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The periodic wave solutions for two systems of nonlinear wave equations*

Wang Ming-Liang(王明亮)[a)b)†], Wang Yue-Ming(王跃明)[a)], and Zhang Jin-Liang(张金良)[a)c)]

[a)] *Department of Mathematics and Physics, Henan University of Science and Technology, Luoyang 471003, China*
[b)] *Department of Mathematics, Lanzhou University, Lanzhou 730000, China*
[c)] *School of Mechanical and Electronic Engineering, Northwestern Polytechnic University, Xi'an 710072, China*

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The periodic wave solutions for the Zakharov system of nonlinear wave equations and a long–short-wave interaction system are obtained by using the F-expansion method, which can be regarded as an overall generalization of Jacobi elliptic function expansion proposed recently. In the limit cases, the solitary wave solutions for the systems are also obtained.

Keywords: high-and low-frequency wave interaction systems, F-expansion method, periodic wave solutions expressed by Jacobi elliptic functions, solitary wave solutions

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1. Introduction

In recent years, exploration of exact solutions to the nonlinear partial differential equations (NPDEs) in mathematical physics has attracted much attention by many authors, and up to now a number of powerful methods to construct exact solutions of NPDEs have been introduced and developed: for example, the inverse scattering transform,[1] Backlund transformations,[2] Hirota's bilinear operators,[3] the truncated Painlevè expansion,[4] the hyperbolic function method (including the tanh method and its extension),[5−10] the sine-cosine method,[11,12] the homogeneous balance method and its extension,[13−30] and the Jacobi elliptic function expansion method,[31−38] etc. More recently, the F-expansion method[39] has been proposed, which can be thought of as any kind of Jacobi elliptic function expansion method and hence an overall generalization of it.

In this paper, we consider two systems of nonlinear wave equations in mathematical physics. The first one is the Zakharov system[40] of nonlinear wave equations in the form

$$\mathrm{i}\psi_t + \psi_{xx} + \psi v = 0, \tag{1}$$

$$v_{tt} - v_{xx} + (|\psi|^2)_{xx} = 0. \tag{2}$$

Equations (1) and (2) are one of the fundamental models governing dynamics of nonlinear waves, and describe the interaction between high- and low-frequency waves. The physically most important example involves the interaction between the Langmuir and ion-acoustic waves in a plasma. $\psi(x,t)$ in Eqs.(1) and (2) is a complex envelope of the high-frequency wave, and $v(x,t)$ the real low-frequency field. The sound velocity as well as the coupling constant in Eq.(2) has been normalized to unity. The second one is a long–short-wave interaction system in the form[41]

$$\mathrm{i}\psi_t + \psi_{xx} - \psi v = 0, \tag{3}$$

$$v_t + v_x + (|\psi|^2)_x = 0, \tag{4}$$

where $\psi(x,t)$ is a complex function and $v(x,t)$ a real function. The sonic-Langmuir solitons of Eqs.(3) and (4) were obtained by using the inverse scattering method. The aim of this paper is to obtain various periodic wave solutions expressed by Jacobi elliptic

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†E-mail: mlwang@mail.haust.edu.cn

functions of Eqs.(1), (2) and (3), (4) in terms of the F-expansion method.

The paper is organized as follows. In Section 2, we briefly describe what is the F-expansion method, and how to use it to derive the periodic wave solutions of a nonlinear wave equation(s) in mathematical physics. As illustrative examples, in Sections 3 and 4, the periodic wave solutions expressed by various Jacobi elliptic functions of Eqs.(1), (2) and Eqs.(3), (4) are obtained by using the F-expansion method. In the limit cases, the solitary wave solutions and the other type of travelling wave solutions are also obtained. Finally, in Section 5 some comments are made on the F-expansion method.

