2 = f''^2\varphi_x^2\varphi_y^2 + 2f'f''\varphi_x\varphi_{xy}\varphi_y + f'^2\varphi_{xy}^2, \quad (15)$$

$$\begin{aligned} w_{xxy}^2 + w_{xyy}^2 =& f'''^2(\varphi_x^4\varphi_y^2 + \varphi_x^2\varphi_y^4) \\ &+ 2f''f'''(\varphi_x^2\varphi_{xx}\varphi_y^2 + \varphi_x^2\varphi_{yy}\varphi_y^2 \\ &+ 2\varphi_x^3\varphi_{xy}\varphi_y + 2\varphi_x\varphi_{xy}\varphi_y^3) \\ &+ f''^2(\varphi_{xx}^2\varphi_y^2 + 4\varphi_x\varphi_{xx}\varphi_{xy}\varphi_y \\ &+ 4\varphi_x^2\varphi_{xy}^2 + \varphi_x^2\varphi_{yy}^2 \\ &+ 4\varphi_x\varphi_{xy}\varphi_{yy}\varphi_y + 4\varphi_{xy}^2\varphi_y^2) \\ &+ 2f'f'''(\varphi_x^2\varphi_{xxy}\varphi_y + \varphi_x\varphi_{xyy}\varphi_y^2) \\ &+ 2f'f''(\varphi_{xx}\varphi_{xxy}\varphi_y + 2\varphi_x\varphi_{xy}\varphi_{xxy} \\ &+ \varphi_x\varphi_{xyy}\varphi_{yy} + 2\varphi_y\varphi_{xy}\varphi_{xyy}) \\ &+ f'^2(\varphi_{xxy}^2 + \varphi_{xyy}^2), \end{aligned} \quad (16)$$

$$\begin{aligned} w_{xy}^2(w_{xx} + w_{yy}) =& f''''^2(\varphi_x^4\varphi_y^2 + \varphi_x^2\varphi_y^4) \\ &+ f'f''^2(2\varphi_x^3\varphi_{xy}\varphi_y + \varphi_x^2\varphi_{xx}\varphi_y^2 \\ &+ \varphi_x^2\varphi_{yy}\varphi_y^2 + 2\varphi_x\varphi_{xy}\varphi_y^3) \\ &+ f'^2f''(\varphi_x^2\varphi_{xy}^2 + 2\varphi_x\varphi_{xx}\varphi_{xy}\varphi_y \\ &+ \varphi_{xy}^2\varphi_y^2 + 2\varphi_x\varphi_{xy}\varphi_{yy}\varphi_y) \\ &+ f'^3(\varphi_{xx}\varphi_{xy}^2 + \varphi_{xy}^2\varphi_{yy}), \end{aligned} \quad (17)$$

$$
\begin{aligned}
&w_{xy}(w_{xxxy}+w_{xyyy})\\
=&f''f^{(4)}(\varphi_x^4\varphi_y^2+\varphi_x^2\varphi_y^4)\\
&+f'f^{(4)}(\varphi_x^3\varphi_{xy}\varphi_y+\varphi_x\varphi_{xy}\varphi_y^3)\\
&+3f''f'''(\varphi_x^3\varphi_{xy}\varphi_y+\varphi_x\varphi_{xy}\varphi_y^3\\
&+\varphi_x^2\varphi_{xx}\varphi_y^2+\varphi_x^2\varphi_{yy}\varphi_y^2)\\
&+3f'f'''(\varphi_x^2\varphi_{xy}^2+\varphi_{xy}^2\varphi_y^2\\
&+\varphi_x\varphi_{xx}\varphi_{xy}\varphi_y+\varphi_x\varphi_{xy}\varphi_{yy}\varphi_y)\\
&+f''^2(3\varphi_x\varphi_{xx}\varphi_{xy}\varphi_y+3\varphi_x\varphi_{xy}\varphi_{yy}\varphi_y\\
&+3\varphi_x^2\varphi_{xxy}\varphi_y+3\varphi_x\varphi_{xyy}\varphi_y^2\\
&+\varphi_x^2\varphi_{yyy}\varphi_y+\varphi_x\varphi_{xxx}\varphi_y^2)\\
&+f'f''(3\varphi_{xx}\varphi_{xy}^2+3\varphi_{xy}^2\varphi_{yy}\\
&+3\varphi_x\varphi_{xy}\varphi_{xxy}+3\varphi_{xy}\varphi_{xyy}\varphi_y\\
&+\varphi_{xxx}\varphi_{xy}\varphi_y+\varphi_x\varphi_{xy}\varphi_{yyy}\\
&+\varphi_x\varphi_{xxxy}\varphi_y+\varphi_x\varphi_{xyyy}\varphi_y)\\
&+f'^2(\varphi_{xy}\varphi_{xxxy}+\varphi_{xy}\varphi_{xyyy}).
\end{aligned}
\tag{18}
$$

