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Improving the prediction of chaotic time series*

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One of the features of deterministic chaos is sensitive to initial conditions. This feature limits the prediction horizons of many chaotic systems. In this paper, we propose a new prediction technique for chaotic time series. In our method, some neighbouring points of the predicted point, for which the corresponding local Lyapunov exponent is particularly large, would be discarded during estimating the local dynamics, and thus the error accumulated by the prediction algorithm is reduced. The model is tested for the convection amplitude of Lorenz systems. The simulation results indicate that the prediction technique can improve the prediction of chaotic time series.

Keywords: chaotic time series, local Lyapunov exponent, neighbouring point, neural network
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1. Introduction

In recent years, there has been considerable advance in the prediction of chaotic time series where the underlying dynamics can be described as low dimensional. There are some new nonlinear models (radial basis function,[1] neural nets,[2] local linear models,[3] filter,[4], etc.) and new methods of fitting the models to experimental data (wavelets,[5] genetic algorithm,[6], etc). There has also been advance in the theory and the application of ideas from information theory.[7,8] Furthermore, the methodology based on the predictability of the systems provides a powerful tool for analysing the nonlinear systems, such as estimating the largest Lyapunov exponent,[9] discriminating between chaos and noise.[10,11]

One of the difficulties arising from the prediction techniques is how to estimate the dynamics from the experimental data. The complex structure of the attractor and the noise prevent predictors from approaching the theoretical limits. Kostelich and Yorke[12] proposed a method first suggested by Eckmann and Ruelle[13] for computing the Lyapunov exponent. Gong and Lai[14] outlined an approach to choose alternative and more relevant neighbouring points (ϵ^p neighbouring points). Alparslan *et al*[15] suggested a local forecasting model based upon the time evolution of the topological neighbouring in the reconstructed phase space. Jiménez *et al*[16] presented an alternative forecasting technique for chaotic time series, which is based on the optimal association concept. Doerner *et al*[17] suggested a method to reject those forecasts with low reliability immediately by calculating the predictability simultaneously with the forecast.

2. Approximation of dynamic systems

A standard procedure in the analysis of chaotic experimental data is to reconstruct the attractor using the time delay method. Takens[18] showed that under suitable hypotheses, the trajectory in phase space made from time series by embedding will have the same topological properties as the original attractor. Suppose a time series $\{x_t\}$, $t = 1, \cdots, N$, is given, the data are represented in m-dimensional space by vectors

$$x_t = [x_t, x_{t+1}, \cdots, x_{t+(m-1)}], \tag{1}$$

where m is the embedding dimension. The reconstructed attractor consists of points of the vectors x_t. Usually, if the original attractor has a correlation dimension γ, then an embedding dimension of $2\gamma + 1$ will be adequate for reconstructing the attractor.[18]

The prediction of chaotic time series is based on the reconstruction of chaotic attractor. Most of them

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include the estimation of a derivative from the data or require the estimation of the location of some portion of the attractor. The prediction model can be constructed by using as much dynamical information as possible in the formulation of the relationship between individual observations. For example, we use the recurrent orbits to derive an accurate model of the dynamics in the vicinity of saddle periodic orbits on an attractor.

While estimating the local dynamics of a chaotic system, one of the important steps is how to choose the neighbours of the predicted point. For a given predicted point in the reconstruction, there exist several neighbouring points around it. Since the deterministic chaos depend sensitively on initial conditions, the trajectories starting at two similar state vectors will diverge exponentially. After one or several iterations, these neighbouring points tend to expand in one direction and contract in others. Some of these neighbouring points will greatly limit one's ability to estimate the local dynamics from experimental data because of the complex structure of the attractor and the noise. We call them false neighbouring points. Figure 1 is an illustration of this situation. In Fig. 1, the predicted point is $x_r(t)$, which is located on one portion of the attractor, $x_{ri}(t)$ and $x_{rj}(t)$ are the neighbouring points of the predicted point. It is evident that the point $x_{ri}(t)$ would decrease the approximation accuracy, since the deviation at the next time is large. Although the false neighbouring points possibly involve the dynamical information of attractor, they are not suitable for the dynamical estimation of the chaotic system. We need eliminate them to raise the estimation accuracy of the chaotic system. In this paper, we present a method to choose neighbours of the prediction point by local Lyapunov exponent, and thus raise the prediction accuracy.

