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New doubly periodic and multiple soliton solutions of the generalized (3+1)-dimensional KP equation with variable coefficients*

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A new generalized Jacobi elliptic function method is used to construct the exact travelling wave solutions of nonlinear partial differential equations (PDEs) in a unified way. The main idea of this method is to take full advantage of the elliptic equation which has more new solutions. More new doubly periodic and multiple soliton solutions are obtained for the generalized (3+1)-dimensional Kronig–Penny (KP) equation with variable coefficients. This method can be applied to other equations with variable coefficients.

Keywords: elliptic equation, Jacobi elliptic function, soliton solution
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1. Introduction

In recent years, directly searching for exact solutions of nonlinear partial differential equations (PDEs) has become more and more attractive partly due to the availability of computer symbolic systems such as Maple and Mathematica, which allow us to perform some complicated and tedious algebraic calculation on a computer as well as help us to find new exact solutions of PDEs.[1−5] A number of methods have been presented, such as inverse scattering theory,[6] Hirota's bilinear method,[7] the truncated Painlevé expansion,[8] the homogeneous balance method,[9−13] the hyperbolic tangent function series method,[14] the sine–cosine method,[15] the Jacobi elliptic function expansion method[16−19] and other methods.[20,21] Some (2+1)-dimensional and (3+1)-dimensional models have been studied recently.[22,23] The purpose of this paper is to present a generalized Jacobi elliptic function method and to solve the generalized (3+1)-dimensional Kronig–Penny (KP) equation with variable coefficients.

2. Our method based on the elliptic equation

The main idea of our method is to take full advantage of the elliptic equation that Jacobi elliptic functions satisfy and to use its solutions to obtain new doubly periodic and multiple soliton solutions of the (3+1)-dimensional KP equation with variable coefficients. The desired elliptic equation[24] reads

$$\phi'^2 = A + B\phi^2 + C\phi^4. \tag{1}$$

Here $' := \mathrm{d}/\mathrm{d}\xi, \xi = \xi(x, y, z, t)$, and A, B, C are constants.

Case 1. If $\begin{cases} A = 1, \\ B = -(1+m^2), \\ C = m^2, \end{cases}$

then Eq. (1) has a solution $\mathrm{sn}\xi, \mathrm{cd}\xi\left(\dfrac{\mathrm{cn}\xi}{\mathrm{dn}\xi}\right)$.

Case 2. If $\begin{cases} A = 1 - m^2, \\ B = 2m^2 - 1, \\ C = -m^2, \end{cases}$

then Eq. (1) has a solution $\mathrm{cn}\xi$.

Case 3. If $\begin{cases} A = m^2 - 1, \\ B = 2 - m^2, \\ C = -1, \end{cases}$

then Eq. (1) has a solution $\mathrm{dn}\xi$.

Case 4. If $\begin{cases} A = -m^2(1-m^2), \\ B = 2m^2 - 1, \\ C = 1, \end{cases}$

then Eq. (1) has a solution $\mathrm{ds}\xi\left(\dfrac{\mathrm{dn}\xi}{\mathrm{sn}\xi}\right)$.

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Case 5. If $\begin{cases} A = 1 - m^2, \\ B = 2 - m^2, \\ C = 1, \end{cases}$ then Eq. (1) has a solution $\mathrm{cs}\xi\left(\frac{\mathrm{cn}\xi}{\mathrm{sn}\xi}\right)$.

Case 6. If $\begin{cases} A = \frac{1}{4}, \\ B = \frac{m^2 - 2}{2}, \\ C = \frac{m^2}{4}, \end{cases}$ then Eq. (1) has a solution $\frac{\mathrm{sn}\xi}{1 \pm \mathrm{dn}\xi}$.

Case 7. If $\begin{cases} A = C = \frac{m^2}{4}, \\ B = \frac{m^2 - 2}{2}, \end{cases}$ then Eq. (1) has a solution $\mathrm{sn}\xi \pm \mathrm{icn}\xi$ $\frac{\mathrm{dn}\xi}{\mathrm{i}\sqrt{1 - m^2}\mathrm{sn}\xi \pm \mathrm{cn}\xi}$.

Case 8. If $\begin{cases} A = C = \frac{1}{4}, \\ B = \frac{1 - 2m^2}{2}, \end{cases}$ then Eq. (1) has a solution $\frac{\mathrm{dn}\xi}{m\mathrm{cn}\xi \pm \mathrm{i}\sqrt{1 - m^2}}, m\mathrm{sn}\xi \pm$ $\mathrm{idn}\xi, \frac{\mathrm{sn}\xi}{1 \pm \mathrm{cn}\xi}, \frac{\mathrm{cn}\xi}{\sqrt{1 - m^2}\mathrm{sn}\xi \pm \mathrm{dn}\xi}$.

