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Cyclic integrals and reduction of rotational relativistic Birkhoffian system*

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The order reduction method of the rotational relativistic Birkhoffian equations is studied. For a rotational relativistic Birkhoffian system, the cyclic integrals can be found by using the perfect differential method. Through these cyclic integrals, the order of the system can be reduced. If the rotational relativistic Birkhoffian system has a cyclic integral, then the Birkhoffian equations can be reduced at least two degrees and the Birkhoffian form can be kept. An example is given to illustrate the application of the results.

Keywords: rotational relativity, Birkhoffian equation, cyclic integral, order reduction method
PACC: 0320, 0330, 0230

1. Introduction

In 1979, Bengtsson and Frauendorf accurately measured the maximum values of spin velocity of 14 kinds of nucleons, and the results showed that those maximum values were different from each other.[1]With the developments of science and technology, more and more experimental facts have been related to the problem of high speed spinning. In 1985, Carmeli constructed the theory of rotational relativistic mechanics.[2] Since 1996, we have developed the theory of analytical mechanics for the rotational relativistic system, and given its basic theoretical frame.[3−11] The Birkhoffian system is more general than the Hamiltonian system, and much attention is paid to it in the field of mathematics, mechanics and physics.[12−14] The Hamiltonian dynamics has been extensively applied to the field of modern physics, so the Birkhoffian dynamics should play an important role in the field of modern physics. Recently, Refs.[15]–[20] gave the basic theory, the theory of symmetry, the geometrical theory, the algebraic structure, the integral method and the relation between the form invariance and the Lie symmetries of the rotational relativistic Birkhoffian system.

The method of integration of dynamical equations of complicated system is an important aspect of the modern development of the field of mathematical mechanics and mathematical physics. In 1877, Routh initiated the order reduction method for a holonomic conservative system by using the cyclic integral.[21] After that, this method was always paid great attention by mathematicians, mechanics and physicists. Since 1988, the Routh-order reduction method has been generalized to the nonholonomic system, Vacco dynamics systems, the variable mass system, the noninertial reference frame and the Birkhoffian system in Refs.[22]–[27]. But, the studies were confined to the classical mechanics system.

In this paper, we try to apply and extend the Routh-order reduction method of the classical mechanics system to the rotational relativistic Birkhoffian system. The cyclic integral and its condition of existence for the rotational relativistic Birkhoffian system are given. By using the cyclic integral, the order of the rotational relativistic Birkhoffian equations can be reduced. Also, an example is given to illustrate the application of the result.

2.The cyclic integral of the rotational relativistic Birkhoffian system

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Here we will study the fixed axis rotation of a mechanical system of N particles. At time t, the angular velocity of the ith particle angular velocity is $\dot{\theta}_i$, the limiting is Γ_i, the classical rotational moment of inertia is I_{0i}, and its relativistic rotational moment of inertia is[2,3]

$$I_i = \frac{I_{0i}}{\sqrt{1-\dot{\theta}_i^2/\Gamma_i^2}}. \tag{1}$$

We can construct the Birkhoffian B^* and the Birkhoff functions $R_\nu^*(\nu=1,\cdots,2n)$ of a rotational relativistic system as

$$\begin{aligned} B^* &= B^*(I_i(t,a^\mu),t,a^\mu),\\ R_\nu^* &= R_\nu^*(I_i(t,a^\mu),t,a^\mu)\\ &(\mu,\nu=1,\cdots,2n), \end{aligned} \tag{2}$$

and let

$$\begin{aligned} \widetilde{B}^* &= \widetilde{B}^*(t,a^\mu) = B^*(I_i(t,a^\mu),t,a^\mu),\\ \widetilde{R}_\nu^* &= \widetilde{R}_\nu^*(t,a^\mu) = R_\nu^*(I_i(t,a^\mu),t,a^\mu). \end{aligned} \tag{3}$$

