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Explicit exact solitary wave solutions for generalized symmetric regularized long-wave equations with high-order nonlinear terms*

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In this paper, we have obtained the bell-type and kink-type solitary wave solutions of the generalized symmetric regularized long-wave equations with high-order nonlinear terms by means of proper transformation and undetermined assumption method.

Keywords: symmetric regularized long-wave equation, undetermined assumption method, solitary wave solution

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1. Introduction

It is well known that many important dynamic processes can be described by specific nonlinear partial differential equations. It is of fundamental physical interest to solve the partial differential equations. In the past several years, both mathematicians and physicists have made attempts in this field (see Refs.[1]–[12]). In Refs.[3,4] and [13]–[15], the authors have studied the following symmetric regularized long-wave equation (SRLWE)

$$\begin{cases} u_{xxt} - u_t = \left(\rho + \dfrac{1}{2}u^2\right)_x, \\ \rho_t + u_x = 0. \end{cases} \tag{1}$$

Eq. (1) was proposed as a model for the promulgation of plasma and sound waves with a weak nonlinearity.[3] It has also been given in other fields such as nonlinear mathematical physics.[4] The solitary wave solutions, conserved laws and interaction among the solitary wave solutions have been studied.[3,4] Moreover, the global and numerical solutions of Eq.(1) have also been investigated (see Refs.[13]–[15]).

In this paper we consider a generalized SRLWE with high-order nonlinear terms

$$\begin{cases} u_{xxt} - u_t = (b_1\rho + b_2u^{p+1} + b_3u^{2p+1})_x, \\ b_i = \text{constant}, \quad p > 0 \\ \rho_t + u_x = 0, \end{cases} \tag{2}$$

and a generalized SRLWE with a dissipation term

$$\begin{cases} u_{xxt} - u_t = ru_{xx} + (b_1\rho + b_2u^{p+1})_x, \\ r, b_i = \text{constant}, \quad p > 0, \\ \rho_t + u_x = 0. \end{cases} \tag{3}$$

By means of appropriate transformations and conjectures, we find the bell-type and the kink-type solitary wave solutions of Eq.(2), and the kink-type solitary wave solutions of Eq.(3). We show that the kink-type solutions of Eq.(3) are different from those of Eq.(2)

2. The bell-type and kink-type solitary wave solutions of the generalized SRLWE (2)

To find the travelling wave solutions of Eq.(2), we make the transformation $u(x,t) = u(\xi) = u(x - vt)$, $\rho(x,t) = \rho(\xi) = \rho(x - vt)$ and change Eq.(2) into the form

$$\begin{cases} -vu'''(\xi) + vu'(\xi) \\ = b_1\rho'(\xi) + (b_2u^{p+1}(\xi) + b_3u^{2p+1}(\xi))_\xi, \\ \rho'(\xi) = \dfrac{1}{v}u'(\xi). \end{cases} \tag{4}$$

In order to seek solitary wave solutions of Eq.(2), we assume that

$$u'(\xi),\ u''(\xi) \to 0, \quad |\xi| \to \infty, \tag{4a}$$

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and the asymptotic values c_+, c_- of $u(\xi)(c_\pm = \lim_{\xi\to\pm\infty} u(\xi))$ satisfy the algebraic equation

$$\frac{b_1 - v^2}{v}x + b_2x^{p+1} + b_3x^{2p+1} = 0, \tag{4b}$$

then after integrating Eq.(4), the $\rho(\xi)$ and $u(\xi)$ satisfy

$$\rho(\xi) = \frac{1}{v}u(\xi) + c,$$

$$u''(\xi) + \frac{b_1 - v^2}{v^2}u(\xi) + \frac{b_2}{v}u^{p+1}(\xi) + \frac{b_3}{v}u^{2p+1}(\xi) = 0. \tag{5}$$

Thus, the problem of finding solitary wave solutions of Eq.(2) leads to solving the ordinary differential Eq.(5).

