Vol 12 No 2, February 2003
1009-1963/2003/12(02)/0136-04

Chinese Physics

© 2003 Chin. Phys. Soc.
and IOP Publishing Ltd

Bound states of Klein–Gordon equation for ring-shaped harmonic oscillator scalar and vector potentials

Qiang Wen-Chao(强稳朝)

Faculty of Science, Xi'an University of Architecture and Technology, Xi'an 710055, China

(Received 15 July 2002; revised manuscript received 14 October 2002)

Solving Klein–Gordon equation with equal ring-shaped harmonic oscillator scalar and vector potentials, we obtain the exact normalized bound-state wavefunction and energy equation.

Keywords: ring-shaped harmonic oscillator, Klein–Gordon equation, bound-states
PACC: 0300, 0365

1. Introduction

It is well known that, when a particle is in a strong potential field, the relativistic effect must be considered, which gives the correction for non-relativistic quantum mechanics. Except a few examples, such as hydrogen atom and electrons in a uniform magnetic field, perhaps because of the difficulties on mathematics, in relativistic quantum mechanics the problems that can be exactly solved for Klein–Gordon or Dirac equations are seldom. In recent years, some researchers assumed that the scalar potential is equal to vector potential, and solved the Klein–Gordon as well as Dirac equations, then obtained the s-wave bound-state solution of particles in some typical potential fields.[1−4] We also solved Klein–Gordon and Dirac equations for equal scalar and vector harmonic oscillator potentials and obtained a non s-wave solution, but the exact solution of a relativistic harmonic oscillator.[5] It is obvious that the studies mentioned above are only aiming at central potential fields. On the other hand, some authors solved the ring-shaped harmonic oscillator problem,[6,7] but it is for non-relativistic quantum mechanics. In this paper, we solve the Klein–Gordon equation with equal ring-shaped harmonic oscillator scalar and vector potentials, and give the completely normalized wavefunction and energy equation. The result shows that some non-central problems of relativistic quantum mechanics can also be solved exactly.

A ring-shaped harmonic oscillator means that a three-dimensional harmonic oscillator surrounded by a ring-shaped inverse square potential. In spherical coordinates, a ring-shaped harmonic oscillator potential is given by[8,9]

$$V(\boldsymbol{r}) = \frac{1}{2}kr^2 + \frac{b}{r^2\sin^2\theta}. \tag{1}$$

In order to solve the relativistic ring-shaped harmonic oscillator, we first separate variables in the Klein–Gordon equation for separable variable potentials. The Klein–Gordon equation with equal scalar and vector potentials is ($\hbar = c = 1$)[10]

$$\boldsymbol{p}^2 - (E - V(\boldsymbol{r})/2)^2 = -(\mu + V(\boldsymbol{r})/2)^2, \tag{2}$$

where $\boldsymbol{P}$ is the momentum operator, E and μ are the energy and rest mass of the harmonic oscillator respectively, k is the elastic coefficient. In spherical coordinates if

$$V(\boldsymbol{r}) = V(r) + \frac{V(\theta)}{r^2} + \frac{V(\phi)}{r^2\sin^2\theta}, \tag{3}$$

then separating variables in Eq.(2) gives

$$K''(\phi) = [(E+\mu)V(\phi) - m^2]K(\phi) = 0, \tag{4}$$

$$H''(\theta) + \cot\theta H(\theta) - [m^2\csc^2\theta + (E+\mu)V(\theta) - s(s+1)]H(\theta) = 0, \tag{5}$$

$$(r^2R'(r))' - [s(s+1) + (E+\mu)r^2V(r) - (E^2-\mu^2)r^2]R(r) = 0, \tag{6}$$

where m and s are separated variable constants. For ring-shaped harmonic oscillator potentials,

$$V(r)=\frac{1}{2}kr^2,\quad V(\phi)=0,\quad V(\theta)=b\csc^2\theta. \tag{7}$$

Eqs. (4)–(6) become respectively

$$K''(\phi)+m^2K(\phi)=0, \tag{8}$$

$$H''(\theta)+\cot\theta H'(\theta) - [(m^2+b(E+\mu))\csc^2\theta-s(s+1)]H(\theta)=0, \tag{9}$$

$$(r^2R'(r))'-\left[s(s+1)+\frac{1}{2}k(E+\mu)r^4 - (E^2-\mu^2)r^2\right]R(r)=0. \tag{10}$$

Eq. (8) has the solution satisfying periodic boundary conditions

$$K(\phi)=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}m\phi}\quad m=0,\pm1,\pm2,\cdots \tag{11}$$

In the next two sections we will solve Eqs.(9) and (10).

