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Classical and quantum wormholes with tachyon field*

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The wormhole equations are presented in the presence of tachyon field. Specializing at some values of ω (the ratio of pressure to energy density), we find a family of classical and quantum wormhole solutions.

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1. Introduction

Wormhole was introduced by Wheeler in the 1950s.[1] It is a Euclidean field configuration in some field theory containing gravity, which consists of two asymptotically flat regions connected by a tube or throat. One specific example is the Schwarzschild bridge which is a slice of the black hole joining two asymptotic regions. Topological structures of space–time were profoundly discussed at the end of the 1980s due to the extensive study of quantum cosmology. The introduction of such concepts as four-dimensional wormholes, baby universe, and their effects on space–time has provoked great interest among theoretical physicists.[2−9] Since this kind of wormholes can endow spaces with different topologies, they represent tiny quantum fluctuations of space. From the point of view of mathematics, the condition that a wormhole exists on a four-dimensional asymptotic flat manifold M_4 is that the Ricci tensor of M_4 has negative eigenvalues somewhere on M_4.[10] In the pure gravitation theory, Hawking discussed wormholes in which two baby universes are connected.[2] The wormhole with a stable Euclidean action was first discovered by Giddings and Strominger[11] in the theory with a spontaneously broken Abelian internal symmetry—the theory of a Goldstone boson, or axion, minimally coupled to Einstein gravity. In the point of view of physics, such wormhole solutions can also be used to study the dimensions of spacetime, quantum chaos[12−14] and the third quantization problem of baby universes.[8,15−18] Up until now, various wormhole solutions have been discovered, the examples are wormholes in complex scalar field both with[19] and without[20] broken symmetry, wormholes in Yang–Mills gauge field,[21−23] Skyrme field,[24] higher-dimensional space–time,[25−27] scalar field,[28] spinors field,[29] the superstring theory,[30] and the non-Abelian Born–Infeld theory.[31]

In this paper, we will discuss the wormholes with the tachyon field, which is introduced in string theory. It is mainly due to Sen's work that our understanding of the role of tachyons in string theory has greatly improved.[32,33] The basic idea is that the usual open string vacuum is unstable but there exists a stable vacuum with zero energy density which is stable. There is evidence that this state is associated with the condensation of electric flux tubes of closed strings. These flux tubes are described successfully using an effective Born–Infeld action.[34,35] This success of effective action methods encourages one to pursue this further and to attempt a description of the cosmology of tachyon rolling. Moreover, not to take into account the effects of gravity during the process is inconsistent, since it involves a spatially uniform distribution of energy. So Gibbons[36] discussed the cosmological effects of the tachyon rolling down to its ground state by coupling a simple effective field theory for the tachyon field to Einstein gravity. As the tachyon rolls

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down to the minimum of its potential the universe expands. Depending upon initial conditions, the scale factor may or may not start off accelerating. It is of great importance for the discussion upon cosmology evolution of tachyon because it reveals that tachyon may play the role of dark energy and give an explanation of the accelerating of the universe. The measurements made on distant supernovas Ia[37−39] and posterior observations and data refinements[40] have led to the increasingly firmer conclusion that our universe is accelerating. It appears well-proved that around two-third of the energy in the universe should be hidden as dark energy.[41] The great progress made in tachyon enables us to consider the possibility of tachyon field to be the dark energy.

This paper is composed of a classical and quantum treatment of wormholes which are endowed with a tachyon field. The former is thought as being responsible for the connection with cosmology and the latter is assumed to account for the content of dark energy needed to justify present cosmological observations. Our study is based on the idea that the universe contains a tachyon field component, whose state equation $p = \omega\rho$ may take on any form, only restricted by $-1 \leq \omega \leq 1$, during its very early quantum evolution. Therefore, the quantum state of the very early universe should be contributed by the semi-classical or quantum paths that correspond to the Euclidean solutions associated with all possible values of ω. We firstly describe the general Euclidean formalism for a homogeneous tachyon field which is minimally coupled with Hilbert–Einstein gravity in the presence of positive and negative cosmological constants. From the equations that define the tachyon field and the conservation law, we finally obtain the wormhole equation or Hamiltonian constraint equation in terms of parameter ω. In the next sections, we deal with in detail the classical and quantum states that correspond to the particular state equations $\omega = -2/3, -1/3, 0, 1/3, 2/3, 1$.