Substituting Eqs.(11)–(18) into the left-hand sides of Eqs.(9) and (10), and collecting the homogeneous terms of partial derivatives for $\varphi=\varphi(x,y,z,t)$ together separately, yield

$$
\begin{aligned}
&w_{xyt}-2\alpha w_{xxy}-2\beta w_{xyy}\\
=&\left[f'''\varphi_x\varphi_y+f''\left(\varphi_y\frac{\partial}{\partial x}+\varphi_x\frac{\partial}{\partial y}+\varphi_{xy}\right)\right.\\
&\left.+f'\frac{\partial^2}{\partial x\partial y}\right](\varphi_t-2\alpha\varphi_x-2\beta\varphi_y),
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&(\lambda-\alpha^2-\beta^2)w_{xy}^2-\frac{1}{4}(w_{xxy}^2+w_{xyy}^2)\\
&+\frac{1}{2}w_{xy}^2(w_{xx}+w_{yy})+\frac{1}{2}w_{xy}(w_{xxxy}+w_{xyyy})\\
=&\left(\frac{1}{2}f''^3-\frac{1}{4}f'''^2+\frac{1}{2}f''f^{(4)}\right)\\
&\times(\varphi_x^4\varphi_y^2+\varphi_x^2\varphi_y^4)+m(x,y,t).
\end{aligned}
\tag{20}
$$

Here $m(x,y,t)$ is a polynomial of various partial derivatives of φ, and the degree of $m(x,y,t)$ is lower than $\varphi_x^4\varphi_y^2+\varphi_x^2\varphi_y^4$. Setting the coefficient of $\varphi_x^4\varphi_y^2+\varphi_x^2\varphi_y^4$ to zero, we obtain the ordinary differential equation (ODE) for f

$$
2f''^3-f'''^2+2f''f^{(4)}=0. \tag{21}
$$

Equation (21) admits a solution

$$
f=4\ln\varphi. \tag{22}
$$

From Eq.(22), we obtain

$$
f''f'''=-\frac{1}{3}f^{(5)},\quad f'f'''=-\frac{4}{3}f^{(4)},\quad f''^2=-\frac{2}{3}f^{(4)}, \tag{23}
$$

$$
f'f''=-2f''',\quad f'f''^2=\frac{2}{3}f^{(5)},\quad f'^2f''=\frac{8}{3}f^{(4)},
$$

$$
f'^3=8f''',\quad f'^2=-4f'',\quad f'f^{(4)}=-f^{(5)}. \tag{24}
$$

Substituting Eqs.(22)–(24) into the left-hand side of Eq.(10), we have

$$
\begin{aligned}
&(\lambda-\alpha^2-\beta^2)w_{xy}^2-\frac{1}{4}(w_{xxy}^2+w_{xyy}^2)\\
&+\frac{1}{2}w_{xy}^2(w_{xx}+w_{yy})+\frac{1}{2}w_{xy}(w_{xxxy}+w_{xyyy})\\
=&\frac{1}{6}f^{(4)}[\varphi_{xx}^2\varphi_y^2+\varphi_x^2\varphi_{yy}^2+2\varphi_x\varphi_{xx}\varphi_{xy}\varphi_y\\
&+2\varphi_x\varphi_{xy}\varphi_{yy}\varphi_y-2\varphi_x^2\varphi_{xxy}\varphi_y\\
&-2\varphi_x\varphi_{xyyy}\varphi_y^2-2\varphi_x\varphi_{xxx}\varphi_y^2-2\varphi_x^2\varphi_{yyy}\varphi_y\\
&-4(\lambda-\alpha^2-\beta^2)\varphi_x^2\varphi_y^2]+f'''[\varphi_{xx}\varphi_{xy}^2+\varphi_{xy}^2\varphi_{yy}\\
&+\varphi_{xx}\varphi_{xxy}\varphi_y+\varphi_x\varphi_{xyy}\varphi_{yy}-\varphi_x\varphi_{xy}\varphi_{xxy}\\
&-\varphi_{xy}\varphi_y\varphi_{xyy}-\varphi_{xxx}\varphi_{xy}\varphi_y-\varphi_x\varphi_{xy}\varphi_{yyy}\\
&-\varphi_x\varphi_{xxxy}\varphi_y-\varphi_x\varphi_{xyyy}\varphi_y\\
&-4(\lambda-\alpha^2-\beta^2)\varphi_x\varphi_{xy}\varphi_y]\\
&+f''[\varphi_{xxy}^2+\varphi_{xyy}^2-2\varphi_{xy}\varphi_{xxxy}\\
&-2\varphi_{xy}\varphi_{xyyy}-4(\lambda-\alpha^2-\beta^2)\varphi_{xy}^2].
\end{aligned}
\tag{25}
$$