2. Description of the F-expansion method

Suppose that a nonlinear wave equation for $u(x,t)$ is given by

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, \cdots) = 0, \qquad (5)$$

in which both nonlinear terms and higher-order derivatives of $u(x,t)$ are all involved. In general, the left-hand side of Eq.(5) is a polynomial in u and its various derivatives. The F-expansion method for solving Eq.(5) proceeds in the following four steps:

Step 1. Seek the travelling wave solutions of Eq.(5) by taking

$$u(x,t) = u(\xi), \qquad \xi = kx + \varpi t + x_0, \qquad (6)$$

where k, ϖ are nonzero constants to be determined later; x_0 is an arbitrary real constant. Substituting Eq.(6) into Eq.(5) yields an ordinary differential equation (ODE) for $u(\xi)$

$$P(u, ku', \varpi u', k^2u'', k\varpi u'', \varpi^2 u'', k^3u''', \cdots) = 0, \quad (7)$$

where $u' = \dfrac{\mathrm{d}u(\xi)}{\mathrm{d}\xi}$, $u'' = \dfrac{\mathrm{d}^2u(\xi)}{\mathrm{d}\xi^2}$, $u''' = \dfrac{\mathrm{d}^3u(\xi)}{\mathrm{d}\xi^3}$, etc.

Step 2. Suppose that $u(\xi)$ can be expressed by a finite power series of $F(\xi)$

$$u(\xi) = \sum_{n=0}^{N} a_n F^n(\xi), \qquad a_N \neq 0, \qquad (8)$$

where $a_0, a_1, \cdots, a_N$ are constants to be determined later, $F(\xi)$ satisfies the first-order nonlinear ODE

$$F'^2(\xi) = PF^4 + QF^2 + R, \qquad (9)$$

in which P, Q, R are real constants, integer N in Eq.(8) can be determined by considering the homogeneous balance between the nonlinear term(s) and the highest-order derivatives of $u(\xi)$ in Eq.(7). In other words, we use a finite power series expansion of $F(\xi)$ which is a solution of ODE (9), instead of any kind of Jacobi elliptic function expansion,[31−37] to seek the solution of Eq.(7). Substituting the F-expansion (8) into ODE (7) and using ODE (9), then the left-hand side of Eq.(7) can be converted into a polynomial in $F(\xi)$. Setting each coefficient of the polynomial to zero yields a set of algebraic equations for $a_0, a_1, \cdots, a_N$ and ϖ (possibly k).

Step 3. Solving the algebraic equations obtained in Step 2, probably with the aid of Mathematica or Maple, then $a_0, a_1, \cdots, a_N$ and ϖ (possibly k) can be expressed by P, Q, R. Substituting these results into F-expansion (8), we can obtain a general form of travelling wave solution of Eq.(5).

Step 4. Choose properly the (P, Q, R) in ODE (9), such that the corresponding solution $F(\xi)$ of Eq.(9) is one of the Jacobi elliptic functions. The relations between the values of (P, Q, R) and the corresponding Jacobi elliptic function solution $F(\xi)$ of Eq.(9) are given in Appendix A. Substituting the values of (P, Q, R) and the corresponding Jacobi elliptic function $F(\xi)$ chosen from Appendix A into the general form of travelling wave solution obtained in Step 3, then an ideal periodic wave solution expressed by a Jacobi elliptic function can be obtained. In this way, we can simultaneously obtain more periodic wave solutions to Eq.(5).

The F-expansion method can be applied not only to a single nonlinear wave equation, but can also be coupled to nonlinear wave equations.

3. The periodic wave solutions for the Zakharov system

In this section, we use the F-expansion method to seek the periodic wave solutions of Eqs.(1) and (2).

Since $\psi(x,t)$ in Eqs.(1), (2) is a complex function, we assume that

$$\psi(x,t) = \mathrm{e}^{\mathrm{i}\eta} u(x,t), \quad \eta = \alpha x + \beta t, \qquad (10)$$

$u(x,t)$ is a real function, the constants α and β are to be determined. Substituting Eq.(10) into Eqs.(1), (2) and cancelling $\mathrm{e}^{\mathrm{i}\eta}$, yields the partial differential equations (PDEs) for $u(x,t)$ and $v(x,t)$,

$$u_t + 2\alpha u_x = 0, \qquad (11)$$

$$u_{xx} + uv - (\alpha^2 + \beta)u = 0, \qquad (12)$$

$$v_{tt} - v_{xx} + (u^2)_{xx} = 0. \tag{13}$$

(i) We seek the travelling wave solutions for PDEs (11)–(13) in the form

$$\begin{aligned} &u(x,t) = u(\xi),\\ &v(x,t) = v(\xi),\\ &\xi = k(x - 2\alpha t), \end{aligned} \tag{14}$$

where k, α are to be determined. Substituting Eqs.(14) into PDEs (11)–(13) yields ODEs for $u(\xi)$ and $v(\xi)$,

$$k^2 u'' + uv - (\alpha^2 + \beta)u = 0, \tag{15}$$

$$(4\alpha^2 - 1)k^2 v'' + k^2 (u^2)'' = 0, \tag{16}$$

where

$$u'' = \frac{\mathrm{d}^2 u(\xi)}{\mathrm{d}\xi^2}, \quad v'' = \frac{\mathrm{d}^2 v(\xi)}{\mathrm{d}\xi^2}.$$