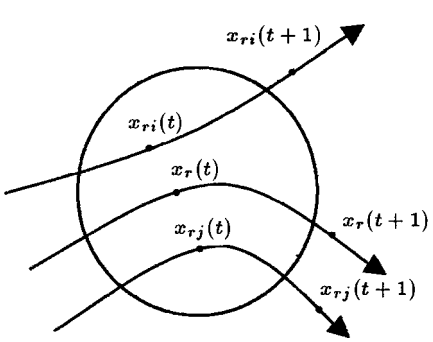


Fig.1. Schematic diagram of diverging trajectories on an attractor.

3. ε^λ neighbouring point

Time series prediction is to deduce the future values of time series according to its past values. The prediction can be described as

$$x_{t+T} = f(x_t), \tag{2}$$

$$\begin{aligned} &f : (x_t, x_{t+1}, \cdots, x_{t+(m-1)}) \\ &\rightarrow (x_{t+T}, x_{t+1+T}, \cdots, x_{t+(m-1)+T}), \end{aligned} \tag{3}$$

where $x_{t+(m-1)+T} = v(x_t, x_{t+1}, \cdots x_{t+(m-1)})$, v is an unknown function and T is the predicted step. The derivative of map function f is the matrix[19]

$$(f')_{x(t)} = \begin{pmatrix} v_m' & v_{m-1}' & v_{m-2}' & \cdots & v_2' & v_1' \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

where $v_m' = \dfrac{\partial v}{\partial x_{t+m-1}}, \cdots, v_1' = \dfrac{\partial v}{\partial x_t}$. For the initial conditions $x_t + \delta x_t$, if δx_t is sufficiently small, we can linearize the map

$$\begin{aligned} x_{t+T} + \delta x_{t+T} =& f(x_t + \delta x_t) \\ \cong& f(x_t) + f'(x_t)\delta x_t + \cdots \end{aligned} \tag{4}$$

Using Eq. (2), we obtain the linearized error propagation equation

$$\delta x_{t+T} \cong f'(x_t)\delta x_t, \tag{5}$$

$$\| \delta x_{t+T} \| \cong \| f'(x_t) \| \| \delta x_t \| . \tag{6}$$

The local Lyapunov exponent reads

$$\lambda(t)^{\text{local}} = \frac{1}{T} \ln \left(\left\| \frac{\delta x_{t+T}}{\delta x_t} \right\| \right) \cong \frac{1}{T} \ln(\| f'(x_t) \|). \tag{7}$$

For a chaotic system, it is known that sensitivity to initial conditions limits the prediction horizons. In most cases, we can only make short-term predictions in which the results are reliable. Lyapunov exponent gives some suggestion of prediction horizons on average over all of the attractors.[20,21] However, Lyapunov exponents are less than satisfactory because the prediction horizons vary with the positions in phase space. Local Lyapunov exponents may be more useful. Since the local Lyapunov exponent varies with its location on the attractor, the predictability of a reconstruction varies with the location accordingly.

