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New periodic solutions to a generalized Hirota–Satsuma coupled KdV system*

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Using expansions in terms of the Jacobi elliptic cosine function and third Jacobi elliptic function, some new periodic solutions to the generalized Hirota–Satsuma coupled KdV system are obtained with the help of the algorithm Mathematica. These periodic solutions are also reduced to the bell-shaped solitary wave solutions and kink-shape solitary solutions. As special cases, we obtain new periodic solution, bell-shaped and kink-shaped solitary solutions to the well-known Hirota–Satsuma equations.

Keywords: periodic solution, Hirota–Satsuma coupled KdV system, Jacobi elliptic function
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1. Introduction

Since the exact solutions to nonlinear wave equations help to understand the characteristics of nonlinear equations, seeking for analytical solutions of nonlinear equations plays an important role in studying nonlinear problems. Recently, new methods for obtaining exact solutions to nonlinear wave equations were initiated, such as the homogeneous balance method, nonlinear transformation method, sine-cosine method and so on.[1,2] These methods can give solitary wave solutions and a few periodic solutions. Some methods for periodic solutions of nonlinear wave equations were proposed in Refs.[3]–[12]. We have also obtained the solitary-wave solutions for a generalized Hirota–Satsuma coupled system. In this paper, we use Jacobi elliptic function expansion method to obtain new periodic solutions for a generalized Hirota–Satsuma coupled KdV system and the well-known Hirota–Satsuma equation, including Jacobi elliptic cosine function solutions and the third kind of Jacobi elliptic function solutions.

2. New periodic solutions

The nonlinear evolution equation

$$F(u, u_t, u_x, u_{tt}, u_{xx}, \cdots) = 0 \tag{1}$$

has the travelling wave solution of the form

$$u = u(\xi), \quad \xi = k(x - ct), \tag{2}$$

where k and c are the wavenumber and wave velocity, respectively. Our idea is that substitution of (2) into (1) yields an ordinary differential equation which is solved by the use of Jacobi elliptic cosine function $\mathrm{cn}\xi$ and the third Jacobi elliptic function $\mathrm{dn}\xi$ expansion. New exact periodic solutions to (1) are obtained.

Consider a generalized Hirota-satsuma coupled KdV system

$$\begin{cases} u_t = \dfrac{1}{4}u_{xxx} + 3uu_x + 3(-v^2 + w)_x, \\ v_t = -\dfrac{1}{2}v_{xxx} - 3uv_x, \\ w_t = -\dfrac{1}{2}w_{xxx} - 3uw_x. \end{cases} \tag{3}$$

If let $w = 0, x \to \sqrt{2}x, t \to \sqrt{2}t$ in (3), the well-known Hirota–Satsuma equation is showed as follows

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$$\begin{cases} u_t = \frac{1}{2}(u_{xxx} + 6uu_x) - 6vv_x, \\ v_t = -v_{xxx} - 3uv_x. \end{cases} \tag{4}$$

First, we seek for exact periodic solutions of (3), which are the forms of expansion in cnξ and dnξ. Then as a special case of (3), new exact periodic solutions of (4) appear.

Let

$$\begin{aligned} &u(x,t) = U(\xi), \quad v(x,t) = V(\xi), \\ &w(x,t) = W(\xi), \quad \xi = k(x - ct). \end{aligned} \tag{5}$$

Substituting (5) into (3) yields the following ordinary differential system

$$\begin{cases} -cU' = \frac{1}{4}k^2U''' + 3UU' + 3(-V^2 + W)', \\ cV' = \frac{1}{2}k^2V''' + 3UV', \\ cW' = \frac{1}{2}k^2W''' + 3UW'. \end{cases} \tag{6}$$

Here " $'$ " denots the derivative with respect to ξ.