(ii) Considering the homogeneous balance between u'' and uv in Eq.(15) as well as v'' and $(u^2)''$ in Eq.(16), we assume that $u(\xi)$ and $v(\xi)$ can be expressed as

$$u(\xi) = a_0 + a_1 F(\xi), \qquad a_1 \neq 0, \tag{17}$$

$$v(\xi) = b_0 + b_1 F(\xi) + b_2 F^2(\xi), \qquad b_2 \neq 0, \tag{18}$$

where a_0, a_1, b_0, b_1, b_2 are constants to be determined, $F(\xi)$ is a solution of ODE (9).

Substituting Eqs.(17), (18) into Eqs.(15), (16), and using ODE (9), then the left-hand sides of Eqs.(15), (16) can be converted into two polynomials in $F(\xi)$; setting each coefficient of the polynomials to zero yields a set of algebraic equation for a_0, a_1, b_0, b_1, b_2

$$F^0 : a_0 b_0 - a_0(\alpha^2 + \beta) = 0, \tag{19}$$

$$F : a_1 Q k^2 + (a_0 b_1 + a_1 b_0) - a_1(\alpha^2 + \beta) = 0, \tag{20}$$

$$F^2 : a_1 b_1 + a_0 b_2 = 0, \tag{21}$$

$$F^3 : 2a_1 P k^2 + a_1 b_2 = 0, \tag{22}$$

$$F^0 : 2(4\alpha^2 - 1)b_2 R + 2a_1^2 R = 0, \tag{23}$$

$$F : (4\alpha^2 - 1)b_1 Q + 2a_0 a_1 Q = 0, \tag{24}$$

$$F^2 : 4(4\alpha^2 - 1)b_2 Q + 4a_1^2 Q = 0, \tag{25}$$

$$F^3 : 2(4\alpha^2 - 1)b_1 P + 4a_0 a_1 P = 0, \tag{26}$$

$$F^4 : 6(4\alpha^2 - 1)b_2 P + 6a_1^2 P = 0. \tag{27}$$

(iii) Solving algebraic equations (19)–(27), we have the solutions

$$\begin{aligned} &a_0 = 0,\\ &a_1 = \pm k\sqrt{2(4\alpha^2 - 1)P},\\ &(if (4\alpha^2 - 1)P > 0),\\ &b_0 = -Qk^2 + \alpha^2 + \beta,\\ &b_1 = 0,\\ &b_2 = -2Pk^2, \end{aligned} \tag{28}$$

where k, β are arbitrary constants.

Substituting Eq.(28) into Eqs.(17), (18) and using Eq.(10), we have a general form of travelling wave solutions of Eqs.(1), (2).

$$\psi(x,t) = \pm k\sqrt{2P(4\alpha^2 - 1)}\mathrm{e}^{\mathrm{i}\eta} F(\xi),$$

$$((4\alpha^2 - 1)P > 0), \tag{29}$$

$$v(x,t) = -2Pk^2 F^2(\xi) - Qk^2 + \alpha^2 + \beta, \tag{30}$$

provided the solution $F(\xi)$ of ODE (9) is known, where

$$\xi = k(x - 2\alpha t), \qquad \eta = \alpha x + \beta t,$$

k, β are arbitrary constants.

(iv) From the first line of Appendix A, we see that the Jacobi sn-function

$$F(\xi) = \mathrm{sn}\xi \tag{31}$$

is a solution of ODE (9) with

$$P = m^2, \quad Q = -(1 + m^2), \quad R = 1. \tag{32}$$

Substituting Eqs.(31), (32) into Eqs.(29), (30), we obtain the periodic wave solutions expressed by the Jacobi sn-function of Eqs.(1), (2)

$$\psi_1(x,t) = \pm mk\sqrt{2(4\alpha^2 - 1)}\mathrm{e}^{\mathrm{i}\eta}\mathrm{sn}\xi, \quad \left(\alpha^2 > \frac{1}{4}\right), \tag{33}$$

$$v_1(x,t) = -2m^2 k^2 \mathrm{sn}^2\xi + (1 + m^2)k^2 + \alpha^2 + \beta, \tag{34}$$

where

$$\xi = k(x - 2\alpha t), \quad \eta = \alpha x + \beta t,$$

k, β are arbitrary constants.