Local Lyapunov exponent describes how the nearby trajectories diverge exponentially in phase space. The larger the local Lyapunov exponent, the faster the nearby trajectories diverge. If the nearby trajectories diverge too fast, the deviation at the next time would be large. From Fig. 1, this will lead to the fast amplification of error in making predictions. So we postulate that some neighbouring points, for which the corresponding local Lyapunov exponent is particularly large, cannot be used to estimate the local dynamics. In this paper, we use the ε^λ neighbouring points instead of the ordinary neighbouring points during the estimation of the local dynamics. In phase space, we let $x_r(t)$ denote the predicted point at time t, $x_{r_i}(t)$ is its i-th neighbour, $i = 1, \cdots, k$, k is the number of the neighbouring points, $D_i(t)$ is the distance from the predicted point $x_r(t)$ to its i-th neighbouring point $x_{r_i}(t)$, $D_i(t) = \|x_r(t) - x_{r_i}(t)\|$. The neighbouring points which are called ε^λ neighbours should satisfy two conditions: (1) $D_i(t) \leq \varepsilon$ and (2) $\lambda_i^{\text{local}}(t) < \sigma$, where ε and σ are arbitrary parameters. The first condition indicates that the ε^λ neighbours must be located within a given region, and the second condition indicates that the local Lyapunov exponent of ε^λ neighbours is smaller than a given value. This means that we will eliminate some neighbouring points for which the corresponding local Lyapunov exponent is larger than a given value and whose location is outside a given region.

4. Results

In Ref. [22], it was pointed out that both a function and its derivatives can be approximated with a single-layer feed-forward network. Here, we use a feed-forward neural network to approximate the unknown function v and its derivation, and to calculate the local Lyapunov exponent using Eq. (7). We test the chaotic time series which is generated by Lorenz equations. The T-step prediction error is measured with the root-mean-square error(RMSE)

$$N_{\text{RMSE}} = \sqrt{\frac{(1/n)\sum_{j=1}^{n}(x_{j+T} - \hat{x}_{j+T})^2}{(1/N)\sum_{j=1}^{N}(x_j - \bar{x})^2}}, \qquad (8)$$

where $\hat{x}_j$ is the desired value of x_j, $\bar{x}$ is the mean value of data, N is the length of the time series, and n is the length of the test set.

The Lorenz equations have been studied extensively, they can produce signals seemingly to be complex and mimic stochastic. The ordinary differential equations are[23]

$$\begin{aligned} \dot{x} &= -ax + ay, \\ \dot{y} &= -xz + bx - y, \\ \dot{z} &= xy - cz. \end{aligned} \qquad (9)$$

The parameters in Eqs. (9) are most commonly taken to be a=10, b=28, and c=8/3. For generating the data, we integrate the Lorenz equations with a step size of 1×10^{-2}. The starting values of the variables x, y and z are x=10, y=10, z=10. A feed-forward neural network is used to predict the Lorenz series which is generated from the x coordinate of the Lorenz system. The network's architecture is one hidden layer with four input neurons, three hidden neurons and one output neuron.

We select randomly the time series with a length of N=1000. The length of the test is set to be 500. One-step predicted results are shown in Figs. 2 and 3. The results show how the predicted errors vary with σ. In Fig. 2, for the case of ε=2, the predicted errors, which decline at first with σ, increase gradually, and then change slowly. In Fig. 3, for the case of ε =3, it can be seen that the predicted errors grow considerably at around σ =2. From Figs. 2 and 3, we can find two points: (1) With σ increasing systematically, more neighbouring points can be included so that the prediction model can utilize more local information to estimate the dynamic system. This is the cause that the predicted errors decline at first in Figs. 2 and 3. (2) As σ is larger in Fig. 3, some neighbouring points, for which the corresponding value of $\lambda_i^{\text{local}}(t)$ is particularly large, would increase the error accumulated by the prediction algorithm. In fact, the predicted results using the ordinary neighbours diverge significantly from the actual continuation when ε is larger. If we remove some neighbouring points for which the corresponding value of $\lambda_i^{\text{local}}(t)$ is larger than a given values, such as $\lambda_i^{\text{local}}(t) > 2$ (i.e. $\sigma = 2$), we can reduce the predicted errors. So the prediction using the ε^λ neighbours is advantageous to the prediction using the ordinary neighbours. Figure 4 shows the predicted values of Lorenz time series compared with the desired values, ε and σ are taken to be ε=3, σ=1.5. The dashed line denotes the predicted values obtained using ε^λ neighbours. The dotted line denotes those using the ordinary neighbours.