2. The solution of θ-dimensional equation

Letting

$$x=\sin^2\theta\qquad v=\sqrt{b(E+\mu)+m^2},$$

$$H(\theta)=\sin^v\theta f(x), \tag{12}$$

Eq.(9) becomes

$$(1-x)xf''(x)+[(1+v)-(3/2+v)x]f'(x) - \frac{1}{4}[v(v+1)-s(s+1)]f(x)=0, \tag{13}$$

which is a hypergeometric equation. Noticing that $v>0$, we obtain the solution of Eq.(13), that is not infinity at $\theta=0$ or π,

$$f(x)=\mathrm{F}[-(s-v)/2,(s+v+1)/2,1+v,x], \tag{14}$$

where $\mathrm{F}[\alpha,\beta,\gamma,x]$ is a hypergeometric function.[11] In order that $f(x)$ is finite in the range $[0,\ \ 1]$, $f(x)$ must terminate as a polynomial. This demands that $-(s-v)/2$ or $(s+v+1)/2$ be equal to zero or a negative integer. If

$$(s+v+1)/2=-l\quad l=0,1,2,\cdots,$$

then $s=-(2l+v+1)<0$. Next section will show that the radial wavefunction $R(r)$ includes a factor r^s, which does not converge at point r=0 when $s<0$. On the other hand, if

$$(s-v)/2=-l\qquad l=0,1,2,\cdots,$$

then $s=(2l+v)>0$, the factor r^s is convergent at point r=0, so the correct choice for values of s is

$$s=2l+v\qquad l=0,1,2,3,\cdots, \tag{15}$$

and the wavefunction of θ-dimension becomes

$$H(\theta)=\sin^v\theta\mathrm{F}[-l,l+v+1/2,1+v,\sin^2\theta]. \tag{16}$$

By means of a formula[11]

$$\{\mathrm{F}(\alpha,\beta,\alpha+\beta+1/2,z)\}^2 =\sum_{n=0}^{\infty}\frac{(2\alpha)_n(2\beta)_n(\alpha+\beta)_n}{n!((\alpha+\beta+1/2)_n(2\alpha+2\beta)_n}z^n, \tag{17}$$

where $(\lambda)_n$ is the Pochhammer symbol,[11] we have

$$\begin{aligned}&\int_0^\pi\sin\theta(H(\theta))^2\mathrm{d}\theta\\ =&\int_0^\pi(\sin\theta)^{2v+1}\langle\mathrm{F}[-l,l+v+1/2,1+v,\sin^2\theta]\rangle^2\mathrm{d}\theta\\ =&\int_0^\pi\sum_{n=0}^{2l+1}\frac{(-2l)_n(2l+2v+1)_n(v+1/2)_n}{n!(1+v)_n(1+2v)_n}\\ &\times(\sin\theta)^{2v+1}\mathrm{d}\theta\\ =&\sum_{n=0}^{2l+1}\frac{(-2l)_n(2l+2v+1)_n(v+1/2)_n\sqrt{\pi}\Gamma(n+v+1)}{n!(1+v)_n(1+2v)_n\Gamma(n+v+3/2)}\\ =&\frac{2^{1+2v}\Gamma(1+2l)(\Gamma(1+v))^2}{(1+4l+2v)\Gamma(1+2l+2v)},\end{aligned} \tag{18}$$

which gives the normalized wavefunction of θ-dimension

$$H_l(\theta)=C_l\sin^v\theta\mathrm{F}[-l,l+v+1/2,1+v,\sin^2\theta], \tag{19}$$

where the normalization constant

$$C_l=\frac{1}{2^{v+1/2}\Gamma(1+v)}\sqrt{\frac{(1+4l+2v)\Gamma(1+2l+2v)}{\Gamma(1+2l)}}. \tag{20}$$

3. The solution of r-dimensional equation

Letting

$$\beta=\sqrt{k(E+\mu)},\quad \rho=\frac{\beta}{\sqrt{2}}r^2, \tag{21}$$

$$R(r)=\mathrm{e}^{-\frac{\beta}{2\sqrt{2}}r^2}r^s f(\rho), \tag{22}$$

converts the r-dimensional equation into

$$\rho f''(\rho)+(3/2+s-\rho)f'(\rho) - \left[\frac{3+2s}{4}-\frac{(E-\mu)\beta}{2\sqrt{2}k}\right]f(\rho)=0, \quad (23)$$

which is a confluent hypergeometric equation and the solution of which is a confluent hypergeometric function

$$f(\rho)=\mathrm{F}\left[\frac{3+2s}{4}-\frac{(E-\mu)\beta}{2\sqrt{2}k},\frac{3}{2}+s,\rho\right]. \quad (24)$$

For bound states, the confluent hypergeometric function must terminate as a polynomial, which demands

$$\frac{3+2s}{4}-\frac{(E-\mu)\beta}{2\sqrt{2}k}=-n \quad n=0,1,2,\cdots \quad (25)$$

This condition, together with Eqs.(15) and (21), gives the energy equation of a relativistic ring-shaped harmonic oscillator

$$(3+4l+2\sqrt{b(E+\mu)+m^2})\sqrt{k} - (E-\mu)\sqrt{2(E+\mu)}=-4n\sqrt{k}. \quad (26)$$

Rationalizing Eq.(26) gives the equation including sixth power of energy E, which has no analytic solution for E.

