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Effects of cross-correlated noises on the relaxation time of the bistable system*

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The stationary correlation function and the associated relaxation time for a general system driven by cross-correlated white noises are derived, by virtue of a Stratonovich-like ansatz. The effects of correlated noises on the relaxation time of a bistable kinetic model coupled to an additive and a multiplicative white noises are studied. It is proved that for small fluctuations the relaxation time T_c as a function of λ (the correlated intensity between noises) exhibits very different behaviours for $\alpha < D$ and for $\alpha > D$ (α and D, respectively, stand for the intensities of additive and multiplicative noises). When $\alpha > D$, T_c increases with increasing λ. But when $\alpha < D$, T_c increases with λ for the case of weak correlated noises and sharply decreases with λ for the case of strong correlated noises, and thus T_c-λ curve behaves with one extremum.

Keywords: correlated noise, relaxation time, bistable system
PACC: 0540

1. Introduction

It has been found that the weak noise acting upon a dynamical system can have far-reaching consequences and can bring about interesting phenomena.[1−10] In most of the previous works, noises that are present simultaneously in a physical process are usually assumed to have different origins and are treated as uncorrelated random variables. However, Fulinski and Telejko[11] pointed out that noises in some stochastic processes may have a common origin and thus will not be independent. Recent studies of noise have revealed that the presence of cross-correlated noises may change the dynamics of a system.

The cross-correlation effects of additive and multiplicative noises, either on the stationary state or on the kinetics of a bistable potential system, have attracted considerable interests. For instance, taking into account the cross-correlation between additive and multiplicative Gaussian white noises, Fulinski A and Telejko,[11] Cao and co-workers[12] and Jia and Li[13], respectively, studied the stationary probability distribution (SPD) and pointed out that the SPD is strongly influenced by the correlation intensity between noises. Jia and Li[14] and Mei *et al*[15] respectively, studied the mean first-passage time (MFPT) of a bistable system for the case of zero cross-correlation time[14] and that of nonzero cross-correlation time.[15] Madureira *et al*[5] proved that as a function of the correlation strength between additive and multiplicative noise sources the activating rate of a bistable system can be suppressed by an order of magnitude. Also we have noticed that Fox discussed N-component stochastic processes with correlations between the noises, and derived an explicit equation for the probability distribution to the first order of the correlation time.[16]

For a system coupled to noises, the two-time correlation function of the state variable and the associated relaxation time are also very important aspects of dynamical behaviour. As indicated in Ref. [17], if we are considering a measurable quantity $x(t)$ which fluctuates with time, in practice we can sometimes determine the distribution of the value of x, though more usually, what is available at one time are the mean and the variance. However, the mean and the variance do not inform a great deal about the underlying dynamics of what is happening. What would be of interest is some quantity which is a measure of the influence of a value of x at time t on the value at time $t+\tau$. Such a quantity is the correlation function. Stratonovich and San Miguel, respectively, studied the correlation function for the bistable model driven by a

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multiplicative noise based on Stratonovich's ansatz.[18] In this paper, our interest is to discuss the effects of the cross-correlated noises on the correlation function of the system state variable and the associated relaxation time. In section 2, we obtain an analytic expression of the stationary correlation function for a general system driven by cross-correlated noises. In section 3, the result obtained is applied to the bistable kinetic model. A discussion of the results concludes this paper in section 4.

2. The stationary correlation function

Considering a one-dimensional Langevin type of equation (LE) with N component noises $\eta_\mu(t)$, we have

$$\dot{x} = h(x) + g_\mu(x)\eta_\mu(t), \tag{1}$$

in which μ=1, 2, ..., N, and the summation over repeated indices is implicit. The dot over x in Eq.(1) denotes the time derivative. $g_\mu(x)$ may be non-linear functions of the process x. The noise components, $\eta_\mu(t)$, are assumed to be Gaussian white noises with zero mean, and to have correlations specified by

$$\theta_{\mu\nu}(t,t') = \langle \eta_\mu(t)\eta_\nu(t')\rangle = 2D_{\mu\nu}\delta(t-t'). \tag{2}$$

$D_{\mu\mu}$ stands for the strength of the μth noise component $\eta_\mu(t)$. If $D_{\mu\nu} \neq 0$ ($\mu \neq \nu$), then the corresponding two noise components are correlated to each other and the parameter $D_{\mu\nu}$ measures the intensity of cross-correlation between them; otherwise, they are not correlated.