2. Wormhole equation

Consider the Euclidean metric with the topology of $R^1 \bigotimes S^3$

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 + a^2(\tau)\,\mathrm{d}\Omega_3^2, \tag{1}$$

where $a(\tau)$ is the radius of a three-dimensional sphere. The effective Lagrangian density of the tachyon field in the Born–Infeld-type is given by[42]

$$L = -V(T)\sqrt{|-\det(g_{\mu\nu} - \partial_\mu T \partial_\nu T)|}, \tag{2}$$

where $T(\tau)$ is the spatially homogenous tachyon field, $V(T)$ is the tachyon potential. Using Eqs. (1) and (2), the Einstein equations can be written as

$$3\frac{\dot{a}^2 - 1}{a^2} = 8\pi\frac{V}{\sqrt{1-\dot{T}^2}} - \lambda, \tag{3}$$

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}^2 - 1}{a^2} = 8\pi V\sqrt{1-\dot{T}^2} - \lambda, \tag{4}$$

for $\dot{T}^2 \leq 1$; and

$$3\frac{\dot{a}^2 - 1}{a^2} = 8\pi\frac{V}{\sqrt{\dot{T}^2-1}} - \lambda, \tag{5}$$

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}^2 - 1}{a^2} = -8\pi V\sqrt{\dot{T}^2-1} - \lambda, \tag{6}$$

for $\dot{T}^2 > 1$. Here dot denotes the derivative with respect to τ. λ is the cosmological constant. Equations (3) and (4) show that the tachyon field contributes energy density and pressure as

$$\rho_T = \frac{V}{\sqrt{1-\dot{T}^2}}, \tag{7}$$

$$p_T = V\sqrt{1-\dot{T}^2}, \tag{8}$$

while Eqs. (5) and (6) show that the tachyon field contributes to the energy density and pressure as

$$\rho_{\mathrm{m}} = \frac{-V}{\sqrt{\dot{T}^2-1}}, \tag{9}$$

$$p_{\mathrm{m}} = -V\sqrt{\dot{T}^2-1}. \tag{10}$$

It is easy to find that ρ_{m} and p_{m} is the energy density and pressure for matter, and ρ_{d} and p_{d} for dark energy provided that $V|_{\dot{T}^2>1} < 0$ and $V|_{\dot{T}^2\leq 1} > 0$. This implies that when $\dot{T}^2 > 1$, the tachyon field behaves like an ordinary matter; and when$\dot{T}^2 \leq 1$, it behaves like a dark energy depending upon the evolution of tachyon field. In other words, the tachyon field can play the role of both matter and dark energy in the evolution of Universe. Whether appearing in the form of matter or dark energy, there exists a universal state equation

$$\omega \equiv \frac{p_{\mathrm{T}}}{\rho_{\mathrm{T}}} = \frac{p_{\mathrm{m}}}{\rho_{\mathrm{m}}} = \dot{T}^2 - 1. \tag{11}$$

It is not difficult to obtain the conservation law of energy from Eqs. (3) and (4) or Eqs. (5) and (6)

$$\rho = \rho_0 \exp\left[-3\int(\omega+1)\,\mathrm{d}\ln a\right]. \tag{12}$$

Here ρ_0 is a constant. Thus Eqs. (3), (4) and Eqs. (5), (6) can be all reduced to one equation

$$\dot{a}^2 = 1 - \frac{8\pi}{3}\rho_0 \exp\left[-\int (3\omega+1)\,\mathrm{d}\ln a\right] - \frac{1}{3}\lambda a^2. \quad (13)$$

It is found that Eq. (13) is also the Euclidean Hamiltonian constraint. Set $8\pi\rho_0/3 = L^2$, $\mid \lambda \mid /3 = H^2$, then Eq. (13) becomes

$$\dot{a}^2 = 1 - L^2 \exp\left[-\int (3\omega+1)\,\mathrm{d}\ln a\right] \mp H^2 a^2, \quad (14)$$

where the plus and minus in the sign $\mp$ correspond to the negative and positive cosmological constant λ. Although we have not been able to find any closed form solutions to Eq. (14) for a generic ω, some analytical solutions for particular values of ω will be obtained and discussed in the next sections.