Then the Birkhoffian equations of rotational relativistic system can be written in the form[15]

$$\begin{aligned} &\left(\frac{\partial\widetilde{R}_\nu^*}{\partial a^\mu}-\frac{\partial\widetilde{R}_\mu^*}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial\widetilde{B}^*}{\partial a^\mu}-\frac{\partial\widetilde{R}_\mu^*}{\partial t}=0\\ &(\mu,\nu=1,\cdots,2n), \end{aligned} \tag{4}$$

where the repeated subscripts represent the summation. In general, it is supposed that the system (4) is non-singular, i.e.

$$\det(\omega_{\mu\nu}^*)\neq 0,\quad \omega_{\mu\nu}^*=\left(\frac{\partial\widetilde{R}_\nu^*}{\partial a^\mu}-\frac{\partial\widetilde{R}_\mu^*}{\partial a^\nu}\right). \tag{5}$$

If $\widetilde{R}_\mu^*$ and $\widetilde{B}^*$ do not include time t explicitly, the system (4) is autonomous. If $\widetilde{R}_\mu^*$ do not include time t explicitly, the system (4) is semi-autonomous.

Definition. If a coordinate a^k does not enter into the Birkhoffian $\widetilde{B}^*(t,a^\mu)$ and Birkhoff's functions $\widetilde{R}_\nu^*(t,a^\mu)$ of the rotational relativistic Birkhoffian system explicitly, i.e.

$$\begin{aligned} \frac{\partial\widetilde{B}^*}{\partial a^k}&=\frac{\partial B^*}{\partial I_i}\frac{\partial I_i}{\partial a^k}+\frac{\partial B^*}{\partial a^k}=0,\\ \frac{\partial\widetilde{R}_\nu^*}{\partial a^k}&=\frac{\partial R_\nu^*}{\partial I_i}\frac{\partial I_i}{\partial a^k}+\frac{\partial R_\nu^*}{\partial a^k}=0\\ &(\nu=1,\cdots,2n;\quad i=1,\cdots,N), \end{aligned} \tag{6}$$

then it is called a cyclic coordinate of the system.

Corollary 1. Let a^k be a cyclic coordinate, then $\widetilde{R}_k^*(t,a^\mu)=$ constant $(\mu=1,\cdots,2n;\quad \mu\neq k)$ is a first integral, called a cyclic integral of the rotational relativistic Birkhoffian system.

Proof: Since a^k is a cyclic coordinate, we have condition (6). So the kth equation of the system (4) takes the form

$$\begin{aligned} &-\frac{\partial\widetilde{R}_k^*}{\partial a^\nu}\dot{a}^\nu-\frac{\partial\widetilde{R}_k^*}{\partial t}=0\\ &(\nu=1,\cdots,2n;\quad \nu\neq k). \end{aligned} \tag{7}$$

That is, $\mathrm{d}\widetilde{R}_k^*/\mathrm{d}t$=0. Therefore,

$$\begin{aligned} &\widetilde{R}_k^*(t,a^\nu)=\text{constant}\\ &(\nu=1,\cdots,2n;\quad \nu\neq k). \end{aligned} \tag{8}$$

3. The order reduction method for the rotational relativistic Birkhoffian system

Using the cyclic integral, we now reduce the order of Eqs.(4).

Corollary 2. If a rotational relativistic Birkhoffian system has a cyclic integral, then the Birkhoffian equations can be reduced at least two degrees and keep the Birkhoffian form.

Proof: Let $\widetilde{R}_k^*(t,a^\nu)=$ constant $(\nu=1,\cdots 2n;\nu\neq k)$ be a cyclic integral of a rotational relativistic Birkhoffian system. Generally, assuming that $\partial\widetilde{R}_k^*/\partial a^\beta\neq 0$, then we can take a^β as a function of other variables

$$\begin{aligned} &a^\beta=M(t,a^\nu),\\ &(\nu=1,\cdots,2n;\quad \nu\neq k,\beta). \end{aligned} \tag{9}$$

Substituting Eq.(9) into Eq.(8), we obtain

$$\widetilde{R}_k^*(t,a^1,\cdots,a^{\beta-1},M(t,a^\nu),a^{\beta+1},\cdots,a^{2n})=h. \tag{10}$$

Differentiating the equation with respect to a^ν, we obtain

$$\frac{\partial M}{\partial a^\nu}=-\frac{\partial\widetilde{R}_k^*}{\partial a^\nu}\left(\frac{\partial\widetilde{R}_k^*}{\partial a^\beta}\right)^{-1}. \tag{11}$$