Our work is based on the Fokker–Planck equation (FPE) corresponding to Eq.(1). The FPE corresponding to LE (1) with Eq.(2) can be written as Fox's equation for the probability distribution function $P(x,t)$[16] in the limit of the correlation time going to zero:

$$\frac{\partial P(x,t)}{\partial t} = L_{\rm FP}(t)P(x,t), \tag{3}$$

with

$$\begin{aligned} L_{\rm FP}(t) =& -\frac{\partial}{\partial x(t)}h[x(t)] \\ &+ D_{\mu\nu}\frac{\partial}{\partial x(t)}g_\mu[x(t)]\frac{\partial}{\partial x(t)}g_\nu[x(t)]. \end{aligned} \tag{4a}$$

The adjoint operator of $L_{\rm FP}$ is given by

$$\begin{aligned} L^+_{\rm FP}(t) =& h[x(t)]\frac{\partial}{\partial x(t)} \\ &+ D_{\mu\nu}g_\mu[x(t)]\frac{\partial}{\partial x(t)}g_\nu[x(t)]\frac{\partial}{\partial x(t)}. \end{aligned} \tag{4b}$$

In this paper we are interested in the stationary two-time correlation function of $x(t)$ governed by LE (1), and the associated relaxation time. For a general model for which a stationary state exists, the stationary correlation function of the system variable x for the different times t and $t+\tau$ ($\tau \geq 0$) may then be defined by

$$C(\tau) = \langle x(t+\tau)x(t)\rangle_s - \langle x(t)\rangle_s^2, \tag{5}$$

where τ denotes the time difference. From Eqs. (5) and (1), we have

$$\begin{aligned} \frac{{\rm d}C(\tau)}{{\rm d}\tau} =& \langle h[x(t+\tau)]x(t)\rangle_s \\ &+ \langle g_\mu[x(t+\tau)]\eta_\mu(t+\tau)x(t)\rangle_s. \end{aligned} \tag{6}$$

The second term in the last equation can be calculated by the Novikov theorem,[19] yielding

$$\begin{aligned} &\langle g_\mu[x(t+\tau)]\eta_\mu(t+\tau)x(t)\rangle_s \\ &= \int_0^{t+\tau} {\rm d}t_1\theta_{\mu\nu}(t+\tau,t_1)\left\langle \frac{\delta g_\mu[x(t+\tau)]}{\delta\eta_\nu(t_1)}x(t)\right\rangle_s \\ &\quad + \int_0^{t} {\rm d}t_1\theta_{\mu\nu}(t+\tau,t_1)\left\langle g_\mu[x(t+\tau)]\frac{\delta x(t)}{\delta\eta_\nu(t_1)}\right\rangle_s \\ &= \int_0^{t+\tau} {\rm d}t_1\theta_{\mu\nu}(t+\tau,t_1) \\ &\quad \times \left\langle \frac{\partial g_\mu[x(t+\tau)]}{\partial x(t+\tau)}\frac{\delta x(t+\tau)}{\delta\eta_\nu(t_1)}x(t)\right\rangle_s, \end{aligned} \tag{7}$$

where we have used causality in the sense that $\delta x(t)/\delta\eta(t_1) \equiv 0$ if $t < t_1$.[20] The response function $\delta x(t+\tau)/\delta\eta_\nu(t_1)$ in Eq.(7) can be given from Eq.(1):

$$\begin{aligned} \frac{\delta x(t+\tau)}{\delta\eta_\nu(t_1)} =& g_\nu[x(t_1)]\exp\left\{\int_{t_1}^{t+\tau}{\rm d}s\Big[h'[x(s)] \right. \\ &\left. + g'_\mu[x(s)]\eta_\mu(s)\Big]\right\}, \end{aligned} \tag{8}$$

in which $\partial/\partial x$ is denoted by a prime. Then, according to Stratonovich integral, the delta function occurring in the limit of an integral has 1/2 its weight counted.[17] We further obtain the following form

$$\begin{aligned} &\langle g_\mu[x(t+\tau)]\eta_\mu(t+\tau)x(t)\rangle_s \\ &= D_{\mu\nu}\lim_{t_1\to t+\tau}\left\langle \frac{\partial g_\mu[x(t+\tau)]}{\partial x(t+\tau)}\frac{\delta x(t+\tau)}{\delta\eta_\nu(t_1)}x(t)\right\rangle_s \\ &= D_{\mu\nu}\left\langle g_\mu[x(t+\tau)]\frac{\partial g_\nu[x(t+\tau)]}{\partial x(t+\tau)}x(t)\right\rangle_s. \end{aligned} \tag{9}$$