In the limit case when $m \to 1$, then $\mathrm{sn}\xi \to \tanh\xi$, thus Eqs.(33), (34) become the solitary wave solutions of Eqs.(1), (2)

$$\psi_1(x,t) = \pm k\sqrt{2(4\alpha^2 - 1)}\mathrm{e}^{\mathrm{i}\eta}\tanh\xi, \quad \left(\alpha^2 > \frac{1}{4}\right), \tag{35}$$

$$\begin{aligned} v_1(x,t) =& -2k^2 \tanh^2\xi + 2k^2 + \alpha^2 + \beta\\ =& 2k^2 \mathrm{sech}^2\xi + \alpha^2 + \beta. \end{aligned} \tag{36}$$

From the second line of Appendix A, we see that the Jacobi cn-function

$$F(\xi) = \mathrm{cn}\xi \tag{37}$$

is a solution of ODE (9) with

$$P = -m^2, \quad Q = 2m^2 - 1, \quad R = 1 - m^2. \tag{38}$$

Substituting Eqs.(37), (38) into Eqs.(29), (30), we obtain the periodic wave solutions expressed by the Jacobi cn-function of Eqs.(1), (2)

$$\psi_2(x,t) = \pm mk\sqrt{2(1-4\alpha^2)}\mathrm{e}^{\mathrm{i}\eta}\mathrm{cn}\xi, \quad \left(\alpha^2 < \frac{1}{4}\right), \tag{39}$$

$$v_2(x,t) = -2m^2k^2\mathrm{cn}^2\xi + (1-2m^2)k^2 + \alpha^2 + \beta. \tag{40}$$

From the third line of Appendix A, we see that the Jacobi dn-function

$$F(\xi) = \mathrm{dn}\xi \tag{41}$$

is a solution of ODE (9) with

$$P = -1, \quad Q = 2 - m^2, \quad R = m^2 - 1. \tag{42}$$

Substituting Eqs.(41), (42) into Eqs.(29), (30), we obtain the periodic wave solutions expressed by the Jacobi dn-function of Eqs.(1), (2)

$$\psi_3(x,t) = \pm k\sqrt{2(1-4\alpha^2)}\mathrm{e}^{\mathrm{i}\eta}\mathrm{dn}\xi, \quad \left(\alpha^2 < \frac{1}{4}\right), \tag{43}$$

$$v_3(x,t) = 2k^2\mathrm{dn}^2\xi + (m^2-2)k^2 + \alpha^2 + \beta. \tag{44}$$

In the limit case when $m \to 1$, both $\mathrm{cn}\xi$ and $\mathrm{dn}\xi \to \mathrm{sech}\xi$, thus Eqs.(39), (40) and (43), (44) become

$$\begin{aligned}\psi_2(x,t) =& \psi_3(x,t)\\ =& \pm k\sqrt{2(1-4\alpha^2)}\mathrm{e}^{\mathrm{i}\eta}\mathrm{sech}\xi, \quad \left(\alpha^2 < \frac{1}{4}\right),\end{aligned} \tag{45}$$

$$v_2(x,t) = v_3(x,t) = 2k^2\mathrm{sech}^2\xi - k^2 + \alpha^2 + \beta. \tag{46}$$

Similarly, we can obtain the other periodic wave solutions of Eqs.(1), (2) in terms of Appendix A as follows:

$$\begin{cases} \psi_4(x,t) = \pm k\sqrt{2(4\alpha^2-1)}\mathrm{e}^{\mathrm{i}\eta}\mathrm{ns}\xi, & \left(\alpha^2 > \frac{1}{4}\right),\\ v_4(x,t) = -2k^2\mathrm{ns}^2\xi + (1+m^2)k^2 + \alpha^2 + \beta; \end{cases}$$