3. $\omega = -2/3$

On the condition that $\lambda < 0$ and $L^4 > 4H^2$, we find a wormhole solution given by

$$a = \frac{L^2}{2H^2} + \frac{1}{2H^2}\sqrt{L^4 - 4H^2}\cosh(H\tau). \quad (15)$$

The minimum scale of the wormhole is

$$a_{\min} = \frac{L^2}{2H^2} + \frac{1}{2H^2}\sqrt{L^4 - 4H^2}. \quad (16)$$

The conceptual shape of this wormhole is shown in Fig.1.

We next consider the quantum state of the above Euclidean wormhole. Under the assumption that when we disregard the factor ordering problem related to the momentum operator, we obtain the corresponding Wheeler-De Witt (WD) equation by replacing the momentum $\pi_a = \dot{a}$ for the quantum mechanical operator $\partial/\partial a$

$$\frac{\partial^2\psi}{\partial a^2} + \left[\left(\frac{L^4}{4H^2} - 1\right) - H^2\left(a - \frac{L^2}{2H^2}\right)^2\right]\psi = 0. \quad (17)$$

It is the equation of motion of a harmonic oscillator. Its solution is the Hermitian function.

$$\psi_n(a) = \left(1/\sqrt{\pi}2^n n!\right)^{1/2} \exp\left[-\frac{1}{2}H\left(a - \frac{L^2}{2H^2}\right)^2\right] \times H_n\left(a - \frac{L^2}{2H^2}\right). \quad (18)$$

Thus we have the following relation

$$\rho_0 = \frac{3H}{4\pi}\left[(2k+1)H + 1\right]^{1/2}, (k = 0, 1, 2, \cdots). \quad (19)$$

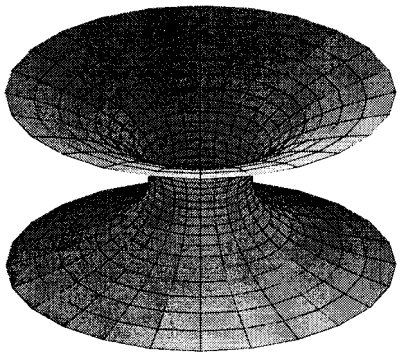


Fig.1. The conceptual shape of a wormhole.

This implies that the energy density or the scale of the wormhole is quantized. For the ground state, the wavefunction is given by

$$\psi_0 = \frac{1}{\pi^{1/4}}\mathrm{e}^{-\frac{1}{2}H\left(a - \frac{L^2}{2H^2}\right)^2}. \quad (20)$$

It is apparent that the quantum wormhole of the ground state has the most probability at $a = \dfrac{L^2}{2H^2}$. Equation (15) tells us that the minimum scale of the classical wormhole is larger than $\dfrac{L^2}{2H^2}$. This is similar to the case of classical oscillator.

The most probable scale of the wormhole for the ground state is

$$a_0 = \frac{1}{2H}\left(\sqrt{H+1} + \sqrt{5H+1}\right). \quad (21)$$

It is of the order of 10^{-25}m provided that $\lambda \sim -10^{-51}\mathrm{m}^{-2}$.

4. $\omega = -1/3$

In such a case, there is a wormhole solution on the condition that $\lambda < 0$ and $1 - L^2 < 0$,

$$a = \frac{1}{H}\sqrt{L^2 - 1}\cosh(H\tau). \quad (22)$$

The minimum scale of the wormhole is given by

$$a_{\min} = \frac{1}{H}\sqrt{L^2 - 1}. \quad (23)$$

It also has the conceptual shape as shown in Fig.1 and the corresponding WD equation is also the equation of motion of a harmonic oscillator

$$\frac{\partial^2\psi}{\partial a^2} + \left[\left(L^2 - 1\right) - H^2 a^2\right]\psi = 0. \quad (24)$$

Similarly, we have the relation

$$\rho_0 = \frac{3}{8\pi}\left[(2k+1)H + 1\right], (k = 0, 1, 2, \cdots). \quad (25)$$

The most probable scale of the wormhole for the ground state is

$$a_0 = \frac{1}{\sqrt{H}}. \tag{26}$$

It is of the order of 10^{-13}m provided that $\lambda \sim -10^{-51}\mathrm{m}^{-2}$. It is larger than that of the $\omega = -2/3$ case.