Substituting Eq. (9) into Eq. (6), and in terms of operator $L^+_{\rm FP}$, Eq. (6) can be reduced to

$$\frac{{\rm d}C(\tau)}{{\rm d}\tau} = \langle x(t)L^+_{\rm FP}(\tau)x(t+\tau)\rangle_s. \tag{10}$$

As a first step toward the understanding of the effects of the correlation between noises on the correlation function, we here use an approximation by Stratonovich, i.e. Stratonovich decoupling ansatz[18]

$$\frac{\langle x(t)L^{+}_{\rm FP}(\tau)x(t+\tau)\rangle_s}{\langle x(t)L^{+}_{\rm FP}(t)x(t)\rangle_s} = \frac{\langle x(t+\tau)x(t)\rangle_s - \langle x(t)\rangle_s^2}{\langle x^2\rangle_s - \langle x\rangle_s^2}. \tag{11}$$

Finally, we find

$$C(\tau) = C(0)\exp[-\gamma(\lambda)\tau], \tag{12}$$

where $C(0)$ represents the value of $C(\tau)$ at the time difference $\tau = 0$. $C(0)=\langle(\delta x)^2\rangle_s=\langle x^2\rangle_s - \langle x\rangle_s^2$. It is clearly seen that the stationary correlation function $C(\tau)$ decays with increasing time difference τ, and the decay rate depends on the factor of $\gamma(\lambda)$,

$$\begin{aligned}\gamma(\lambda) &= -\frac{\langle x(t)L^{+}_{\rm FP}(t)x(t)\rangle_s}{\langle(\delta x)^2\rangle_s} \\ &= -\frac{\langle[h(x)+D_{\mu\nu}g_\mu(x)g'_\nu(x)]x\rangle_s}{\langle(\delta x)^2\rangle_s} \\ &= \frac{D_{\mu\nu}\langle g_\mu(x)g_\nu(x)\rangle_s}{\langle(\delta x)^2\rangle_s}.\end{aligned} \tag{13}$$

In the stationary state, $T_{\rm c}$, known as the relaxation time, can be defined for arbitrary processes x as

$$T_{\rm c}(\lambda) = \int_0^{+\infty}\frac{C(\tau)}{\langle(\delta x)^2\rangle_s}{\rm d}\tau, \tag{14}$$

which is independent of the precise function form of the correlation function. Substituting Eq. (12) into Eq. (14) and integrating, we obtain

$$T_{\rm c}(\lambda) = \frac{1}{\gamma(\lambda)} = \frac{\langle x^2\rangle_s - \langle x\rangle_s^2}{D_{\mu\nu}\langle g_\mu(x)g_\nu(x)\rangle_s}. \tag{15}$$

3. Time correlation function for a bistable kinetic system driven by correlated noises

Now we apply the results obtained above to an important one-dimensional (1D) system, the bistable kinetic system driven by additive and multiplicative Gaussian white noises with zero mean, and assume the dimensionless form

$$\dot{x} = x - x^3 + xp(t) + q(t), \tag{16}$$

where

$$\langle p(t)p(t')\rangle = 2D\delta(t-t'), \tag{17a}$$

$$\langle q(t)q(t')\rangle = 2\alpha\delta(t-t'), \tag{17b}$$

$$\langle p(t)q(t')\rangle = 2\lambda\sqrt{D\alpha}\delta(t-t'). \tag{17c}$$

The parameter λ reflects the degree of the cross-correlation between $p(t)$ and $q(t)$. Setting $g_1(x) = x$, $g_2(x) = 1$, and then substituting them into Eqs. (3), (4a) and (12), (13) as well as Eq. (15), thus we obtain the FPE corresponding to Eq. (16)