$$\begin{cases} \psi_5(x,t) = \pm k\sqrt{2(1-m^2)(4\alpha^2-1)}\mathrm{e}^{\mathrm{i}\eta}\mathrm{nc}\xi, & \left(\alpha^2 > \frac{1}{4}\right),\\ v_5(x,t) = -2(1-m^2)k^2\mathrm{nc}^2\xi + (1-2m^2)k^2 + \alpha^2 + \beta; \end{cases}$$

$$\begin{cases} \psi_6(x,t) = \pm k\sqrt{2(m^2-1)(4\alpha^2-1)}\mathrm{e}^{\mathrm{i}\eta}\mathrm{nd}\xi, & \left(\alpha^2 < \frac{1}{4}\right),\\ v_6(x,t) = 2(1-m^2)k^2\mathrm{nd}^2\xi + (m^2-2)k^2 + \alpha^2 + \beta; \end{cases}$$

$$\begin{cases} \psi_7(x,t) = \pm k\sqrt{2(1-m^2)(4\alpha^2-1)}\mathrm{e}^{\mathrm{i}\eta}\mathrm{sc}\xi, & \left(\alpha^2 > \frac{1}{4}\right),\\ v_7(x,t) = -2(1-m^2)k^2\mathrm{sc}^2\xi + (m^2-2)k^2 + \alpha^2 + \beta; \end{cases}$$

$$\begin{cases} \psi_8(x,t) = \pm mk\sqrt{2(m^2-1)(4\alpha^2-1)}\mathrm{e}^{\mathrm{i}\eta}\mathrm{sd}\xi, & \left(\alpha^2 < \frac{1}{4}\right),\\ v_8(x,t) = 2m^2(1-m^2)k^2\mathrm{sd}^2\xi + (1-2m^2)k^2 + \alpha^2 + \beta; \end{cases}$$

$$\begin{cases} \psi_9(x,t) = \pm k\sqrt{2(4\alpha^2-1)}\mathrm{e}^{\mathrm{i}\eta}\mathrm{cs}\xi, & \left(\alpha^2 > \frac{1}{4}\right),\\ v_9(x,t) = -2k^2\mathrm{cs}^2\xi + (m^2-2)k^2 + \alpha^2 + \beta; \end{cases}$$

$$\begin{cases} \psi_{10}(x,t) = \pm k\sqrt{2(4\alpha^2-1)}\mathrm{e}^{\mathrm{i}\eta}\mathrm{ds}\xi, & \left(\alpha^2 > \frac{1}{4}\right),\\ v_{10}(x,t) = -2k^2\mathrm{ds}^2\xi + (1-m^2)k^2 + \alpha^2 + \beta. \end{cases}$$

In the limit case when $m \to 0$, then the nontrivial results are obtained as follows:

$$\begin{cases} \psi_4(x,t) = \psi_{10}(x,t) = \pm k\sqrt{2(4\alpha^2-1)}\mathrm{e}^{\mathrm{i}\eta}\mathrm{csc}\xi, & \left(\alpha^2 > \frac{1}{4}\right),\\ v_4(x,t) = v_{10}(x,t) = -2k^2\mathrm{csc}^2\xi + k^2 + \alpha^2 + \beta; \end{cases}$$

$$\begin{cases} \psi_5(x,t) = \pm k\sqrt{2(4\alpha^2-1)}\mathrm{e}^{\mathrm{i}\eta}\mathrm{sec}\xi, & \left(\alpha^2 > \frac{1}{4}\right),\\ v_5(x,t) = -2k^2\mathrm{sec}^2\xi + k^2 + \alpha^2 + \beta; \end{cases}$$

$$\begin{cases} \psi_7(x,t) = \pm k\sqrt{2(4\alpha^2-1)}\mathrm{e}^{\mathrm{i}\eta}\tan\xi, & \left(\alpha^2 > \frac{1}{4}\right), \\ v_7(x,t) = -2k^2\sec^2\xi + \alpha^2 + \beta; \end{cases}$$

$$\begin{cases} \psi_9(x,t) = \pm k\sqrt{2(4\alpha^2-1)}\mathrm{e}^{\mathrm{i}\eta}\cot\xi, & \left(\alpha^2 > \frac{1}{4}\right), \\ v_9(x,t) = -2k^2\csc^2\xi + \alpha^2 + \beta. \end{cases}$$