5. $\omega = 0$

5.1. $\lambda = 0$

In such a case, the wormhole equation is reduced to

$$\dot{a}^2 = 1 - \frac{L^2}{a}. \tag{27}$$

Then we obtain the wormhole solution

$$\tau = \sqrt{a^2 - aL^2} + \frac{L^2}{2}\ln\left(-\frac{L^2}{2} + a + \sqrt{a^2 - aL^2}\right). \tag{28}$$

The minimum scale is

$$a_{\min} = L^2. \tag{29}$$

This wormhole also has the conceptual shape as shown in Fig.1. The corresponding WD equation can be written as

$$\frac{\partial^2\psi}{\partial a^2} - \left(1 - \frac{L^2}{a}\right)\psi = 0. \tag{30}$$

Set $a = 1/L^2 x$, Eq. (30) becomes

$$\frac{\partial^2\psi}{\partial x^2} + \left(-\frac{1}{L^4} + \frac{1}{x}\right)\psi = 0. \tag{31}$$

This equation is equivalent to that describing the motion of a particle in a potential $-1/x$ with a total energy $-1/L^4$. Using the Wenzel–Kramers–Brillouin (WKB) approximation, we have the wavefunction

$$\psi \sim x^{-1/4}\sin\left[\int_0^x \sqrt{-\frac{1}{L^4} + \frac{1}{x}}\mathrm{d}x - \frac{\pi}{4}\right] \tag{32}$$

for the classical region $0 < x < x_{\min}$; $x_{\min}$ is related to $a_{\min}$ by $a_{\min} = 1/L^2 x_{\min}$. The energy density is quantized as

$$\rho_0 = \frac{3}{4\pi}(k+1), \qquad (k = 0, 1, 2, \cdots). \tag{33}$$

For the classical forbidden region $x > x_{\min}$ we have the solution

$$\psi = \frac{1}{p}\exp\left(-\int_{x_{\min}}^x p\mathrm{d}x\right), \tag{34}$$

where p is defined by $p = |-1/L^4 + 1/x|$.

5.2. $\lambda \neq 0$

In this case, the wormhole equation is reduced to

$$\dot{a}^2 = 1 - \frac{L^2}{a} \mp H^2 a^2. \tag{35}$$

We find that their solution can be written as

$$\begin{aligned}\tau =& \frac{1}{2H^2(1 \mp 8H^2L^2)^{1/4}} \\ &\times [\mp 2\sqrt{1 \mp 8H^2L^2}E(m,n) + F(m,n) \\ &- (\mp)F(m,n)\sqrt{1 \mp 8H^2L^2}],\end{aligned} \tag{36}$$

where F and E are the elliptic integrals of the first and the second kinds, respectively, and m, n are defined by

$$\begin{aligned} m &\equiv \sqrt{1 + \frac{4H^2 a}{\mp 1 + \sqrt{1 \mp 8H^2L^2}}}, \\ n &\equiv \frac{\sqrt{2}}{2}\sqrt{1 \mp \frac{1}{\sqrt{1 \mp 8H^2L^2}}}. \end{aligned} \tag{37}$$

For $4/(9L^4) > \lambda > 0$, the wormhole has a maximum and minimum scales given by

$$a_{\min} = \frac{2}{3}\sqrt{3}\frac{1}{H}\cos(\theta), \tag{38}$$

$$a_{\max} = \frac{2}{3}\sqrt{3}\frac{1}{H}\cos\left(\theta + \frac{4}{3}\pi\right), \tag{39}$$

where θ is defined by

$$\theta = \frac{1}{3}\arccos\left(-\frac{\sqrt{3}L^2}{2H}\right). \tag{40}$$

Its conceptual shape is shown in Fig.2. For $\lambda < 0$, the wormhole has a minimum scale

$$\begin{aligned} a_{\min} =& \sqrt[3]{\frac{L^2}{2H^2} + \sqrt{\frac{L^4}{4H^4} + \frac{1}{27H^6}}} \\ &+ \sqrt[3]{\frac{L^2}{2H^2} - \sqrt{\frac{L^4}{4H^4} + \frac{1}{27H^6}}}. \end{aligned} \tag{41}$$

The corresponding WD equation is

$$\frac{\partial^2\psi}{\partial a^2} - \left(1 - \frac{L^2}{a} \mp H^2 a^2\right)\psi = 0. \tag{42}$$

Setting $a = 1/L^2 x$, then Eq. (42) becomes

$$\frac{\partial^2\psi}{\partial x^2} + \left(-\frac{1}{L^4} + \frac{1}{x} \mp \frac{H^2}{L^8}x^2\right)\psi = 0. \tag{43}$$

It is equivalent to the equation of motion of a particle in the potential of $-1/x - x^2H^2/L^4$.