Setting the right-hand sides of Eqs.(19) and (25) to zero, and noticing that f', f'', $f^{(3)}$ and $f^{(4)}$ are linearly independent, we obtain the BT of Eqs.(9) and (10) as follows:

$$
w=f(\varphi), \tag{11}
$$

$$
\varphi_t-2\alpha\varphi_x-2\beta\varphi_y=0, \tag{26}
$$

$$
\begin{aligned}
&\varphi_{xx}^2\varphi_y^2+\varphi_x^2\varphi_{yy}^2+2\varphi_x\varphi_{xx}\varphi_{xy}\varphi_y+2\varphi_x\varphi_{xy}\varphi_{yy}\varphi_y\\
&-2\varphi_x^2\varphi_{xxy}\varphi_y-2\varphi_x\varphi_{xyyy}\varphi_y^2\\
&-2\varphi_x\varphi_{xxx}\varphi_y^2-2\varphi_x^2\varphi_{yyy}\varphi_y\\
&-4(\lambda-\alpha^2-\beta^2)\varphi_x^2\varphi_y^2=0,
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
&\varphi_{xx}\varphi_{xy}^2+\varphi_{xy}^2\varphi_{yy}+\varphi_{xx}\varphi_{xxy}\varphi_y+\varphi_x\varphi_{xyy}\varphi_{yy}\\
&-\varphi_x\varphi_{xy}\varphi_{xxy}-\varphi_{xy}\varphi_y\varphi_{xyy}-\varphi_{xxx}\varphi_{xy}\varphi_y\\
&-\varphi_x\varphi_{xy}\varphi_{yyy}-\varphi_x\varphi_{xxxy}\varphi_y-\varphi_x\varphi_{xyyy}\varphi_y\\
&-4(\lambda-\alpha^2-\beta^2)\varphi_x\varphi_{xy}\varphi_y=0,
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
&\varphi_{xxy}^2+\varphi_{xyy}^2-2\varphi_{xy}\varphi_{xxxy}-2\varphi_{xy}\varphi_{xyyy}\\
&-4(\lambda-\alpha^2-\beta^2)\varphi_{xy}^2=0.
\end{aligned}
\tag{29}
$$

Noticing the homogeneity of Eqs.(26)–(29), we obtain the solution of Eqs.(26)–(29)

$$
\varphi=1+\exp(lx+my+nt), \tag{30}
$$

where

$$n = 2\alpha l + 2\beta m, \tag{31}$$

At the same time, we obtain

$$\lambda = -\frac{1}{4}(l^2 + m^2) + \alpha^2 + \beta^2, \tag{32}$$

where l, m, α and β are all constants.

Therefore, we can derive the solution to Eq.(1)

$$q = \mathrm{e}^{\mathrm{i}\eta}\sqrt{lm\mathrm{sech}^2(\omega/2)}. \tag{33}$$

Here

$$\eta = \alpha x + \beta y - \frac{1}{4}(l^2 + m^2)t + (\alpha^2 + \beta^2)t, \tag{34}$$

and

$$\omega = lx + my + (2\alpha l + 2\beta m)t. \tag{35}$$

3. The NNV equation

In this section, we consider the NNV equation

$$u_t + [u_{xxx} + u_{yyy} - 3(u\partial_y^{-1}u_x)_x - 3(u\partial_x^{-1}u_y)_y] = 0. \tag{36}$$

In Ref.[46], the coherent structures of the NNV equation are discussed, and the dromion to the NNV equation are given in Ref.[47]. Ruan and Chen[48] used the variable separation approach to study the localized coherent structures of the generalized NNV equation. In this paper, we use the homogeneous balance principle to derive BT to the NNV equation, and then with the aid of BT, give the solitary wave solution to the NNV equation.