In the limit case when $m \to 1$, then

$$\begin{cases} \psi_4(x,t) = \pm k\sqrt{2(4\alpha^2-1)}\mathrm{e}^{\mathrm{i}\eta}\coth\xi, & \left(\alpha^2 > \frac{1}{4}\right), \\ v_4(x,t) = -2k^2\mathrm{csch}^2\xi + \alpha^2 + \beta; \end{cases}$$

$$\begin{cases} \psi_9(x,t) = \psi_{10}(x,t) = \pm k\sqrt{2(4\alpha^2-1)}\mathrm{e}^{\mathrm{i}\eta}\mathrm{csch}\xi, & \left(\alpha^2 > \frac{1}{4}\right), \\ v_9(x,t) = v_{10}(x,t) = -2k^2\mathrm{csch}^2\xi - k + \alpha^2 + \beta. \end{cases}$$

4. The periodic wave solutions for a long–short-wave interaction system

In this section, we will use the F-expansion method to seek the periodic wave solutions of Eqs.(3), (4).

Since $\psi = \psi(x,t)$ in Eqs.(3), (4) is a complex function, we assume that

$$\psi(x,t) = \mathrm{e}^{\mathrm{i}\eta}u(x,t), \quad \eta = \alpha x + \beta t, \tag{47}$$

where α, β are constants to be determined, $u = u(x,t)$ -real function. Substituting Eq.(47) into Eqs.(3), (4) and cancelling $\mathrm{e}^{\mathrm{i}\eta}$ yields

$$u_t + 2\alpha u_x = 0, \tag{48}$$

$$u_{xx} - uv - (\alpha^2 + \beta)u = 0, \tag{49}$$

$$v_t + v_x + 2uu_x = 0. \tag{50}$$

(i) We seek travelling wave solutions for PDEs (48)–(50) in the form

$$\begin{aligned} u(x,t) &= u(\xi), \\ v(x,t) &= v(\xi), \\ \xi &= k(x - 2\alpha t), \end{aligned} \tag{51}$$

where k, α are to be determined. Substituting Eq.(51) into PDEs (48)–(50), then we have ODEs for $u(\xi)$ and $v(\xi)$

$$k^2u'' - uv - (\alpha^2 + \beta)u = 0, \tag{52}$$

$$k(1-2\alpha)v' + 2kuu' = 0, \tag{53}$$

where

$$u'' = \frac{\mathrm{d}^2u(\xi)}{\mathrm{d}\xi^2}, \quad v'' = \frac{\mathrm{d}v(\xi)}{\mathrm{d}\xi}.$$

(ii) Considering the homogeneous balance between u'' and uv in Eq.(52) as well as v' and uu' in Eq.(53), we assume that $u(\xi)$ and $v(\xi)$ can be expressed by

$$u(\xi) = a_0 + a_1F(\xi), \qquad a_1 \neq 0, \tag{54}$$

$$v(\xi) = b_0 + b_1F(\xi) + b_2F^2(\xi), \qquad b_2 \neq 0, \tag{55}$$

where a_0, a_1, b_0, b_1, b_2 are constants to be determined, $F(\xi)$ is a solution of ODE (9).

Substituting Eqs.(54), (55) into Eqs.(52), (53), and using ODE (9), then the left hand sides of Eqs.(52), (53) are converted into two polynomials in $F(\xi)$; setting each coefficient of the polynomials to zero yields a set of algebraic equation for a_0, a_1, b_0, b_1, b_2

$$F^0 : k(1-2\alpha)b_1 + 2ka_0a_1 = 0, \tag{56}$$

$$F : 2k(1-2\alpha)b_2 + 2ka_1^2 = 0, \tag{57}$$

$$F^0 : -a_0b_0 - (\alpha^2 + \beta)a_0 = 0, \tag{58}$$

$$F : a_1Qk^2 - (a_0b_1 + a_1b_0) - (\alpha^2 + \beta)a_1 = 0, \tag{59}$$

$$F^2 : a_0b_2 + a_1b_1 = 0, \tag{60}$$

$$F^3 : 2a_1k^2P - a_1b_2 = 0. \tag{61}$$

(iii) Solving algebraic equations (56)–(61), we

have the solutions

$$\begin{aligned} &b_2 = 2k^2P, \\ &b_1 = 0, \\ &b_0 = Qk^2 - (\alpha^2+\beta), \\ &a_1 = \pm k\sqrt{2(2\alpha-1)P}, \\ &(\text{if } (2\alpha-1)P > 0), \\ &a_0 = 0, \end{aligned} \tag{62}$$

(63)

k, β are arbitrary constants.