$$\begin{aligned}\partial_t P(x,t) =& -\partial_x(x - x^3 + Dx + \lambda\sqrt{D\alpha})P(x,t) \\ &+ \partial_x^2(Dx^2 + 2\lambda\sqrt{D\alpha} + \alpha)P(x,t),\end{aligned} \tag{18}$$

and the expressions of two-time correlation function of x and the associated relaxation time:

$$C(\tau) = \langle(\delta x)^2\rangle_s\exp[-\tau/T_{\rm c}(\lambda)], \tag{19}$$

$$T_{\rm c}(\lambda) = \frac{\langle x^2\rangle_s - \langle x\rangle_s^2}{D\langle x^2\rangle_s + 2\lambda\sqrt{D\alpha}\langle x\rangle_s + \alpha}. \tag{20}$$

To compute the stationary time correlation function $C(\tau)$ (19) and the relaxation time $T_{\rm c}$ (20), we have to acquire the SPD of the bistable system described by Eq. (16). According to Eq. (18) this leads to the following expressions of SPD

$$P_{\rm st}(x) = NG(x)^{-1/2}\exp\left[-U(x)/\alpha\right]. \tag{21}$$

Here the generalized potential $U(x)$ is[5,12]

$$\begin{aligned}U(x) =& -\int^x\frac{y-y^3}{Ry^2 + 2\lambda\sqrt{R}y + 1}{\rm d}y \\ =& f(x) + \frac{\widetilde{E}}{\sqrt{(1-\lambda^2)R}} \\ &\times\arctan\frac{\sqrt{R}x+\lambda}{\sqrt{(1-\lambda^2)}} + \widetilde{K}\ln G(x),\end{aligned} \tag{22}$$

in which

$$\begin{aligned}f(x) =& -\frac{2\lambda}{R\sqrt{R}}x + \frac{x^2}{2R}, \\ \widetilde{E} =& \frac{\lambda}{\sqrt{R}}\left[1 - \frac{(4\lambda^2-3)}{R}\right], \\ \widetilde{K} =& \frac{1}{2R}\left[\frac{(4\lambda^2-1)}{R} - 1\right],\end{aligned} \tag{23a}$$

$$G(x) = Rx^2 + 2\lambda\sqrt{R}x + 1, \quad R = \frac{D}{\alpha}. \tag{23b}$$

In Eq. (21), N is the normalization constant.

4. Discussion and conclusion

In order to illustrate the effects of the cross-correlated degree of additive and multiplicative noises λ upon the relaxation time $T_{\rm c}$, we plot the curves of

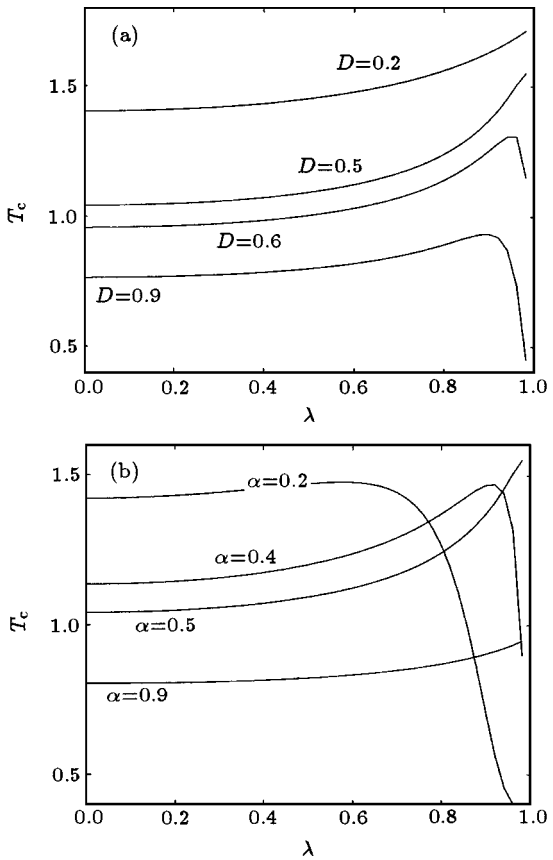


Fig.1. The relaxation time, T_c, as a function of λ. (a) $\alpha = 0.5$ is fixed and $D = 0.2$, 0.5, 0.6 and 0.9, respectively. (b)$D = 0.5$ is fixed and $\alpha = 0.2$, 0.4, 0.5 and 0.9, respectively.