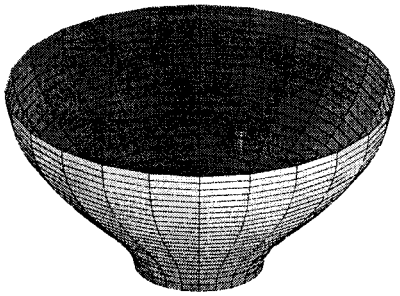


Fig.2. The conceptual shape of another wormhole.

Using WKB approximation, we obtain the wormhole wavefunction. When $\lambda > 0$, we have

$$\psi = \frac{2}{\sqrt{p}} \sin\left[\int_x^{x_{\min}} p\mathrm{d}x + \frac{\pi}{4}\right] \tag{44}$$

for the classical region $x < x_{\min}$; and

$$\psi = -\frac{1}{\sqrt{p}} \exp\left[-\int_{x_{\min}}^{x} p\mathrm{d}x\right] \tag{45}$$

for the classical forbidden region $x_{\min} < x < x_{\max}$; and

$$\psi = -\frac{1}{\sqrt{p}} \exp\left[\mathrm{i}\int_{x_{\max}}^{x} p\mathrm{d}x + \mathrm{i}\frac{\pi}{4}\right] \tag{46}$$

for the classical region $x_{\max} < x$. Here p is defined by

$$p = \sqrt{\left|-\frac{1}{L^4} + \frac{1}{x} - \frac{H^2}{L^8}x^2\right|}, \tag{47}$$

and $x_{\min}, x_{\max}$ are related to $a_{\min}, a_{\max}$ by $a_{\min} = 1/L^2 x_{\min}$ and $a_{\max} = 1/L^2 x_{\max}$. The quantum behaviour of the wormhole for $\lambda < 0$ is the same as that for $\lambda = 0$.

6. $\omega = 1/3$

6.1. $\lambda = 0$

The wormhole equation is

$$\dot{a}^2 = 1 - \frac{L^2}{a^2}. \tag{48}$$

It is easy to find the wormhole solution

$$a = \sqrt{\tau^2 + L^2}, \tag{49}$$

and the minimum scale

$$a_{\min} = L. \tag{50}$$

It has the conceptual shape shown as in Fig.1. The corresponding WD equation of the wormhole is given by

$$\frac{\partial^2\psi}{\partial a^2} - \left(1 - \frac{L^2}{a^2}\right)\psi = 0. \tag{51}$$

The quantum behaviour of this wormhole is the same as that in the case of $\omega = 0, \lambda = 0$.

6.2. $\lambda > 0$

The wormhole equation in this case can be written as

$$\dot{a}^2 = 1 - \frac{L^2}{a^2} - H^2 a^2. \tag{52}$$

Then we obtain the wormhole solution under the condition that $0 < \lambda < \sqrt{3}/(2L)$ as follows:

$$a = \frac{1}{\sqrt{2}H}\left[1 - \sqrt{1 - 4H^2L^2}\cos\left(2H\tau\right)\right]^{1/2}. \tag{53}$$

This kind of wormholes has both a minimum and a maximum scales

$$a_{\min} = \frac{1}{\sqrt{2}H}\left[1 - \sqrt{1 - 4H^2L^2}\right]^{1/2}, \tag{54}$$

$$a_{\max} = \frac{1}{\sqrt{2}H}\left[1 + \sqrt{1 - 4H^2L^2}\right]^{1/2}. \tag{55}$$

The conceptual shape is shown in Fig.2. The corresponding WD equation is

$$\frac{\partial^2\psi}{\partial a^2} - \left(1 - \frac{L^2}{a^2} - H^2a^2\right)\psi = 0. \tag{56}$$

The quantum behaviour of this wormhole is the same as the case of $\omega = 0, \lambda > 0$.