2.1. Periodic solutions in terms of Jacobi elliptic cosine functions

Taking the form solutions of (6) as follows

$$\begin{aligned} U(\xi) &= \sum_{j=0}^{n} a_j \mathrm{cn}^j\xi, \\ V(\xi) &= \sum_{j=0}^{n} b_j \mathrm{cn}^j\xi, \\ W(\xi) &= \sum_{j=0}^{n} c_j \mathrm{cn}^j\xi. \end{aligned}$$

Balancing both nonlinear term and the term of derivatives of higher order, we find that $n = 2$. Therefore,

$$\begin{cases} U = a_0 + a_1\mathrm{cn}\xi + a_2\mathrm{cn}^2\xi, \\ V = b_0 + b_1\mathrm{cn}\xi + b_2\mathrm{cn}^2\xi, \\ W = c_0 + c_1\mathrm{cn}\xi + c_2\mathrm{cn}^2\xi. \end{cases} \tag{7}$$

It is easy to find that

$$U' = -(a_1 + 2a_2\mathrm{cn}\xi)\mathrm{sn}\xi\mathrm{dn}\xi,$$

$$\begin{aligned} U'' =& 2(1 - m^2)a_2 + (2m^2 - 1)a_1\mathrm{cn}\xi \\ &+ 4(2m^2 - 1)a_2\mathrm{cn}^2\xi - 2m^2a_1\mathrm{cn}^3\xi - 6m^2a_2\mathrm{cn}^4\xi, \\ U''' =& [(1 - 2m^2)a_1 + 8(1 - 2m^2)a_2\mathrm{cn}\xi \\ &+ 6m^2a_1\mathrm{cn}^2\xi + 24m^2a_2\mathrm{cn}^3\xi]\mathrm{sn}\xi\mathrm{dn}\xi, \end{aligned}$$

where m is a model number, and $0 < m < 1$. The derivatives of V and W are similar to those of U. Substituting (7) into (6) gives

$$\begin{aligned} &ca_1\mathrm{sn}\xi\mathrm{dn}\xi + 2ca_2\mathrm{cn}\xi\mathrm{sn}\xi\mathrm{dn}\xi \\ =&\frac{1}{4}k^2a_1(1 + 4m^2)\mathrm{sn}\xi\mathrm{dn}\xi - \frac{3}{2}k^2a_1m^2\mathrm{sn}^3\xi\mathrm{dn}\xi \\ &+ 4k^2a_2m^2\mathrm{sn}\xi\mathrm{cn}^3\xi\mathrm{dn}\xi - 2k^2a_2m^2(1 - \mathrm{cn}^2\xi)\mathrm{cn}\xi\mathrm{sn}\xi\mathrm{dn}\xi + 2k^2a_2(1 - m^2)\mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}\xi \\ &+ 3k[-a_0a_1\mathrm{sn}\xi\mathrm{dn}\xi - 2a_0a_2\mathrm{cn}\xi\mathrm{sn}\xi\mathrm{dn}\xi - a_1^2\mathrm{cn}\xi\mathrm{sn}\xi\mathrm{dn}\xi - 3a_1a_2(1 - \mathrm{sn}^2\xi)\mathrm{sn}\xi\mathrm{dn}\xi \\ &- 2a_2^2\mathrm{cn}^3\xi\mathrm{sn}\xi\mathrm{dn}\xi] + 3(2b_0b_1\mathrm{sn}\xi\mathrm{dn}\xi + 4b_0b_2\mathrm{cn}\xi\mathrm{sn}\xi\mathrm{dn}\xi + 2b_1^2\mathrm{cn}\xi\mathrm{sn}\xi\mathrm{dn}\xi + 4b_1b_2\mathrm{cn}^2\xi\mathrm{sn}\xi\mathrm{dn}\xi \\ &+ 2b_1b_2\mathrm{cn}^2\xi\mathrm{sn}\xi\mathrm{dn}\xi + 4b_2^2\mathrm{cn}^3\xi\mathrm{sn}\xi\mathrm{dn}\xi - c_1\mathrm{sn}\xi\mathrm{dn}\xi - 2c_2\mathrm{cn}\xi\mathrm{sn}\xi\mathrm{dn}\xi), \end{aligned} \tag{8}$$

$$\begin{aligned} &cb_1\mathrm{sn}\xi\mathrm{dn}\xi - 2cb_2\mathrm{cn}\xi\mathrm{sn}\xi\mathrm{dn}\xi \\ =&\frac{1}{2}k^2b_1(1 + 4m^2)\mathrm{sn}\xi\mathrm{dn}\xi - 3k^2b_1m^2\mathrm{sn}^3\xi\mathrm{dn}\xi \\ &+ 8k^2b_2m^2\mathrm{sn}\xi\mathrm{cn}^3\xi\mathrm{dn}\xi - 4k^2b_2m^2\mathrm{sn}^3\xi\mathrm{cn}\xi\mathrm{dn}\xi + 4k^2b_2(1 - m^2)\mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}\xi \\ &+ 3(-a_0b_1\mathrm{sn}\xi\mathrm{dn}\xi - 2a_0b_2\mathrm{cn}\xi\mathrm{sn}\xi\mathrm{dn}\xi - a_1b_1\mathrm{cn}\xi\mathrm{sn}\xi\mathrm{dn}\xi - 2a_1b_2\mathrm{cn}^2\xi\mathrm{sn}\xi\mathrm{dn}\xi \\ &- a_2b_1\mathrm{cn}^2\xi\mathrm{sn}\xi\mathrm{dn}\xi - 2a_2b_2\mathrm{cn}^3\xi\mathrm{sn}\xi\mathrm{dn}\xi), \end{aligned} \tag{9}$$