T_c-λ in Fig.(1). It is interesting to note that the relaxation time T_c as a function of λ exhibits very different behaviours for $\alpha < D$ and for $\alpha > D$. When $\alpha > D$, T_c increases with increasing λ. λ always brings about a rise of T_c. But when $\alpha < D$, T_c increases with λ for the case of weak cross-correlation, and decreases sharply with λ for the case of strong cross-correlation, and thus T_c-λ curve behaves with one extremum. In fact, the model described by Eq. (16) represents the overdamped motion of a Brownian particle at position x in a double-well potential. For the bistable situation the relaxation of fluctuation implies an equilibration process between the populations of the wells.[18,21] When the noises are correlated, the symmetry of the stationary probability distribution of the bistable system is destroyed.[12] For small noise intensities, in the generalized potential the well at the neighbour of $x_1 = -1$ is deeper than the other at the neighbour of $x_2 = 1$. This means that the barrier for transition from x_1 to x_2 is much higher than that from x_2 to x_1 (see Fig. 2). As a result, the relaxation time is fundamentally related to the escape time for $x_1 \to x_2$. With the cross-correlated noises, the generalized force is[5]

$$\frac{x - x^3}{Rx^2 + 2\lambda\sqrt{R}x + 1}, \tag{24}$$

and it strongly depends on λ and so λ plays an important role. It is obvious that the generalized force increases within the interval $x \in (-1, 0)$ with increasing $\lambda(\lambda > 0)$, and the barrier over which a particle surmounts from x_1 to x_2 increases, as can be seen from Fig. 2. Therefore, the relaxation time increases with λ. However, we also note that in the case of $\alpha > D$, with increasing λ the height change of the barrier for transition from x_1 to $x_0 = 0$ is so small that T_c gradually grows. On the contrary, in the case of $D > \alpha$ the height of the barrier increases sharply with λ. For the strongly correlated noises (e.g. $\lambda = 0.9$, $D = 0.5$, $\alpha = 0.2$), the barrier is so high that the stochastic trajectories cannot cross it. Consequently, the relaxation of small fluctuations corresponds to an equilibration process in a single potential well. When λ increases, the potential well becomes deeper and therefore T_c decreases.

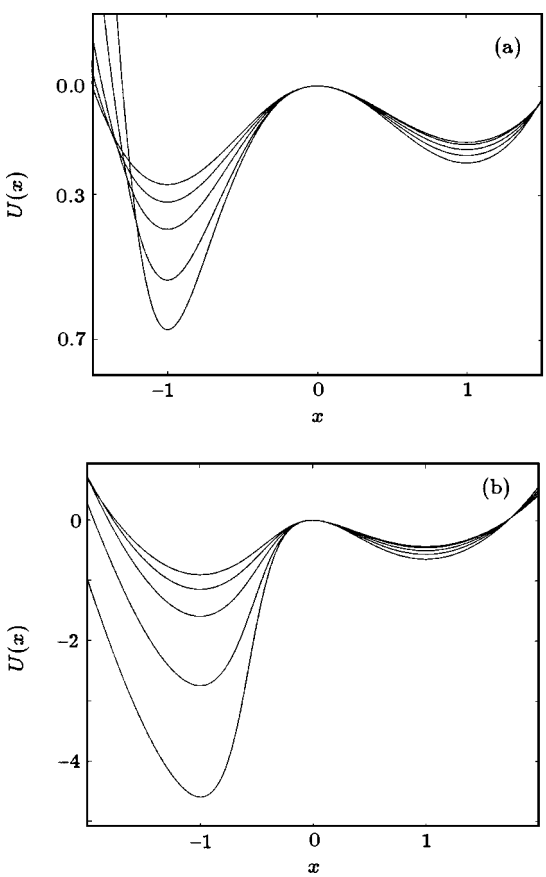


Fig.2. The generalized potential, $U(x)$, with different values of λ. λ=0.2, 0.4, 0.6, 0.8 and 0.9 from top to bottom at $x = -1$. (a)The case of $\alpha > D$. $\alpha = 0.9$ and $D = 0.5$. (b)The case of $\alpha < D$. $\alpha = 0.2$ and $D = 0.5$.

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