In Eq.(36), there are two highest derivative terms and two nonlinear terms. According to the idea of the homogeneous balance principle, considering the homogeneous balance between u_{xxx} and $(u\partial_y^{-1}u_x)_x$, as well as u_{yyy} and $(u\partial_x^{-1}u_y)_y$ simultaneously, we suppose that the solution of Eq.(36) is of the form

$$u = f''\varphi_x\varphi_y + f'\varphi_{xy}, \tag{37}$$

where $f = f(\varphi)$ is a function of one argument only, and $\varphi(x, y, t)$ is to be determined later.

From Eq.(37), we have

$$\begin{aligned} u_{xxx} =& f^{(5)}\varphi_x^5\varphi_y + f^{(4)}(4\varphi_x^3\varphi_{xy} + 6\varphi_x^2\varphi_{xx}\varphi_y) \\ &+ f'''(12\varphi_x\varphi_{xx}\varphi_{xy} + 6\varphi_x^2\varphi_{xxy} \\ &+ 3\varphi_{xx}^2\varphi_y + 4\varphi_x\varphi_{xxx}\varphi_y) \\ &+ f''(6\varphi_{xx}\varphi_{xxy} + 4\varphi_{xxx}\varphi_{xy} \\ &+ 4\varphi_x\varphi_{xxxy} + \varphi_{xxxx}\varphi_y) + f'\varphi_{xxxxy}, \end{aligned} \tag{38}$$

$$\begin{aligned} u_{yyy} =& f^{(5)}\varphi_x\varphi_y^4 + f^{(4)}(4\varphi_{xy}\varphi_y^3 + 6\varphi_x\varphi_{yy}\varphi_y^2) \\ &+ f'''(12\varphi_{xy}\varphi_y\varphi_{yy} + 6\varphi_{xyy}\varphi_y^2 \\ &+ 3\varphi_x\varphi_{yy}^2 + 4\varphi_x\varphi_y\varphi_{yyy}) \\ &+ f''(6\varphi_{xyy}\varphi_{yy} + 4\varphi_{xy}\varphi_{yyy} \\ &+ 4\varphi_{xyyy}\varphi_y + \varphi_x\varphi_{yyyy}) + f'\varphi_{xyyyy}, \end{aligned} \tag{39}$$

$$\begin{aligned} (u\partial_y^{-1}u_x)_x =& 2f''f'''\varphi_x^4\varphi_y + f''^2(4\varphi_x^2\varphi_{xx}\varphi_y + 2\varphi_x^3\varphi_{xy}) \\ &+ f'f'''(\varphi_x^2\varphi_{xx}\varphi_y + \varphi_x^3\varphi_{xy}) \\ &+ f'f''(\varphi_{xx}^2\varphi_y + \varphi_x\varphi_{xxx}\varphi_y \\ &+ 5\varphi_x\varphi_{xx}\varphi_{xy} + \varphi_x^2\varphi_{xxy}) \\ &+ f'^2(\varphi_{xxx}\varphi_{xy} + \varphi_{xx}\varphi_{xxy}), \end{aligned} \tag{40}$$

$$\begin{aligned} (u\partial_x^{-1}u_y)_y =& 2f''f'''\varphi_x\varphi_y^4 + f''^2(4\varphi_x\varphi_{yy}\varphi_y^2 + 2\varphi_{xy}\varphi_y^3) \\ &+ f'f'''(\varphi_x\varphi_{yy}\varphi_y^2 + \varphi_{xy}\varphi_y^3) \\ &+ f'f''(\varphi_x\varphi_{yy}^2 + \varphi_x\varphi_{yyy}\varphi_y \\ &+ 5\varphi_{xy}\varphi_y\varphi_{yy} + \varphi_{xyy}\varphi_y^2) \\ &+ f'^2(\varphi_{xyy}\varphi_{yy} + \varphi_{xy}\varphi_{yyy}), \end{aligned} \tag{41}$$

$$\begin{aligned} u_t =& f'''\varphi_x\varphi_y\varphi_t + f''(\varphi_x\varphi_{yt} + \varphi_{xt}\varphi_y \\ &+ \varphi_{xy}\varphi_t) + f'\varphi_{xyt}. \end{aligned} \tag{42}$$