From the βth equation of rotational relativistic Birkhoffian system (4), we obtain

$$\begin{aligned} \dot{a}^k=&\left[\left(\frac{\partial\widetilde{B}^*}{\partial a^\beta}+\frac{\partial\widetilde{R}_\beta^*}{\partial a^\nu}\right)-\left(\frac{\partial\widetilde{R}_\nu^*}{\partial a^\beta}-\frac{\partial\widetilde{R}_\beta^*}{\partial a^\nu}\right)\dot{a}^\nu\right]\left(\frac{\partial\widetilde{R}_k^*}{\partial a^\beta}\right)^{-1}\\ &(\nu=1,\cdots,2n;\quad \nu\neq k). \end{aligned} \tag{12}$$

Substituting Eq.(12) into the remaining $(2n$–$2)$ equations of Eq.(4), and considering that

$$\dot{a}^\beta=\frac{\partial M}{\partial a^\nu}\dot{a}^\nu+\frac{\partial M}{\partial t}$$

$$(\nu=1,\cdots,2n;\quad \nu\neq k,\beta), \tag{13}$$

we have

$$\begin{aligned}
&\left[\left(\frac{\partial\widetilde{R}^*_\nu}{\partial a^\rho}+\frac{\partial\widetilde{R}^*_\nu}{\partial a^\beta}\frac{\partial M}{\partial a^\rho}\right)-\left(\frac{\partial\widetilde{R}^*_\rho}{\partial a^\nu}+\frac{\partial\widetilde{R}^*_\rho}{\partial a^\beta}\frac{\partial M}{\partial a^\nu}\right)\right.\\
&+\left(\frac{\partial\widetilde{R}^*_\beta}{\partial a^\rho}+\frac{\partial\widetilde{R}^*_\beta}{\partial a^\beta}\frac{\partial M}{\partial a^\rho}\right)\frac{\partial M}{\partial a^\nu}\\
&\left.-\left(\frac{\partial\widetilde{R}^*_\beta}{\partial a^\nu}+\frac{\partial\widetilde{R}^*_\beta}{\partial a^\beta}\frac{\partial M}{\partial a^\nu}\right)\frac{\partial M}{\partial a^\rho}\right]\dot{a}^\nu\\
&-\left(\frac{\partial\widetilde{B}^*}{\partial a^\rho}+\frac{\partial\widetilde{B}^*}{\partial a^\beta}\frac{\partial M}{\partial a^\rho}\right)-\left(\frac{\partial\widetilde{R}^*_\rho}{\partial t}+\frac{\partial\widetilde{R}^*_\rho}{\partial a^\beta}\frac{\partial M}{\partial t}\right)\\
&+\left(\frac{\partial\widetilde{R}^*_\beta}{\partial a^\rho}+\frac{\partial\widetilde{R}^*_\beta}{\partial a^\beta}\frac{\partial M}{\partial a^\rho}\right)\frac{\partial M}{\partial t}\\
&-\left(\frac{\partial\widetilde{R}^*_\beta}{\partial t}+\frac{\partial\widetilde{R}^*_\beta}{\partial a^\beta}\frac{\partial M}{\partial t}\right)\frac{\partial M}{\partial a^\rho}=0\\
&(\nu,\rho=1,\cdots,2n;\quad \nu,\rho\neq k,\beta).
\end{aligned}\tag{14}$$

Let

$$\begin{aligned}
&\widetilde{R}^*_\nu(t,a^1,\cdots,a^{\beta-1},M,a^{\beta+1},\cdots,a^{2n})\\
=&\widetilde{R}^{*'}_\nu(t,a^1,\cdots,a^{\beta-1},a^{\beta+1},\cdots,a^{2n}),\\
&\widetilde{B}^*(t,a^1,\cdots,a^{\beta-1},M,a^{\beta+1},\cdots,a^{2n})\\
=&\widetilde{B}^{*'}(t,a^1,\cdots,a^{\beta-1},a^{\beta+1},\cdots,a^{2n}),
\end{aligned}\tag{15}$$