$$\begin{aligned} &c(-c_1\mathrm{sn}\xi\mathrm{dn}\xi - 2c_2\mathrm{cn}\xi\mathrm{sn}\xi\mathrm{dn}\xi) \\ =&\frac{1}{2}k^2[c_1(1 + 4m^2)\mathrm{sn}\xi\mathrm{dn}\xi - 6c_1m^2\mathrm{sn}^3\xi\mathrm{dn}\xi \\ &+ 16c_2m^2\mathrm{sn}\xi\mathrm{cn}^3\xi\mathrm{dn}\xi - 8c_2m^2\mathrm{sn}^3\xi\mathrm{cn}\xi\mathrm{dn}\xi + 8c_2(1 - m^2)\mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}\xi] \\ &+ 3(a_0 + a_1\mathrm{cn}\xi + a_2\mathrm{cn}^2\xi)(-c_1\mathrm{sn}\xi\mathrm{dn}\xi - 2c_2\mathrm{cn}\xi\mathrm{sn}\xi\mathrm{dn}\xi). \end{aligned} \tag{10}$$

From (8)–(10), we have

$$a_1c-\frac{1}{4}k^2a_1(1+4m^2)+3a_0a_1+9a_1a_2-6b_0b_1+3c_1-18b_1b_2=0,$$

$$2a_2c+2k^2a_2m^2-2k^2a_2(1-m^2)+6a_0a_2+3a_1^2-12b_0b_2-6b_1^2+6c_2=0,$$

$$-\frac{3}{2}k^2a_1m^2+9a_1a_2-18b_1b_2=0,\quad 6k^2a_2m^2-6a_2^2+12b_2^2=0,$$

$$b_1c+\frac{1}{2}k^2b_1(1+4m^2)-3a_0b_1-6a_1b_2-3a_2b_1=0,\quad -4k^2b_2m^2=0,$$

$$2b_2c+4k^2b_2(1-m^2)-6a_0b_2-3a_1b_1=0,\quad 8k^2b_2m^2-6a_2b_2=0,$$

$$-3k^2b_1m^2+6a_1b_2+3a_2b_1=0,$$

$$cc_1+\frac{1}{2}k^2c_1(1+4m^2)-3a_0c_1-6a_1c_2-3a_2c_1=0,$$

$$2cc_2+4k^2c_2(1-m^2)-6a_0c_2-3a_1c_1=0,\quad -3k^2c_1m^2+6a_1c_2+3a_2c_1=0,$$

$$8k^2c_2m^2-6a_2c_2=0,\quad -4k^2c_2m^2=0,$$

from which we obtain, using Mathematica

$$a_1=c_2=b_2=0,\quad a_2=k^2m^2,\quad a_0=\frac{1}{6}(2c+k^2-2k^2m^2),\quad c_1=2b_0b_1,$$

$$b_1=\pm km\sqrt{\frac{1}{6}(2c+2kc-2k^2+k^3+4k^2m^2-2k^3m^2)}.$$

Thus, we obtain new periodic solutions in the form of Jacobi elliptic cosine function

$$\begin{aligned}u(x,t)=&\frac{1}{6}(2c+k^2-2k^2m^2)+k^2m^2\mathrm{cn}^2k(x-ct),\\ v(x,t)=&b_0\pm km\sqrt{\frac{1}{6}(2c+2kc-2k^2+k^3+4k^2m^2-2k^3m^2)}\mathrm{cn}k(x-ct),\\ w(x,t)=&c_0\pm 2b_0km\sqrt{\frac{1}{6}(2c+2kc-2k^2+k^3+4k^2m^2-2k^3m^2)}\mathrm{cn}k(x-ct),\end{aligned}\tag{11}$$

where b_0, c_0 are two arbitrary parameters.