Substituting Eqs.(38)–(42) into the left-hand side of Eq.(36), and collecting the homogeneous terms of partial derivatives for $\varphi = \varphi(x, y, z, t)$ together separately, we have

$$\begin{aligned} &u_t + [u_{xxx} + u_{yyy} - 3(u\partial_y^{-1}u_x)_x - 3(u\partial_x^{-1}u_y)_y] \\ =&(f^{(5)} - 6f''f''')(\varphi_x\varphi_y^4 + \varphi_x^4\varphi_y) + m(x, y, t), \end{aligned} \tag{43}$$

where $m(x, y, t)$ is a polynomial of partial derivatives of $\varphi(x, y, t)$, and the degree of $m(x, y, t)$ is lower than $\varphi_x\varphi_y^4 + \varphi_x^4\varphi_y$. Setting the coefficient of $\varphi_x\varphi_y^4 + \varphi_x^4\varphi_y$ to zero, we have the ODE for $f(\varphi)$,

$$f^{(5)} - 6f''f''' = 0. \tag{44}$$

Eq. (44) admits a solution as follows:

$$f = -2\ln\varphi. \tag{45}$$

Then we have

$$f''^2 = \frac{1}{3}f^{(4)}, \quad f'f''' = \frac{2}{3}f^{(4)}, \tag{46}$$

$$f'f'' = f''', \quad f'^2 = 2f''. \tag{47}$$

Substituting Eqs.(45)–(47) into the left-hand side of Eq.(1) yields

$$\begin{aligned} & u_t + \alpha(t)[u_{xxx} + u_{yyy} - 3(u\partial_y^{-1}u_x)_x - 3(u\partial_x^{-1}u_y)_y] \\ =& f'''[\varphi_x\varphi_y(\varphi_t + (\varphi_{xxx} + \varphi_{yyy})) \\ & + 3(\varphi_x^2\varphi_{xxy} - \varphi_x\varphi_{xx}\varphi_{xy} + \varphi_{xyy}\varphi_y^2 - \varphi_{xy}\varphi_y\varphi_{yy})] \\ & + f''\left[\left(\varphi_y\frac{\partial}{\partial x} + \varphi_x\frac{\partial}{\partial y} + \varphi_{xy}\right)(\varphi_t + (\varphi_{xxx} + \varphi_{yyy})) \right. \\ & \left. + 3(\varphi_{xyyy}\varphi_y - \varphi_{xy}\varphi_{yyy} + \varphi_x\varphi_{xxxy} - \varphi_{xy}\varphi_{xxx})\right] \\ & + f'\left[\frac{\partial^2}{\partial x\partial y}(\varphi_t + (\varphi_{xxx} + \varphi_{yyy}))\right]. \end{aligned} \tag{48}$$

Setting the right-hand side of Eq.(48) to zero, and considering the linearly independence of f', f'' and f''', we obtain the BT to the NNV equation as follows:

$$u = -2(\ln\varphi)_{xy}, \tag{37}$$

$$\varphi_t + (\varphi_{xxx} + \varphi_{yyy}) = 0, \tag{49}$$

$$\varphi_x^2\varphi_{xxy} - \varphi_x\varphi_{xx}\varphi_{xy} + \varphi_{xyy}\varphi_y^2 - \varphi_{xy}\varphi_y\varphi_{yy} = 0, \tag{50}$$

$$\varphi_{xyyy}\varphi_y - \varphi_{xy}\varphi_{yyy} + \varphi_x\varphi_{xxxy} - \varphi_{xy}\varphi_{xxx} = 0. \tag{51}$$

Considering the homogeneity of Eqs.(50) and (51), we gain the solution for Eqs.(37), (50) and (51)

$$\varphi = 1 + \exp(lx + my - (l^3 + m^3)t + \eta), \tag{52}$$

where l, m and η are constants.

Therefore, the solitary wave solution to the NNV equation is obtained

$$u = \frac{lm}{4}\mathrm{sech}^2(lx + my - (l^3 + m^3)t + \eta). \tag{53}$$

4. Conclusion

From the derivations of the BT, and exact solutions to the Davey–Stewartson equation and the NNV equation, we can see that the homogeneous balance principle not only can be applied to the nonlinear equations which have one highest derivative term, but also can be applied to the nonlinear equations which have more highest derivative terms. It has been shown that the homogeneous balance principle is very powerful.

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