Case 9. If $\begin{cases} A = C = \frac{m^2 - 1}{4}, \\ B = \frac{m^2 + 1}{2}, \end{cases}$ then Eq. (1) has a solution $\frac{\mathrm{dn}\xi}{1 \pm m\mathrm{sn}\xi}$.

Case 10. If $\begin{cases} A = C = \frac{1 - m^2}{4}, \\ B = \frac{1 + m^2}{2}, \end{cases}$ then Eq. (1) has a solution $\frac{\mathrm{cn}\xi}{1 \pm \mathrm{sn}\xi}$.

Case 11. If $\begin{cases} A = -\frac{(1 - m^2)^2}{4}, \\ B = \frac{1 + m^2}{2}, \\ C = -\frac{1}{4}, \end{cases}$ then Eq. (1) has a solution $m\mathrm{cn}\xi \pm \mathrm{dn}\xi$.

Case 12. If $\begin{cases} A = \frac{1}{4}, \\ B = \frac{1 + m^2}{2}, \\ C = \frac{(1 - m^2)^2}{4}, \end{cases}$ then Eq. (1) has a solution $\frac{\mathrm{sn}\xi}{\mathrm{dn}\xi \pm \mathrm{cn}\xi}$.

Case 13. If $\begin{cases} A = \frac{1}{4}, \\ B = \frac{m^2 - 2}{2}, \\ C = \frac{m^4}{4}, \end{cases}$ then Eq. (1) has a solution $\frac{\mathrm{cn}\xi}{\sqrt{1 - m^2} \pm \mathrm{dn}\xi}$.

Here m denotes the modulus of the Jacobi elliptic functions.

For a given PDE with $u(x, y, z, t)$ in four independent variables x, y, z, t

$$H(u, u_t, u_x, u_y, u_z, u_{xt}, u_{yt}, u_{zt}, u_{tt}, u_{xx}, \cdots) = 0. \quad (2)$$

We assume that the solutions of Eq. (2) can be expressed in the form

$$u(x, y, z, t) = a_0(x, y, z, t) + \sum_{i=1}^{n} [a_i(x, y, z, t)\phi^i(\xi) + b_i(x, y, z, t)\phi^{-i}(\xi)], \quad (3)$$

where $\xi = \xi(x, y, z, t)$. The process take the following steps.

Step 1. Determine n in Eq. (3) by balancing the linear terms of the highest order with the nonlinear terms in Eq. (2).

Step 2. Substituting Eqs. (3) and (1) into Eq. (2) yields a set of PDEs for $a_0(x, y, z, t)$, $a_1(x, y, z, t)$, $\cdots$, $a_n(x, y, z, t)$ because all coefficients of ϕ^i have to vanish.

Step 3. From the relations in step 2, $a_0(x, y, z, t)$, $a_1(x, y, z, t)$, $\cdots$, $a_n(x, y, z, t)$ can be determined by solving the set of PDEs.

Step 4. Substituting the obtained coefficients $a_0(x, y, z, t)$, $a_1(x, y, z, t)$, $\cdots$, $a_n(x, y, z, t)$ into Eq. (3) and using special solutions of Eq. (1), we obtain the explicit and exact solutions of Eq. (2).

In the following we illustrate the method by considering the generalized (3+1)-dimensional KP equation with variable coefficients.

3. The generalized (3+1)-dimensional KP equation with variable coefficients

The KP equation reads

$$[u_t + s(t)uu_x + p(t)u_{xxx}]_x + h(t)u_{yy} + g(t)u_{zz} = 0, \quad (4)$$

where $s(t)$, $p(t)$, $h(t)$ and $g(t)$ are arbitrary functions of t with $s(t) \neq= 0$, $p(t) \neq= 0$.

The special cases of Eq. (4) have been studied by several authors. If $g(t)$=0, Eq. (4) becomes (2+1)-dimensional KP equation. If $g(t) = h(t) = 0$, Eq. (4) becomes Korteweg-de Vris (KdV) equation. When $s(t)$, $p(t)$, $h(t)$ and $g(t)$ are arbitrary constants with $s(t) = p(t) = 6$, Eq. (4) has been studied recently by Li.[25]

In order to obtain exact solutions of Eq. (4), we first balance u_{xxxx} with uu_{xx} to gain n=2. Therefore, we choose the following ansatz:

$$\begin{aligned} u =& a_0 + a_1\phi(\alpha x + \beta(t)y + \gamma(t)z + \omega t) \\ &+ \frac{b_1}{\phi[\alpha x + \beta(t)y + \gamma(t)z + \omega t]} \\ &+ a_2\phi^2[\alpha x + \beta(t)y + \gamma(t)z + \omega t] \\ &+ \frac{b_2}{\phi^2[\alpha x + \beta(t)y + \gamma(t)z + \omega t]}, \end{aligned} \tag{5}$$

where $\beta(t)$ and $\gamma(t)$ are arbitrary functions of t; α and ω are arbitrary nonzero constants.