In particular, we find new periodic solutions for the well-known Hirota–Satsuma equation(4)

$$\begin{aligned}u(x,t)=&\frac{1}{6}(2c+k^2-2k^2m^2)+k^2m^2\mathrm{cn}^2\sqrt{2}k(x-ct),\\ v(x,t)=&b_0\pm km\sqrt{\frac{1}{6}(2c+2kc-2k^2+k^3+4k^2m^2-2k^3m^2)}\mathrm{cn}\sqrt{2}k(x-ct).\end{aligned}\tag{12}$$

Since when $m\to 1, \mathrm{cn}\xi\to\mathrm{sech}\xi$, thus as far as (11) is concerned, we obtain the solitary wave solution with bell-shape for $u(x,t)$ but kink-shape for $v(x,t)$ and $w(x,t)$

$$\begin{aligned}u(x,t)=&\frac{1}{6}(2c+k^2-2k^2m^2)+k^2m^2\mathrm{sech}^2k(x-ct),\\ v(x,t)=&b_0\pm km\sqrt{\frac{1}{6}(2c+2kc-2k^2+k^3+4k^2m^2-2k^3m^2)}\mathrm{sech}k(x-ct),\\ w(x,t)=&c_0\pm 2b_0km\sqrt{\frac{1}{6}(2c+2kc-2k^2+k^3+4k^2m^2-2k^3m^2)}\mathrm{sech}k(x-ct).\end{aligned}$$

Similarly, we have solitary-wave solution with bell-shape forms for $u(x,t)$ but kink-shape forms for $v(x,t)$ to the Hirota–Satsuma (4) in terms of (12)

$$u(x,t)=\frac{1}{6}(2c+k^2-2k^2m^2)+k^2m^2\mathrm{sech}^2\sqrt{2}k(x-ct),$$

$$v(x,t)=b_0\pm km\sqrt{\frac{1}{6}(2c+2kc-2k^2+k^3+4k^2m^2-2k^3m^2)}\mathrm{sech}\sqrt{2}k(x-ct).$$

2.2. Periodic solutions in terms of the third Jacobi elliptic function

We at once give the solutions to (3) by the use of homogeneous balance method in the form

$$\begin{cases} U = a_0 + a_1 \mathrm{dn}\xi + a_2 \mathrm{dn}^2\xi, \\ V = b_0 + b_1 \mathrm{dn}\xi + b_2 \mathrm{dn}^2\xi, \\ W = c_0 + c_1 \mathrm{dn}\xi + c_2 \mathrm{dn}^2\xi. \end{cases} \tag{13}$$

It is easy to find that

$$U' = -m^2 a_1 \mathrm{sn}\xi\mathrm{cn}\xi - 2m^2 a_2 \mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}\xi,$$

$$\begin{aligned} U'' =& m^2 a_1 \mathrm{dn}\xi + 2m^2 a_2 \mathrm{dn}^2\xi - 2m^2 a_1 \mathrm{cn}^2\xi\mathrm{dn}\xi \\ & - 4m^2 a_2 \mathrm{cn}^2\xi + 6m^4 a_2 \mathrm{cn}^2\xi\mathrm{sn}^2\xi, \\ U''' =& 4m^2 a_1 \mathrm{cn}\xi\mathrm{sn}\xi + 8m^2 a_2 \mathrm{cn}\xi\mathrm{sn}\xi\mathrm{dn}\xi \\ & - 4m^4 a_1 \mathrm{cn}\xi\mathrm{sn}^3\xi - 12m^4 a_2 \mathrm{cn}\xi\mathrm{sn}^3\xi\mathrm{dn}\xi \\ & + 2m^4 a_1 \mathrm{sn}\xi\mathrm{cn}^3\xi + 12m^4 a_2 \mathrm{cn}^3\xi\mathrm{sn}\xi\mathrm{dn}\xi \\ & - m^4 a_1 \mathrm{sn}\xi\mathrm{cn}\xi - 4m^4 a_2 \mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}\xi. \end{aligned}$$