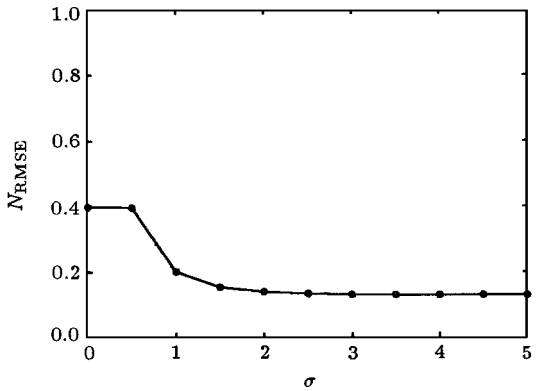


Fig.2. Normalized root-mean-square errors as functions of σ.

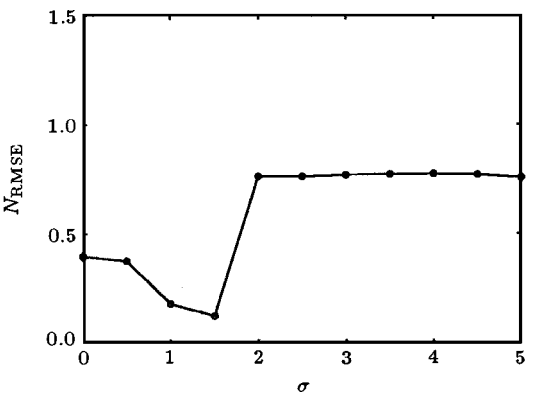


Fig.3. Normalized root-mean-square errors as functions of σ.

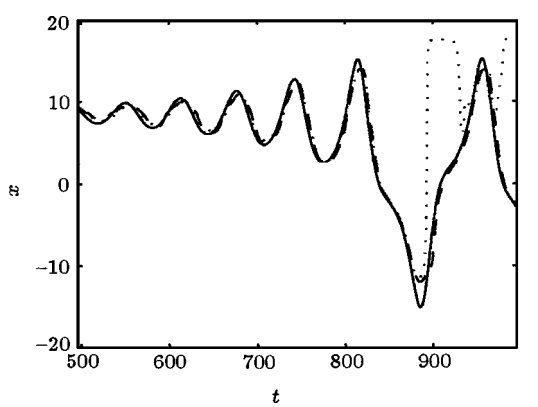


Fig.4. The predicted values of Lorenz time series compared with the desired values.

Other two examples are shown in Figs. 5 and 6. The results show the predicted values of Lorenz time series compared with the actual continuation. The dashed line denotes the predictions using ε^{λ} neighbours, and the dotted line denotes those using the ordinary neighbours. In making predictions, we set ε=3, σ=2.1 in Fig. 5, and ε =4.2, σ=1.5 in Fig. 6. The predicted step is set to be 1. From Figs. 5 and 6, it can be seen that the predictions using the ordinary neighbouring points follow the actual trajectory for a certain period of time, and then start diverging from the actual one. If we eliminate some neighbouring points for which the corresponding local Lyapunov exponent is the larger value of $\lambda_i^{\mathrm{local}}(t)$, the predicted results are close to the actual continuation.

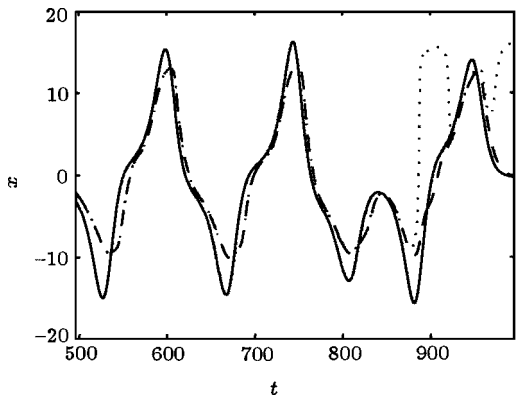


Fig.5. Predicted and desired values of the x coordinate of the Lorenz system, ε=3, σ=2.1.