6.3. $\lambda < 0$

The wormhole equation becomes

$$\dot{a}^2 = 1 - \frac{L^2}{a^2} + H^2a^2. \tag{57}$$

We obtain the wormhole solution as follows:

$$\tau = \frac{1}{2H}\ln\frac{1 + 2H^2a^2 + 2\sqrt{H^4a^4 + H^2a^2 - H^2L^2}}{2H}, \tag{58}$$

which has a minimum scale

$$a_{\min} = \sqrt{\frac{-1 + \sqrt{1 + 4H^2L^2}}{2H^2}}. \tag{59}$$

The conceptual shape is the same as that in Fig.1. This solution was first studied by González-Diaz[43] and Barceló *et al.*[44] The corresponding WD equation is

$$\frac{\partial^2\psi}{\partial a^2} - \left(1 - \frac{L^2}{a^2} + H^2a^2\right)\psi = 0. \tag{60}$$

Solutions to the wavefunction can be generally given in terms of the confluent hypergeometric functions.

They can provide satisfactory quantum states for a wormhole.

7. $\omega = 2/3$

7.1. $\lambda = 0$

The wormhole equation becomes

$$\dot{a}^2 = 1 - \frac{L^2}{a^3}. \tag{61}$$

Letting $a = 1/u$, then we obtain the wormhole solution as follows:

$$\begin{aligned}\tau = &- \frac{\sqrt{1 - L^2u^3}}{u} \\ &- L^{2/3}\sqrt{-\frac{3}{2} - \frac{\mathrm{i}}{2}\sqrt{3}}E(m,n) \\ &- L^{2/3}\frac{1}{\sqrt{-\frac{3}{2} - \frac{\mathrm{i}}{2}\sqrt{3}}}F(m,n),\end{aligned} \tag{62}$$

where F and E are the elliptic integrals of the first and the second kinds, respectively, and m, n are defined by

$$m \equiv \frac{\sqrt{3}}{3}\sqrt{-\mathrm{i}\sqrt{3}\left(uL^{2/3} + \frac{1}{2} + \frac{\mathrm{i}}{2}\sqrt{3}\right)},$$

$$n \equiv \frac{\mathrm{i} + \sqrt{3}}{2}. \tag{63}$$

This wormhole also has a minimum scale

$$a_{\min} = L^{2/3}. \tag{64}$$

The conceptual shape is the same as in Fig.1. The corresponding WD equation is

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}a^2} - \left(1 - \frac{L^2}{a^3}\right)\psi = 0. \tag{65}$$

The discussion on the quantum behaviour is same as the above.

7.2. $\lambda \neq 0$

In this case, the wormhole equation can be written as

$$\dot{a}^2 = 1 - \frac{L^2}{a^3} \mp H^2a^2. \tag{66}$$

We are unable to find exact solutions for these equations. So we give the numerical results in Fig.3, in which we set $L^2 = 1$ and $H^2 = 0.1$. The solid and the dotted curves correspond to the negative and positive cosmological constants, respectively. For the positive cosmological constant, there exist a minimum $a_{\min}$ and a maximum $a_{\max}$ for the scale of the wormhole provided that $0 < \lambda < \lambda_{\mathrm{crit}}$, where $\lambda_{\mathrm{crit}} = 9/(5L)\sqrt[3]{4/L}$ is a critical value. When $\lambda > \lambda_{\mathrm{crit}}$, there is no wormhole solution to Eq. (66). $a_{\min}$ and $a_{\max}$ are the positive roots of the equation

$$1 - \frac{L^2}{a^3} - H^2a^2 = 0. \tag{67}$$

For $L^2 = 1$ and $H^2 = 0.1$, we have $a_{\min} = 1.0$, $a_{\max} = 3.1$. Thus the solid curve is a line segment. The conceptual shape of the wormhole is shown in Fig.2. For the negative cosmological constant, there is only one minimum scale $a_{\min}$ for the wormhole, which is the positive root of the equation

$$1 - \frac{L^2}{a^3} + H^2a^2 = 0. \tag{68}$$

For $L^2 = 1$ and $H^2 = 0.1$, we have $a_{\min} = 0.97$. The conceptual shape of this wormhole is shown in Fig.1.