The derivatives of V and W are similar to those of U. Substituting (13) into (6) yields

$$\begin{aligned} & - c(-m^2 a_1 \mathrm{sn}\xi\mathrm{cn}\xi - 2m^2 a_2 \mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}\xi) \\ =& \frac{1}{4}k^2(4m^2 a_1 \mathrm{cn}\xi\mathrm{sn}\xi + 8m^2 a_2 \mathrm{cn}\xi\mathrm{sn}\xi\mathrm{dn}\xi \\ & - 4m^4 a_1 \mathrm{cn}\xi\mathrm{sn}^3\xi - 12m^4 a_2 \mathrm{cn}\xi\mathrm{sn}^3\xi\mathrm{dn}\xi + 2m^4 a_1 \mathrm{sn}\xi\mathrm{cn}^3\xi + 12m^4 a_2 \mathrm{cn}^3\xi\mathrm{sn}\xi\mathrm{dn}\xi \\ & - m^4 a_1 \mathrm{sn}\xi\mathrm{cn}\xi - 4m^4 a_2 \mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}\xi) - 3m^2 a_0 a_1 \mathrm{sn}\xi\mathrm{cn}\xi + (-6m^2 a_0 a_2 - 3m^2 a_1^2)\mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}\xi \\ & - 9m^2 a_1 a_2 \mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}^2\xi - 6m^2 a_2^2 \mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}^3\xi + (6m^2 b_0 b_1 - m^2 c_1)\mathrm{sn}\xi\mathrm{cn}\xi + (6b_0 b_2 m^2 \\ & + 3m^2 b_1^2 - 2m^2 c_2)\mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}\xi + 9b_1 b_2 m^2 \mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}^2\xi + 6b_2^2 m^2 \mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}^3\xi, \end{aligned} \tag{14}$$

$$\begin{aligned} & c(-m^2 b_1 \mathrm{sn}\xi\mathrm{cn}\xi - 2m^2 b_2 \mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}\xi) \\ =& \frac{1}{2}k^2(4m^2 b_1 \mathrm{cn}\xi\mathrm{sn}\xi + 8m^2 b_2 \mathrm{cn}\xi\mathrm{sn}\xi\mathrm{dn}\xi \\ & - 4m^4 b_1 \mathrm{cn}\xi\mathrm{sn}^3\xi - 12m^4 b_2 \mathrm{cn}\xi\mathrm{sn}^3\xi\mathrm{dn}\xi + 2m^4 b_1 \mathrm{sn}\xi\mathrm{cn}^3\xi + 12m^4 b_2 \mathrm{cn}^3\xi\mathrm{sn}\xi\mathrm{dn}\xi \\ & - m^4 b_1 \mathrm{sn}\xi\mathrm{cn}\xi - 4m^4 b_2 \mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}\xi) - 3m^2 a_0 b_1 \mathrm{sn}\xi\mathrm{cn}\xi + (-6m^2 a_0 b_2 - 3m^2 a_1 b_1)\mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}\xi \\ & + (-6m^2 a_1 b_2 - 3m^2 a_2 b_1)\mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}^2\xi - 6m^2 a_2 b_2 \mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}^3\xi, \end{aligned} \tag{15}$$

$$\begin{aligned} & c(-m^2 c_1 \mathrm{sn}\xi\mathrm{cn}\xi - 2m^2 c_2 \mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}\xi) \\ =& \frac{1}{2}k^2(4m^2 c_1 \mathrm{sn}\xi\mathrm{cn}\xi + 8m^2 c_2 \mathrm{cn}\xi\mathrm{sn}\xi\mathrm{dn}\xi \\ & - 4m^4 c_1 \mathrm{cn}\xi\mathrm{sn}^3\xi - 12m^4 c_2 \mathrm{cn}\xi\mathrm{sn}^3\xi\mathrm{dn}\xi + 2m^4 c_1 \mathrm{sn}\xi\mathrm{cn}^3\xi + 12m^4 c_2 \mathrm{cn}^3\xi\mathrm{sn}\xi\mathrm{dn}\xi \\ & - m^4 c_1 \mathrm{sn}\xi\mathrm{cn}\xi - 4m^4 c_2 \mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}\xi) - 3m^2 a_0 c_1 \mathrm{sn}\xi\mathrm{cn}\xi + (-6m^2 a_0 c_2 - 3m^2 a_1 c_1)\mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}\xi \\ & + (-6m^2 a_1 c_2 - 3m^2 a_2 c_1)\mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}^2\xi - 6m^2 a_2 c_2 \mathrm{sn}\xi\mathrm{cn}\xi\mathrm{dn}^3\xi. \end{aligned} \tag{16}$$