Let

$$u(\xi) = [\varphi(\xi)]^{1/p}, \tag{6}$$

then it follows from Eq.(5) that

$$\frac{1}{p}\varphi(\xi)\varphi''(\xi) + \frac{1-p}{p^2}\varphi'^2(\xi) + \frac{b_1 - v^2}{v^2}\varphi^2(\xi) + \frac{b_2}{v}\varphi^3(\xi) + \frac{b_3}{v}\varphi^4(\xi) = 0. \tag{7}$$

Now, we assume that the solution of Eq.(7) has the following form

$$\varphi(\xi) = \frac{A\mathrm{e}^{\alpha(\xi+\xi_0)}}{(1+\mathrm{e}^{\alpha(\xi+\xi_0)})^2 + B\mathrm{e}^{\alpha(\xi+\xi_0)}} = \frac{A\mathrm{sech}^2\frac{\alpha}{2}(\xi+\xi_0)}{4 + B\mathrm{sech}^2\frac{\alpha}{2}(\xi+\xi_0)}, \tag{8}$$

where A, B, α are constants to be determined. Substituting Eq.(8) into Eq.(7), we obtain a set of algebraic equations

$$\begin{cases} \dfrac{1}{p^2}\alpha^2 + \dfrac{b_1 - v^2}{v^2} = 0, \\ -\dfrac{1}{p}\alpha^2(2+B) + 2(2+B)\dfrac{b_1 - v^2}{v^2} + \dfrac{b_2}{v}A = 0, \\ -\dfrac{2(2p+1)}{p^2}\alpha^2 + [2 + (2+B)^2] \\ \times\dfrac{b_1 - v^2}{v^2} + \dfrac{b_2}{v}A(2+B) + \dfrac{b_3}{v}A^2 = 0. \end{cases} \tag{9}$$

Solving Eq.(9) leads to the following groups of solutions

$$\begin{cases} \alpha = \dfrac{p}{|v|}\sqrt{v^2 - b_1}, \\ A_{1,2} = \pm\dfrac{2(p+2)\sqrt{p+1}(v^2 - b_1)}{\sqrt{v[(p+1)\ b_2^2v - b_3(b_1 - v^2)(p+2)^2]}}, \\ B_{1,2} = -2 \pm \dfrac{2\sqrt{p+1}\ b_2v}{\sqrt{v[(p+1)b_2^2v - b_3(b_1 - v^2)(p+2)^2]}}. \end{cases} \tag{10}$$

Remark: The function sech x is even in the solution (8), which is invariant when $\alpha = \pm\dfrac{p}{|v|}\sqrt{v^2 - b_1}$. Hence, only $\alpha = \dfrac{p}{|v|}\sqrt{v^2 - b_1}$ is taken in Eq.(10).

Substituting Eq.(10) into Eq.(8) we have two solutions of Eq.(7)

$$\varphi_1(\xi) = \frac{\dfrac{(p+2)\sqrt{p+1}(v^2 - b_1)}{\sqrt{v[(p+1)b_2^2v - b_3(b_1 - v^2)(p+2)^2]}}\mathrm{sech}^2\dfrac{p}{2v}\sqrt{v^2 - b_1}(\xi+\xi_0)}{2 + \left(-1 + \dfrac{\sqrt{p+1}b_2v}{\sqrt{v[(p+1)b_2^2v - b_3(b_1 - v^2)(p+2)^2]}}\right)\mathrm{sech}^2\dfrac{p}{2v}\sqrt{v^2 - b_1}(\xi+\xi_0)}, \tag{11}$$

$$\varphi_2(\xi) = \frac{-\dfrac{(p+2)\sqrt{p+1}(v^2 - b_1)}{\sqrt{v[(p+1)b_2^2v - b_3(b_1 - v^2)(p+2)^2]}}\mathrm{sech}^2\dfrac{p}{2v}\sqrt{v^2 - b_1}(\xi+\xi_0)}{2 - \left(1 + \dfrac{\sqrt{p+1}b_2v}{\sqrt{v[(p+1)b_2^2v - b_3(b_1 - v^2)(p+2)^2]}}\right)\mathrm{sech}^2\dfrac{p}{2v}\sqrt{v^2 - b_1}(\xi+\xi_0)}. \tag{12}$$

It is easy to verify that

$v[(p+1)b_2^2v - b_3(b_1 - v^2)(p+2)^2] > 0$, if $v^2 - b_1 > 0$ and $b_3v \geq 0$;

$\varphi_1(\xi) > 0, \forall\xi \in R$, if $b_3v > 0$ or $b_3 = 0$ and $b_2v > 0$;

$\varphi_2(\xi) < 0, \forall\xi \in R$, if $b_3v > 0$ or $b_3 = 0$ and $b_2v < 0$.