Using the relationship of the confluent hypergeometric function with Laguerre polynomial and the normalization condition of Laguerre polynomial[11] as well as Eqs.(22) and (24), we obtain the normalized radial wavefunction

$$R_{n,l}(r)=C_{n,l}\mathrm{e}^{-\frac{\beta}{2\sqrt{2}}r^2}r^{2l+v}\mathrm{F}\left[-n,2l+v+\frac{3}{2},\frac{\beta}{\sqrt{2}}r^2\right], \quad (27)$$

where the normalization constant is

$$C_{n,l}=\frac{\beta^{l+\frac{v}{2}+\frac{3}{4}}}{2^{\frac{l}{2}+\frac{v}{4}-\frac{1}{8}}\Gamma(2l+v+3/2)}\sqrt{\frac{\Gamma(2l+v+n+3/2)}{n!}}. \quad (28)$$

The above result shows that the Klein–Gordon equation with equal scalar and vector potentials, both of which are equal to half of the ring-shaped harmonic oscillator potential, can be exactly solved. The state of a relativistic ring-shaped harmonic oscillator has been determined completely by three quantum numbers n, l, m. After giving n, l, m as well as constants μ, k and b, solving Eq.(26) gives the energy level values.

4. Discussion

It can be proved that we have $H(\theta)=\mathrm{P}_l^v(\cos\theta)$ in Eq.(16). In fact, letting

$$x=\cos\theta, \quad v=\sqrt{b(E+\mu)+m^2},$$

$$H(\theta)=f(x), \quad (29)$$

Eq.(9) becomes

$$(1-x^2)f''(x)-2xf'(x)+\left[s(s+1)-\frac{v^2}{1-x^2}\right]f(x)=0, \quad (30)$$

which is an associate Legendre equation, the solution of which is the associate Legendre function

$$f(x)=\mathrm{P}_s^v(x). \quad (31)$$

So we have

$$H_s(\theta)=\mathrm{P}_s^v(\cos\theta). \quad (32)$$

As soon as b=0, the ring-shaped harmonic oscillator returns to an ordinary harmonic oscillator and

$$v=|m|=0,1,2,\cdots,$$

$$s=2l+v=2l+|m|=0,1,2,\cdots, \quad (33)$$

$$H_s(\theta)=\mathrm{P}_s^v(\cos\theta)=\mathrm{P}_s^{|m|}(\cos\theta). \quad (34)$$

Eq. (34) is just the wavefunction in θ-dimension of a non-relativistic or relativistic particle in a central field. On the other hand, because of $v=|m|=0,1,2,\cdots$ when b=0, Eqs.(26)–(28) becomes respectively

$$(3+2s)\sqrt{k}-(E-\mu)\sqrt{2(E+\mu)}=-4n\sqrt{k}, \quad (35)$$

$$R_{n,s}(r)=C_{n,s}\mathrm{e}^{-\frac{\beta}{2\sqrt{2}}r^2}r^s\mathrm{F}\left[-n,s+\frac{3}{2},\frac{\beta}{\sqrt{2}}r^2\right], \quad (36)$$

$$C_{n,s}=\frac{\beta^{\frac{s}{2}+\frac{3}{4}}}{2^{\frac{s}{4}-\frac{1}{8}}\Gamma(s+3/2)}\sqrt{\frac{\Gamma(s+n+3/2)}{n!}}. \quad (37)$$

In the above three equations, $s=0,1,2,\cdots$ and $n=0,1,2,\cdots$. Eq.(35) is no other than the energy equation of a harmonic oscillator in the Klein–Gordon theory, but Eqs.(36) and (37) are the corresponding normalized wavefunction.[5]

In summary we have the Klein–Gordon equation for a ring-shaped harmonic oscillator, which gives the exact wavefunction and energy equation. The research of the ring-shaped harmonic oscillator in Dirac theory is underway.

References

[1] Hu S Z and Su R K 1991 *Acta Phys. Sin.* **40** 1201 (in Chinese)

[2] Hou C F, Li Y and Zhou Z X 1999 *Acta Phys. Sin.* **48 385** (in Chinese)

[3] Hou C F and Zhou Z X 1999 *Acta Phys. Sin.* (Overseas Edition) **8** 561

[4] Chen G 2001 *Acta Phys. Sin.* **50** 1651 (in Chinese)

[5] Qiang W C 2002 *Chin. Phys.* **11** 757

[6] Chen C Y, Sun D S and Liu Y W 2002 *Acta Phys. Sin.* **51** 468 (in Chinese)

[7] Qiang S W, Huang B W, Wang D Y and Gu Z Y 1999 *Acta Scientiarum Naturalium Universitatis Pekinsis* **36** 346

[8] Quesne C A 1988 *J. Phys. A: Math. Gen.* **21** 3093

[9] Victoria M 1989 *Phys. Lett.* A **134** 395

[10] Wu T Y and Hwang P W Y 1991 *Relativistic Quantum Mechanics and Quantum Fields* (Singapore: World Scientific) p 20

[11] Wang Z X and Guo D R 2000 *Introduction to Special Function* (Beijing: Peking University Press) (in Chinese) p203