we have

$$\begin{aligned}
&\left[\left(\frac{\partial\widetilde{R}^{*'}_\nu}{\partial a^\rho}+\frac{\partial\widetilde{R}^{*'}_\beta}{\partial a^\rho}\frac{\partial M}{\partial a^\nu}\right)-\left(\frac{\partial\widetilde{R}^{*'}_\rho}{\partial a^\nu}+\frac{\partial\widetilde{R}^{*'}_\beta}{\partial a^\nu}\frac{\partial M}{\partial a^\rho}\right)\right]\dot{a}^\nu\\
&-\left[\left(\frac{\partial\widetilde{B}^{*'}}{\partial a^\rho}-\frac{\partial\widetilde{R}^{*'}_\beta}{\partial a^\rho}\frac{\partial M}{\partial t}\right)+\left(\frac{\partial\widetilde{R}^{*'}_\rho}{\partial t}+\frac{\partial\widetilde{R}^{*'}_\beta}{\partial t}\frac{\partial M}{\partial a^\rho}\right)\right]=0\\
&(\nu=1,\cdots,2n;\quad \nu\neq k,\beta).
\end{aligned}\tag{16}$$

Let

$$\begin{aligned}
\frac{\partial\widetilde{\widetilde{R}}^*_\nu}{\partial a^\rho}&=\frac{\partial\widetilde{R}^{*'}_\nu}{\partial a^\rho}+\frac{\partial\widetilde{R}^{*'}_\beta}{\partial a^\rho}\frac{\partial M}{\partial a^\nu}\\
&(\nu,\rho=1,\cdots,2n;\quad \nu,\rho\neq k,\beta),\\
\frac{\partial\widetilde{\widetilde{B}}^*}{\partial a^\rho}&=\frac{\partial\widetilde{B}^{*'}}{\partial a^\rho}-\frac{\partial\widetilde{R}^{*'}_\beta}{\partial a^\rho}\frac{\partial M}{\partial t}\\
&(\rho=1,\cdots,2n;\quad \rho\neq k,\beta),
\end{aligned}\tag{17}$$

we obtain

$$\begin{aligned}
&\left(\frac{\partial\widetilde{\widetilde{R}}^*_\nu}{\partial a^\rho}-\frac{\partial\widetilde{\widetilde{R}}^*_\rho}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial\widetilde{\widetilde{B}}^*}{\partial a^\rho}-\frac{\partial\widetilde{\widetilde{R}}^*_\rho}{\partial t}=0\\
&(\nu,\rho=1,\cdots,2n;\quad \nu,\rho\neq k,\beta).
\end{aligned}\tag{18}$$

Through the above procedure, the Birkhoffian equations of the rotational relativistic system are reduced by two orders while the Birkhoffian form is still kept. After solving Eqs.(18), if Eqs.(8) and (12) are supplanted, we can obtain the solution of the original rotational relativistic Birkhoffian system(4).

4. Deductions

In this paper, if R^*_μ does not include time t, for rotational relativistic Birkhoffian system (4), let

$$\begin{aligned}
a^\mu&=\begin{cases}q^\mu & (\mu=1,\cdots,n);\\ p_{\mu-n} & (\mu=n+1,\cdots,2n),\end{cases}\\
\widetilde{R}^*_\mu&=\begin{cases}p_\mu & (\mu=1,\cdots,n);\\ 0 & (\mu=n+1,\cdots,2n),\end{cases}\\
\widetilde{B}^*&=H_r,
\end{aligned}\tag{19}$$

then the Birkhoffian tensors of the rotational relativistic system are

$$(\omega^*_{\mu\nu})=\begin{bmatrix}0_{n\times n} & +1_{n\times n}\\ -1_{n\times n} & 0_{n\times n}\end{bmatrix},\tag{20}$$

we can obtain immediately the canonical equations of the rotational relativistic Hamiltonian system

$$\dot{q}^s=\frac{\partial H_r}{\partial p_s},\quad \dot{p}_s=-\frac{\partial H_r}{\partial q^s}\quad (s=1,\cdots,n)\tag{21}$$