From the above discussion, we obtain the following theorem.

Theorem 1. Let $v^2 - b_1 > 0$, then we can conclude that:

1. If $b_3v > 0$ or $b_3 = 0$ and $b_2v > 0$, then the generalized SRLWE (2) has a bell-type solitary wave solution

$$\begin{cases} u(x,t) = [\varphi_1(x - vt)]^{1/p}, \\ \rho(x,t) = \dfrac{1}{v}[\varphi_1(x - vt)]^{1/p} + c; \end{cases} \tag{13}$$

2. If p makes $(\mathrm{e})^{1/p}$ meaningful for any negative number e, and the condition $b_3v > 0$ or $b_3 = 0$ and $b_2v < 0$ holds, then the generalized SRLWE (2) has

another bell-type solitary wave solution

$$\begin{cases} u(x,t) = [\varphi_2(x-vt)]^{1/p}, \\ \rho(x,t) = \dfrac{1}{v}[\varphi_2(x-vt)]^{1/p} + c, \end{cases} \tag{14}$$

where $\varphi_1(\xi)$ and $\varphi_2(\xi)$ in (13), (14) are given by Eqs.(11) and (12), respectively.

Letting $b_3 = 0$ in Eqs.(13) and (14), we obtain the following proposition.

Proposition Suppose that p makes $(b_2 v)^{1/p}$ meaningful and $v^2 - b_1 > 0$, then the equation

$$\begin{cases} u_{xxt} - u_t = (b_1\rho + b_2 u^{p+1})_x, \\ \rho_t + u_x = 0, \end{cases} \tag{15}$$

has a bell-type solitary wave solution

$$\begin{cases} u(x,t) = \left[\dfrac{(p+2)(v^2-b_1)}{2b_2 v} \right. \\ \qquad \left. \times \operatorname{sech}^2 \dfrac{p}{2v}\sqrt{v^2-b_1}(x-vt+\xi_0)\right]^{1/p}, \\ \rho(x,t) = \dfrac{1}{v}\left[\dfrac{(p+2)(v^2-b_1)}{2b_2 v} \right. \\ \qquad \left. \times \operatorname{sech}^2 \dfrac{p}{2v}\sqrt{v^2-b_1}(x-vt+\xi_0)\right]^{1/p} + c. \end{cases} \tag{16}$$

Letting $p = 1$, $b_1 = 1$, $b_2 = \dfrac{1}{2}$ and $c = 0$ in Eq.(16), we gain a bell-type solitary wave solution of Eq.(1)

$$\begin{cases} u(x,t) = \dfrac{3(v^2-1)}{v}\operatorname{sech}^2 \dfrac{1}{2v}\sqrt{v^2-1}(x-vt+\xi_0), \\ \rho(x,t) = \dfrac{3(v^2-1)}{v^2}\operatorname{sech}^2 \dfrac{1}{2v}\sqrt{v^2-1}(x-vt+\xi_0), \end{cases} \tag{17}$$

which is the same as the solution given in Ref.[3].