In terms of (14)–(16), we find that

$$\begin{aligned} & - m^2 c a_1 + k^2 m^2 a_1 - 3m^2 a_0 a_1 + 6m^2 b_0 b_1 \\ & - m^2 c_1 - \frac{1}{4}k^2 m^4 a_1 + 9m^2(b_1 b_2 - a_1 a_2) = 0, \end{aligned}$$

$$\begin{aligned} & - 2m^2 c a_2 + 2k^2 m^2 a_2 - 6m^2 a_0 a_2 - 3m^2 a_1^2 \\ & + 6b_0 b_2 m^2 + 3m^2 b_1^2 - 2m^2 c_2 - k^2 m^4 a_2 \\ & + 6m^2(b_2^2 - a_2^2) + 3m^4 a_2 = 0, \end{aligned}$$

$$-k^2 m^4 a_1 + 9m^4(a_1 a_2 - b_1 b_2) = 0, \quad 2m^4 a_1 = 0,$$

$$-3k^2 m^4 a_2 - 6m^4(b_2^2 - a_2^2) - 3m^4 a_2 = 0,$$

$$\begin{aligned} & m^2 c b_1 + 2k^2 m^2 b_1 - \frac{1}{2}k^2 m^4 b_1 \\ & - 3m^2 a_0 b_1 - 6m^2 a_1 b_2 - 3m^2 a_2 b_1 = 0, \end{aligned}$$

$$-2k^2 m^4 b_1 + 6m^4 a_1 b_2 + 3m^4 a_2 b_1 = 0,$$

$$\begin{aligned} & 2m^2 c b_2 + 4k^2 m^2 b_2 - 2k^2 m^4 b_2 - 6m^2 a_0 b_2 \\ & - 3m^2 a_1 b_2 - 6m^2 a_2 b_2 + 6k^2 m^4 b_2 = 0, \end{aligned}$$

$$-6k^2 m^4 b_2 + 6m^4 a_2 b_2 - 6k^2 m^4 b_2 = 0, \quad 2m^4 b_1 = 0,$$

$$m^2cc_1 + 2k^2m^2c_1 - \frac{1}{2}k^2m^4c_1$$
$$- 3m^2a_0c_1 - 6m^2a_1c_2 - 3m^2a_2c_1 = 0,$$
$$-2k^2m^4c_1 + 6m^4a_1c_2 + 3m^4a_2c_1 = 0,$$
$$2m^2cc_2 + 4k^2m^2c_2 - 2k^2m^4c_2$$
$$- 6m^2a_0c_2 - 3m^2a_1c_2 - 6m^2a_2c_2 = 0,$$
$$-6k^2m^4c_2 + 6m^4a_2c_2 - 6k^2m^4c_2 = 0, \quad 2m^4c_1 = 0,$$

solving the above equations via Mathematica yields

$$a_1 = b_1 = c_1 = 0, \quad a_2 = -k^2, \quad b_2 = \mp k^2,$$
$$a_0 = \frac{1}{3}(c + 8k^2 - k^2m^2),$$
$$c_2 = \frac{k^2}{2}(18k^2 - k^2m^2 \mp 6b_0).$$

Thus, we obtain new periodic solutions to the generalized Hirota–Satsuma coupled KdV system (3) in the form of the third Jacobi elliptic function

$$u(x,t) = \frac{1}{3}(c + 8k^2 - k^2m^2) - k^2\mathrm{dn}^2k(x - ct),$$
$$v(x,t) = b_0 \mp k^2\mathrm{dn}^2k(x - ct),$$
$$w(x,t) = c_0 + \frac{k^2}{2}(18k^2 - k^2m^2 \mp 6b_0)\mathrm{dn}^2k(x - ct),$$

where b_0, c_0 are arbitrary parameters.
Since when $m \to 1, \mathrm{dn}\xi \to \mathrm{sech}\xi$, it follows that the bell-shaped solution for the system (3) reads

$$u(x,t) = \frac{1}{3}(c + 8k^2 - k^2m^2) - k^2\mathrm{sech}^2k(x - ct),$$
$$v(x,t) = b_0 \mp k^2\mathrm{sech}^2k(x - ct),$$
$$w(x,t) = c_0 + \frac{k^2}{2}(18k^2 - k^2m^2 \mp 6b_0)\mathrm{sech}^2k(x - ct).$$

From (13)–(16), it is easy to obtain new periodic solutions with the third Jacobi elliptic function and bell-shaped solitary wave solutions for the system (4).

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