The corresponding WD equation reads

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}a^2} - \left(1 - \frac{L^2}{a^3} \mp H^2a^2\right)\psi = 0. \tag{69}$$

The quantum behaviour is the same as in the case of $\omega = 0$.

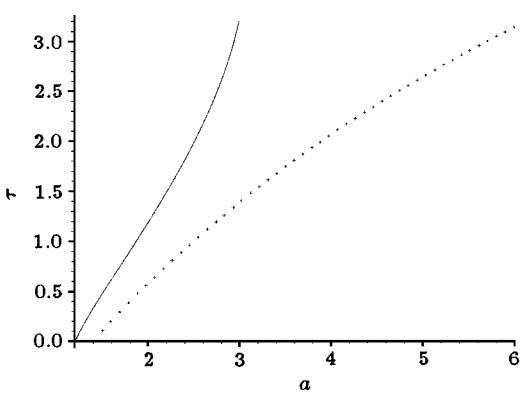


Fig.3. Numerical results for the a-τ relation.

8. $\omega = 1$

8.1. $\lambda = 0$

The wormhole equation becomes

$$\dot{a}^2 = 1 - \frac{L^2}{a^4}, \tag{70}$$

and the solution of it is

$$\begin{aligned}\frac{\tau}{\sqrt{L}} = &\frac{1}{\sqrt{2}}F\left[\arccos\frac{\sqrt{L}}{a}, \frac{1}{\sqrt{2}}\right] \\ &- \sqrt{2}E\left[\arccos\frac{\sqrt{L}}{a}, \frac{1}{\sqrt{2}}\right] \\ &+ \frac{1}{\sqrt{L}a}\sqrt{a^4 - L^4},\end{aligned} \tag{71}$$

where F and E are elliptic integrals of the first and the second kinds, respectively. This is just the Giddings–Strominger wormhole. The corresponding $a_{\min}$ is

$$a_{\min} = L. \tag{72}$$

The corresponding WD equation is

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}a^2} - \left(1 - \frac{L^2}{a^4}\right)\psi = 0. \tag{73}$$

The quantum behaviour is the same as in the case of $\omega = 0$ and $\lambda = 0$.

8.2. $\lambda > 0$

In this case, there exists a critical value of λ; when $\lambda > \lambda_{\mathrm{crit}}$, the requirement for the existence of $a_{\min}$ cannot be satisfied. Therefore, the wormhole solution does not exist. The λ_{crit} can be deduced from the equation

$$\dot{a}^2 = 1 - \frac{L^2}{a^4} - H^2a^2, \tag{74}$$

and we have

$$\lambda_{\mathrm{crit}} = \frac{2}{\sqrt{3}L}. \tag{75}$$

When $0 < \lambda < \lambda_{\mathrm{crit}}$, from Eq. (74) one has $\ddot{a} > 0$, so $a_{\min}$ exists. One thus has the solution of Eq. (74)

$$\tau = \frac{\beta}{H\left[\alpha\left(\beta-\gamma\right)\right]^{1/2}} \times \Pi\left[h\left(\frac{a^2}{L^2}\right), \frac{\alpha-\beta}{\alpha}, \left[\frac{\gamma\left(\beta-\alpha\right)}{\alpha\left(\beta-\gamma\right)}\right]^{1/2}\right], \tag{76}$$

where

$$h\left(\chi\right) = \arcsin\left[\frac{\alpha\left(\chi-\beta\right)}{\chi\left(\alpha-\beta\right)}\right]^{1/2}, \tag{77}$$

Π is the elliptic integral of the third kind, while $\alpha > \beta > 0 > \gamma$; α, β and γ are three real roots of the equation

$$x^3 - \frac{1}{H^2L^2}x^2 + \frac{1}{H^2L^2} = 0. \tag{78}$$

In this case, there exist an $a_{\max}$ and an $a_{\min}$ of the scale of the wormhole. $a_{\max}$ and $a_{\min}$ are two positive real roots of the equation

$$1 - \frac{L^2}{a^4} - H^2a^2 = 0. \tag{79}$$

The conceptual shape is the same in Fig.2. The corresponding WD equation is

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}a^2} - \left(1 - \frac{L^2}{a^4} - H^2a^2\right)\psi = 0. \tag{80}$$

The quantum behaviour is the same as in the case of $\omega = 0$ and $\lambda \neq 0$.