To find the kink-type solitary wave solution of the generalized SRLWE (2), we suppose that the solution of Eq.(7) has the following form

$$\varphi(\xi) = \frac{A\mathrm{e}^{\alpha(\xi+\xi_0)}}{1+\mathrm{e}^{\alpha(\xi+\xi_0)}} = \frac{A}{2}\left[1+\tanh\frac{\alpha}{2}(\xi+\xi_0)\right]. \tag{18}$$

Substituting Eq.(18) into Eq.(7), we obtain a set of algebraic equations about A, α, v

$$\begin{cases} \alpha^2 + p^2\dfrac{b_1-v^2}{v^2} = 0, \\ -\alpha^2 + 2p\dfrac{b_1-v^2}{v^2} + pA\dfrac{b_2}{v} = 0, \\ \dfrac{b_3}{v}A^2 + \dfrac{b_2}{v}A + \dfrac{b_1-v^2}{v^2} = 0. \end{cases} \tag{19}$$

Solving Eq.(19), we obtain

$$\alpha = \pm\frac{p}{v}\sqrt{v^2-b_1}, \quad A = -\frac{(p+1)b_2}{(p+2)b_3}, \tag{20}$$

where the wave velocity v satisfies $(p+2)^2 b_3 v^2 + (p+1)b_2^2 v - (p+2)^2 b_3 b_1 = 0$. Hence, the following theorem is obtained.

Theorem 2. If p makes $(-b_2 b_3)^{1/p}$ meaningful, and the wave velocity v satisfies $(p+2)^2 b_3 v^2 + (p+1)b_2^2 v - (p+2)^2 b_3 b_1 = 0$ and $v^2 - b_1 > 0$, then the generalized SRLWE (2) has kink-type solitary wave solutions

$$\begin{cases} u(x,t) = \left\{ -\dfrac{(p+1)b_2}{2(p+2)b_3}\left[1 \pm \tanh\dfrac{p}{2v}\sqrt{v^2-b_1} \right. \right. \\ \qquad \left. \left. \times (x-vt+\xi_0)\right]\right\}^{1/p}, \\ \rho(x,t) = \dfrac{1}{v}\left\{ -\dfrac{(p+1)b_2}{2(p+2)b_3}\left[1 \pm \tanh\dfrac{p}{2v}\sqrt{v^2-b_1} \right. \right. \\ \qquad \left. \left. \times (x-vt+\xi_0)\right]\right\}^{1/p} + c. \end{cases} \tag{21}$$

3. The solitary wave solutions of the generalized SRLWE (3) with a dissipation term

Assume that $u(\xi)$ satisfies

$$u'(\xi) \to 0, \quad u''(\xi) \to 0, \quad |\xi| \to \infty, \tag{22a}$$

and the asymptotic values c_+, c_- of $u(\xi)$ ($c_\pm = \lim\limits_{\xi\to\pm\infty} u(\xi)$) satisfy the algebraic equation

$$\frac{b_1 - v^2}{v}x + b_2 x^{p+1} = 0, \tag{22b}$$

then the solitary wave solutions $\rho(\xi)$ and $u(\xi)$ of the generalized SRLWE (3) satisfy

$$\rho(\xi) = \frac{1}{v}u(\xi) + c,$$

and

$$u''(\xi) + \frac{r}{v}u'(\xi) + \frac{b_1-v^2}{v^2}u(\xi) + \frac{b_2}{v}u^{p+1}(\xi) = 0. \tag{23}$$

By using the transformation

$$u(\xi) = [\varphi(\xi)]^{1/p},$$

Eq. (23) can be rewritten as

$$\frac{1}{p}\varphi(\xi)\varphi''(\xi) + \frac{1-p}{p^2}\varphi'^2(\xi) + \frac{r}{pv}\varphi(\xi)\varphi'(\xi) + \frac{b_1-v^2}{v^2}\varphi^2(\xi) + \frac{b_2}{v}\varphi^3(\xi) = 0. \tag{24}$$

We can show that Eq.(24) does not possess the solution in the forms of (8) and (18). Now, we assume that the solution of Eq.(24) has the following form

$$\varphi(\xi)=\frac{A}{(1+e^{\alpha(\xi+\xi_0)})^2}, \tag{25}$$

where A,α,v are undetermined constants. Substituting Eq.(25) into Eq.(24), we obtain a set of algebraic equations

$$\begin{cases} \dfrac{b_2}{v}A+\dfrac{b_1-v^2}{v^2}=0, \\ -\dfrac{\alpha^2}{p}-\dfrac{r}{pv}\alpha+\dfrac{b_1-v^2}{v^2}=0, \\ \dfrac{4}{p^2}\alpha^2-\dfrac{2r}{pv}\alpha+\dfrac{b_1-v^2}{v^2}=0. \end{cases} \tag{26}$$

Solving Eq.(26) leads to the following solution

$$\alpha=\frac{pr}{(p+4)v},\quad A=-\frac{2(p+2)r^2}{(p+4)^2b_2v}, \tag{27}$$

where $v^2=b_1-\dfrac{2(p+2)}{(p+4)^2}r^2$.