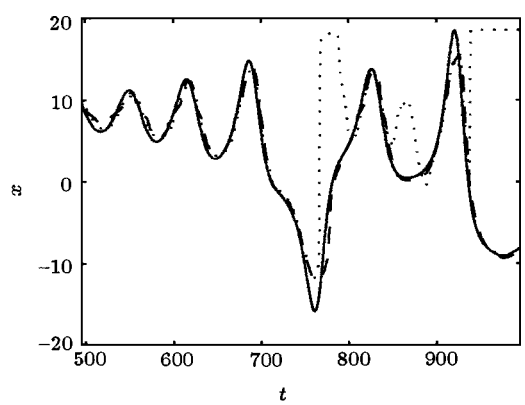


Fig.6. Predicted and desired values of the x coordinate of the Lorenz system, ε=4.2, σ=1.5.

5. Conclusion

Chaotic system is sensitive to its initial conditions. As a result, the errors accumulated by the prediction algorithm grow exponentially. In the reconstruction of attractor, the divergence of nearby trajectories varies with their locations on the attractor. Local Lyapunov exponent characterizes the feature. This means that the predictability of a reconstruction is related to the local Lyapunov exponent. In this paper, we eliminate some neighbouring points for which the corresponding local Lyapunov exponent is

particularly large during estimating the local dynamics of a chaotic system, and thus reduce the influence of these points. In fact, these neighbouring points are true neighbours of the prediction point and possible involve some local dynamics information, but they are not suitable for the local estimation of the chaotic system.

Parameters ε and σ are arbitrary in our approach. One can formulate an algorithm by choosing their values. On many occasions, σ is the larger value of $\lambda_i^{\mathrm{local}}(t)$. Our model has been tested for the x coordinate of the Lorenz system. The simulation results indicate that the predictions of chaotic time series can be improved if we choose suitable parameters ε and σ. So, we think that it is a worthy mechanism for the prediction of chaotic time series.

References

[1] Casdagli M 1989 *Physica* D **35** 335
[2] Barron A R 1993 *IEEE Trans. Inform. Theory* **39** 930
[3] Jimenez J, Moreno J A and Ruggeri G J 1992 *Phys. Rev.* A **45** 3553
[4] Zhang J S and Xiao X C 2000 *Acta Phys. Sin.* **49** 1221 (in Chinese)
[5] Parlitz U and Mayer-Kress G 1995 *Phys. Rev.* E **51** 2709
[6] Szpiro G G 1997 *Phys. Rev.* E **55** 2557
[7] Rissanen J 1989 *Stochastic Complexity in Statistical Inquiry* Vol.15 (Singapore: World Scientific)
[8] Feng G L, Dai X G, Wang A H and Chou J F 2001 *Acta Phys. Sin.* **50** 606 (in Chinese)
[9] Walse D J 1991 *Nature* **350** 485
[10] Ikeguchi T and Ahara K 1997 *Phys. Rev.* E **55** 2530
[11] Barahona M and Poon C S 1996 *Nature* **381** 215
[12] Kostelich E J and Yorke J A 1990 *Physica* D **41** 183
[13] Eckmann J P and Ruelle D 1985 *Rev. Mod. Phys.* **57** 617
[14] Gong X F and Lai C H 1999 *Phys. Rev.* E **60** 5463
[15] Alparslan A K, Sayar M and Atilgan A R 1998 *Phys. Rev.* E **58** 2640
[16] Jiménez J, Moreno J A and Ruggeri G J 1992 *Phys. Rev.* A **45** 3553
[17] Doerner R, Hübinger B and Martienssen W 1994 *Phys. Rev.* E **50** R12
[18] Takens F 1981 *Lecture Notes in Mathematics* Vol. 898 (New York: Springer) p366
[19] Gencay R and Dechert W D 1992 *Physica* D **59** 142
[20] Rosenstein M T, Collins J J and De Luca C J 1993 *Physica* D **65** 117
[21] Froyland G, Judd K and Mees A I 1995 *Phys. Rev.* E **51** 2844
[22] Gallant A R and White H 1992 *Neural Networks* **5** 129
[23] Lorenz E N and Atmos J 1963 *Science* **20** 130