8.3. $\lambda < 0$

Letting $u = 1/a^2$, we obtain the wormhole solution

$$\begin{aligned}\tau =& \frac{2\mathrm{i}}{\sqrt{3}}\left[\left(u + \frac{\beta}{2} + \frac{\alpha\sqrt{3}\mathrm{i}}{2}\right)\sqrt{3}\right]^{1/2} \\ &\times \left(\frac{2u-2\beta}{-3\beta-\sqrt{3}\alpha\mathrm{i}}\right)^{1/2} \\ &\times \left[\sqrt{3}\left(u + \frac{\beta}{2} - \frac{\sqrt{3}\alpha\mathrm{i}}{2}\right)\right]^{1/2} \\ &\times \Pi\left\{\frac{\sqrt{3}}{3}\left[\frac{-\mathrm{i}\sqrt{3}}{\alpha}\left(u + \frac{\beta}{2} + \frac{\alpha\sqrt{3}\mathrm{i}}{2}\right)\right]^{1/2}, \right. \\ &\left. \frac{2\alpha\sqrt{3}\mathrm{i}}{\beta+\sqrt{3}\alpha\mathrm{i}}, \left[\frac{2\alpha\sqrt{3}\mathrm{i}}{\beta+\sqrt{3}\alpha\mathrm{i}}\right]^{1/2}\right\} \\ &\times \frac{-2}{\left(\beta+\alpha\sqrt{3}\mathrm{i}\right)\sqrt{u - L^2u^3 + H^2}},\end{aligned} \tag{81}$$

where α and β are defined by

$$\begin{aligned}\alpha =& \frac{\left(108H^2L + 12\sqrt{3}\sqrt{-4+27H^4L^2}\right)^{1/3}}{6L} \\ &- \frac{2}{\left(108H^2L + 12\sqrt{3}\sqrt{-4+27H^4L^2}\right)^{1/3}}, \\ \beta =& \frac{\left(108H^2L + 12\sqrt{3}\sqrt{-4+27H^4L^2}\right)^{1/3}}{6L} \\ &+ \frac{2}{\left(108H^2L + 12\sqrt{3}\sqrt{-4+27H^4L^2}\right)^{1/3}}.\end{aligned} \tag{82}$$

There exists a minimum scale $a_{\min}$ for this wormhole. $a_{\min}$ satisfies the equation

$$1 - \frac{L^2}{a^4} + H^2a^2 = 0. \tag{83}$$

The conceptual shape is shown in Fig.1. The corresponding WD equation is

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}a^2} - \left(1 - \frac{L^2}{a^4} + H^2a^2\right)\psi = 0. \tag{84}$$

The quantum behaviour is the same as in the case of $\omega = 0$ and $\lambda \neq 0$.

9. $-1/3 < \omega \leq 1$

In general, under the condition that $\lambda = 0$ and $-1/3 < \omega \leq 1$, the wormhole equation becomes

$$\dot{a}^2 = 1 - \frac{L^2}{a^{1+3\omega}}. \tag{85}$$

From Eq. (85) we obtain a family of wormhole solutions

$$\tau = aF\left[\frac{1}{2}, -\frac{1}{1+3\omega}; \frac{3\omega}{3\omega+1}; L^2 a^{-1-3\omega}\right], \qquad (86)$$

where F is the generalized hypergeometric function. The minimum scale of the wormhole is given by

$$a_{\min} = L^{\frac{2}{3\omega+1}}. \qquad (87)$$

The conceptual shape of these wormholes is shown in Fig.1. The quantum behaviour is the same as in the case of $\omega = 0$.

10. Conclusion and discussion

In summary, starting from the Einstein field equations, we obtain the wormhole equation with a tachyon field which is minimally coupled with Hilbert–Einstein gravity in the presence of positive and negative constants. Then specializing at several particular values of ω, we obtain a series of exact solutions for the classical wormholes. It is shown that the classical wormhole solutions belong to only two broad categories: they represent either two back-to-back speakers or a no-bottom bowel. The former has a minimal scale and the latter has both a minimal and a maximal scales. This is closely related to the presence of cosmological constant. Therefore, we should consider them separately. The discussion on quantum states of some wormholes reveals that the energy density ρ_0 of the tachyon field is quantized.

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