Substituting Eq.(27) into Eq.(25), we obtain the following result.

Theorem 3. Let $b_1-\dfrac{2(p+2)r^2}{(p+4)^2}>0$, $v_1=\sqrt{b_1-\dfrac{2(p+2)r^2}{(p+4)^2}}$, $v_2=-\sqrt{b_1-\dfrac{2(p+2)r^2}{(p+4)^2}}$. If $v=v_1$ or $v=v_2$ makes $(-b_2v)^{1/p}$ meaningful, then the generalized SRLWE (3) has kink-type solitary wave solutions

$$u(x,t)=\left[\frac{-\dfrac{2(p+2)r^2}{(p+4)^2b_2v}}{(1+\mathrm{e}^{\frac{pr}{(p+4)v}(x-vt+\xi_0)})^2}\right]^{1/p}, \tag{28}$$

$$\rho(x,t)=\frac{1}{v}\left[\frac{-\dfrac{2(p+2)r^2}{(p+4)^2b_2v}}{(1+\mathrm{e}^{\frac{pr}{(p+4)v}(x-vt+\xi_0)})^2}\right]^{1/p}+c. \tag{29}$$

Since Eq.(25) can be rewritten as

$$\varphi(\xi)=\frac{A}{4}\left[2-2\tanh\frac{\alpha}{2}(\xi+\xi_0)-\operatorname{sech}^2\frac{\alpha}{2}(\xi+\xi_0)\right], \tag{25}'$$

Eqs.(28) and (29) can be rewritten respectively as

$$u(x,t)=\left\{-\frac{(p+2)r^2}{2(p+4)^2b_2v}\left[2-2\tanh\frac{pr(x-vt+\xi_0)}{2(p+4)v}-\operatorname{sech}^2\frac{pr(x-vt+\xi_0)}{2(p+4)v}\right]\right\}^{1/p}, \tag{28}'$$

$$\rho(x,t)=\frac{1}{v}\left\{-\frac{(p+2)r^2}{2(p+4)^2b_2v}\left[2-2\tanh\frac{pr(x-vt+\xi_0)}{2(p+4)v}-\operatorname{sech}^2\frac{pr(x-vt+\xi_0)}{2(p+4)v}\right]\right\}^{1/p}+c. \tag{29}'$$

It is easy to verify that the solutions (28)′ and (29)′ of Eq.(3) is different from the solutions (21) of Eq.(2).

Let $p=1$, $b_1=1$ and $b_2=\dfrac{1}{2}$ in (28)′ and (29)′, we find that the SRLWE with a dissipation term

$$\begin{cases} u_{xxt}-u_t=ru_{xx}+\left(\rho+\dfrac{1}{2}u^2\right)_x, \\ \rho_t+u_x=0, \end{cases} \tag{30}$$

has kink-type solitary wave solutions

$$\begin{cases} u(x,t)=-\dfrac{3r^2}{25v}\left[2-2\tanh\dfrac{r}{10v}(x-vt+\xi_0)\right. \\ \qquad\left.-\operatorname{sech}^2\dfrac{r}{10v}(x-vt+\xi_0)\right], \\ \rho(x,t)=-\dfrac{3r^2}{25v^2}\left[2-2\tanh\dfrac{r}{10v}(x-vt+\xi_0)\right. \\ \qquad\left.-\operatorname{sech}^2\dfrac{r}{10v}(x-vt+\xi_0)\right]+c, \end{cases}$$

where $v=\sqrt{1-\dfrac{6}{25}r^2}$ or $v=-\sqrt{1-\dfrac{6}{25}r^2}$.

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