and the Lagrangian equations of rotational relativistic system

$$\frac{d}{dt}\frac{\partial L_r}{\partial\dot{q}^s}-\frac{\partial L_r}{\partial q^s}=0\quad (s=1,\cdots,n),\tag{22}$$

where

$$\begin{aligned}
H_r(t,q^s,p_s)&=p_s\dot{q}^s-L_r,\quad p_s=\frac{\partial L_r}{\partial\dot{q}_s},\\
L_r(t,q^s,\dot{q}^s)&=T^*_r-V,\\
T^*_r&=I_{0i}\Gamma_i^2(1-\sqrt{1-\dot{\theta}_i^2/\Gamma_i^2})\\
&(s=1,\cdots,n;\quad i=1,\cdots,N).
\end{aligned}\tag{23}$$

Therefore, the corollaries 1 and 2 give the cyclic integral and reduction of the rotational relativistic Hamiltonian system and the rotational relativistic Lagrangian system.

Under the classical approximation $\dot{\theta}_i\ll\Gamma_i$,

$$I_i=\frac{I_{0i}}{\sqrt{1-\dot{\theta}_i^2/\Gamma_i^2}}\approx I_{0i}\quad (i=1,\cdots,N),\tag{24}$$

$$\begin{aligned}
\widetilde{B}^*&=B^*((I_i(t,a^\mu),t,a^\mu),t,a^\mu)\approx B(t,a^\mu),\\
\widetilde{R}^*_\nu&=R^*_\nu((I_i(t,a^\mu),t,a^\mu),t,a^\mu)\approx R_\nu(t,a^\mu),
\end{aligned}\tag{25}$$

the corollaries 1 and 2 give cyclic integral and reduction of the classical rotational Birkhoffian system.

5. Example

As an example, we study a rotational relativistic Birkhoffian system, in which the Birkhoffian is

$$\widetilde{B}^* = I_{01}\Gamma_1^2(a^1)^2 + I_{02}\Gamma_2^2(a^2)^2 + I_{03}\Gamma_3^2(a^3)^2 - I_{01}\Gamma_1^2 - I_{02}\Gamma_2^2 - I_{03}\Gamma_3^2, \tag{26}$$

and Birkhoff's functions are

$$\begin{aligned} \widetilde{R}_1^* &= \widetilde{R}_2^* = 0, \\ \widetilde{R}_3^* &= 2I_{01}\Gamma_1^2 a^1, \\ \widetilde{R}_4^* &= 2I_{02}\Gamma_2^2 a^2, \end{aligned} \tag{27}$$

where

$$\begin{aligned} a^1 &= (1 - \dot{q}_1^2/\Gamma_1^2)^{-1/4}, \\ a^2 &= (1 - \dot{q}_2^2/\Gamma_2^2)^{-1/4}, \\ a^3 &= (1 - \dot{q}_3^2/\Gamma_3^2)^{-1/4}. \end{aligned} \tag{28}$$

The Birkhoffian equations are

$$\dot{a}^3 - a^1 = 0, \quad \dot{a}^4 - a^2 = 0,$$

$$I_{01}\Gamma_1^2\dot{a}^1 + I_{03}\Gamma_3^2 a^3 = 0,$$

$$\dot{a}^2 = 0. \tag{29}$$

Obviously, a^4 is a cyclic coordinate, and $\widetilde{R}_4^* = 2I_{02}\Gamma_2^2 a^2 = h$ = constant is a cyclic integral. So according to corollary 2, the rotational relativistic Birkhoffian system can be reduced two degrees. From the reduced equations

$$\left(\frac{\partial \widetilde{\widetilde{R}}_\nu^*}{\partial a^\rho} - \frac{\partial \widetilde{\widetilde{R}}_\rho^*}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial \widetilde{\widetilde{B}}^*}{\partial a^\rho} - \frac{\partial \widetilde{\widetilde{R}}_\rho^*}{\partial t} = 0 \quad (\rho, \nu = 1, 3), \tag{30}$$

we obtain

$$I_{01}\Gamma_1^2\dot{a}^1 + I_{03}\Gamma_3^2 a^3 = 0, \quad \dot{a}^3 - a^1 = 0. \tag{31}$$

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