Substituting Eqs.(62), (63) into Eqs.(54), (55) and using Eq.(47), we have a general form of travelling wave solutions of Eqs.(3), (4):

$$\psi(x,t) = \pm k\sqrt{2P(2\alpha-1)}\mathrm{e}^{\mathrm{i}\eta}F(\xi), \tag{64}$$

$$v(x,t) = 2Pk^2F^2(\xi) - (\alpha^2+\beta) + Qk^2, \tag{65}$$

provided that $P(2\alpha-1) > 0$ and the solution $F(\xi)$ of ODE (9) is known, where

$$\xi = k(x - 2\alpha t), \qquad \eta = \alpha x + \beta t,$$

k, β are arbitrary constants.

(iv) From the first line of Appendix A, we see that

$$P = m^2, \quad Q = -(1+m^2),$$

$$R = 1, \quad \text{and} \quad F(\xi) = \mathrm{sn}\xi \tag{66}$$

is a solution of ODE (9). Substituting Eq.(66) into Eqs.(64), (65), we obtain the periodic wave solutions expressed by the Jacobi sn-function of Eqs.(3), (4):

$$\psi(x,t) = \pm k\sqrt{2m^2(2\alpha-1)}\mathrm{e}^{\mathrm{i}\eta}\mathrm{sn}\xi, \quad \left(\alpha > \frac{1}{2}\right), \tag{67}$$

$$v(x,t) = 2m^2k^2\mathrm{sn}^2\xi - (1+m^2)k^2 - (\alpha^2+\beta), \tag{68}$$

where

$$\xi = k(x - 2\alpha t), \qquad \eta = \alpha x + \beta t,$$

k, β are arbitrary constants.

In the limit case when $m \to 1$, then $\mathrm{sn}\xi \to \tanh\xi$, thus Eqs.(67), (68) become the solitary wave solutions of Eqs.(3), (4)

$$\psi(x,t) = \pm k\sqrt{2(2\alpha-1)}\mathrm{e}^{\mathrm{i}\eta}\tanh\xi, \quad \left(\alpha^2 > \frac{1}{2}\right), \tag{69}$$

$$\begin{aligned} v(x,t) =& 2k^2(\tanh^2\xi - 1) - (\alpha^2+\beta) \\ =& -2k^2\mathrm{sech}^2\xi - (\alpha^2+\beta). \end{aligned} \tag{70}$$

From the second line of Appendix A, we see that

$$P = -m^2, \quad Q = 2m^2 - 1,$$

$$R = 1 - m^2, \quad \text{and} \quad F(\xi) = \mathrm{cn}\xi \tag{71}$$

is a solution of ODE (9). Substituting Eq.(71) into Eqs.(64), (65), we obtain the periodic wave solutions expressed by the Jacobi cn-function of Eqs.(3), (4)

$$\psi(x,t) = \pm k\sqrt{2m^2(1-2\alpha)}\mathrm{e}^{\mathrm{i}\eta}\mathrm{cn}\xi, \quad \left(\alpha < \frac{1}{2}\right), \tag{72}$$

$$v(x,t) = -2m^2k^2\mathrm{cn}^2\xi + (2m^2-1)k^2 - (\alpha^2+\beta), \tag{73}$$

where

$$\xi = k(x - 2\alpha t), \qquad \eta = \alpha x + \beta t,$$

α, β are constants.

In the limit case when $m \to 1$, then $\mathrm{cn}\xi \to \mathrm{sech}\xi$, thus Eqs.(72), (73) become the solitary wave solutions of Eqs.(3), (4)

$$\psi(x,t) = \pm k\sqrt{2(1-2\alpha)}\mathrm{e}^{\mathrm{i}\eta}\mathrm{sech}\xi, \quad \left(\alpha < \frac{1}{2}\right), \tag{74}$$

$$v(x,t) = -2k^2\mathrm{sech}^2\xi + k^2 - (\alpha^2+\beta). \tag{75}$$

From the third line of Appendix A, we see that

$$P = -1, \quad Q = 2 - m^2,$$

$$R = m^2 - 1, \quad \text{and} \quad F(\xi) = \mathrm{dn}\xi \tag{76}$$

is a solution of ODE (9). Substituting Eq.(76) into Eqs.(64), (65), we obtain the periodic wave solutions expressed by the Jacobi dn-function of Eqs.(3), (4)

$$\psi(x,t) = \pm k\sqrt{2(1-2\alpha)}\mathrm{e}^{\mathrm{i}\eta}\mathrm{dn}\xi, \quad \left(\alpha < \frac{1}{2}\right), \tag{77}$$

$$v(x,t) = -2k^2\mathrm{dn}^2\xi + (2-m^2)k^2 - (\alpha^2+\beta), \tag{78}$$

where

$$\xi = k(x - 2\alpha t), \qquad \eta = \alpha x + \beta t,$$

k, β are arbitrary constants.

In the limit case when $m \to 1$, then $\mathrm{dn}\xi \to \mathrm{sech}\xi$, thus Eqs.(77), (78) become the solitary wave solutions Eqs.(74), (75) as well.

We can obtain the other solutions to Eqs.(3), (4) in terms of Appendix A, but we omit them here for simplicity.

5. Conclusion

The F-expansion method includes every one of various Jacobi elliptic function expansion methods proposed recently. In other words, F here stands for

every one of various Jacobi elliptic functions, and so F-expansion is an overall generalization of Jacobi elliptic function expansion. The main advantages of the F-expansion method are that firstly, one only calculate the function $F(\xi)$, a solution of ODE (9), the coefficients of which can be selected, instead of calculating various Jacobi elliptic functions; secondly, the relations between the coefficients and the corresponding Jacobi elliptic function solution of ODE (9) have been known and given in Appendix A at the end of the paper, so one can simultaneously obtain more periodic wave solutions expressed by various Jacobi elliptic functions in terms of Appendix A. It seems that the F-expansion method is more effective and simpler than the Jacobi elliptic function expansion method.

Appendix A

Relations between values of (P, Q, R) and corresponding $F(\xi)$ in ODE

$$F'^2 = PF^4 + QF^2 + R.$$

P	Q	R	$F'^2 = PF^4 + QF^2 + R$	$F(\xi)$
m^2	$-(1+m^2)$	1	$F'^2 = (1-F^2)(1-m^2F^2)$	$\mathrm{sn}\xi$, $\mathrm{cd}\xi = \dfrac{\mathrm{cn}\xi}{\mathrm{dn}\xi}$
$-m^2$	$2m^2-1$	$1-m^2$	$F'^2 = (1-F^2)(m^2F^2+1-m^2)$	$\mathrm{cn}\xi$
-1	$2-m^2$	m^2-1	$F'^2 = (1-F^2)(F^2+m^2-1)$	$\mathrm{dn}\xi$
1	$-(1+m^2)$	m^2	$F'^2 = (1-F^2)(m^2-F^2)$	$\mathrm{ns}\xi=(\mathrm{sn}\xi)^{-1}$, $\mathrm{dc}\xi = \dfrac{\mathrm{dn}\xi}{\mathrm{cn}\xi}$
$1-m^2$	$2m^2-1$	$-m^2$	$F'^2 = (1-F^2)[(m^2-1)F^2-m^2]$	$\mathrm{nc}\xi=(\mathrm{cn}\xi)^{-1}$
m^2-1	$2-m^2$	-1	$F'^2 = (1-F^2)[(1-m^2)F^2-1]$	$\mathrm{nd}\xi=(\mathrm{dn}\xi)^{-1}$
$1-m^2$	$2-m^2$	1	$F'^2 = (1+F^2)[(1-m^2)F^2+1]$	$\mathrm{sc}\xi = \dfrac{\mathrm{sn}\xi}{\mathrm{cn}\xi}$
$-m^2(1-m^2)$	$2m^2-1$	1	$F'^2 = (1+m^2F^2)[(m^2-1)F^2+1]$	$\mathrm{sd}\xi = \dfrac{\mathrm{sn}\xi}{\mathrm{dn}\xi}$
1	$2-m^2$	$1-m^2$	$F'^2 = (1+F^2)[F^2+(1-m^2)]$	$\mathrm{cs}\xi = \dfrac{\mathrm{cn}\xi}{\mathrm{sn}\xi}$
1	$2m^2-1$	$-m^2(1-m^2)$	$F'^2 = (m^2+F^2)[F^2+(m^2-1)]$	$\mathrm{ds}\xi = \dfrac{\mathrm{dn}\xi}{\mathrm{sn}\xi}$

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