Substituting Eq. (5) into Eq. (4) along with Eq. (1) and using Maple yields a system of equations in ϕ^i. Setting the coefficients of ϕ^i in the obtained system of equations to zero,we can deduce the following set of equations with the respect unknowns a_0, a_1, a_2, b_1, b_2, namely:

$$12sa_1C\alpha^2a_2 + 24pa_1\alpha^4C^2 = 0, \tag{6a}$$

$$24pb_1\alpha^4A^2 + 12sb_1A\alpha^2b_2 = 0, \tag{6b}$$

$$10sb_2^2A\alpha^2 + 120pb_2\alpha^4A^2 = 0, \tag{6c}$$

$$10sa_2^2C\alpha^2 + 120pa_2C^2\alpha^4 = 0, \tag{6d}$$

$$\begin{aligned} &2sa_0b_2\alpha^2C + 2b_2\alpha C\gamma' y + 2sa_0a_2\alpha^2A \\ &+ 2b_2\alpha C\gamma' z + sa_1^2A\alpha^2 + 2b_2\alpha C\omega \\ &+ 2a_2\alpha A\gamma' z + 2a_2\alpha A\beta' y + 2a_2\alpha A\omega \\ &+ 8pa_2\alpha^4BA + sb_1^2C\alpha^2 + 8pb_2\alpha^4BC \\ &+ h(2b_2C\beta^2 + 2a_2A\beta^2) \\ &+ g(2b_2C\gamma^2 + 2a_2A\gamma^2) = 0, \end{aligned} \tag{6e}$$

$$\begin{aligned} &sb_1B\alpha^2a_2 + 6sa_1A\alpha^2a_2 + ha_1B\beta^2 \\ &+ ga_1B\gamma^2 + a_1\alpha B\omega + a_1\alpha B\gamma' z \\ &+ pa_1\alpha^4B^2 + 12pa_1\alpha^4CA + a_1\alpha B\beta' y \\ &+ sa_0a_1\alpha^2B = 0, \end{aligned} \tag{6f}$$

$$\begin{aligned} &4a_2\alpha B\beta' y + 4ga_2B\gamma^2 + 4a_2\alpha B\gamma' z \\ &+ 4a_2\alpha B\omega + 6sa_2^2A\alpha^2 + 16pa_2B^2\alpha^4 \\ &+ 2sa_1^2B\alpha^2 + 72pa_2A\alpha^4C + 4ha_2B\beta^2 \\ &+ 4sa_0a_2\alpha^2B = 0, \end{aligned} \tag{6g}$$

$$\begin{aligned} &6a_2\alpha C\gamma' z + 8sa_2^2B\alpha^2 + 120pa_2C\alpha^4B \\ &+ 6sa_0a_2\alpha^2C + 6a_2\alpha C\beta' y + 6ga_2C\gamma^2 \\ &+ 3sa_1^2C\alpha^2 + 6ha_2C\beta^2 + 6a_2\alpha C\omega \\ &+ 3sb_1^2A\alpha^2 + 8sb_2^2B\alpha^2 + 6sa_0b_2\alpha^2A \\ &+ 120pb_2\alpha^4AB + 6gb_2A\gamma^2 + 6hb_2A\beta^2 \\ &+ 6b_2\alpha A\gamma' z + 6b_2\alpha A\omega + 6b_2\alpha A\beta' y = 0, \end{aligned} \tag{6h}$$

$$\begin{aligned} &9sa_1B\alpha^2a_2 + 2ga_1C\gamma^2 + 20pa_1\alpha^4BC \\ &+ 2a_1\alpha C\beta' y + 2sa_0a_1\alpha^2C + 2a_1\alpha C\omega \\ &+ 2sb_1C\alpha^2a_2 + 2ha_1C\beta^2 + 2a_1\alpha C\gamma' z = 0, \end{aligned} \tag{6i}$$

$$\begin{aligned} &2hb_1A\beta^2 + 2b_1\alpha A\gamma' z + 20pb_1\alpha^4AB \\ &+ 9sb_1B\alpha^2b_2 + 2b_1\alpha A\beta' y + 2sa_1A\alpha^2b_2 \\ &+ 2gb_1A\gamma^2 + 2sa_0b_1\alpha^2A + 2b_1\alpha A\omega = 0, \end{aligned} \tag{6j}$$

$$\begin{aligned} &pb_1\alpha^4B^2 + b_1\alpha B\omega + gb_1B\gamma^2 + sa_0b_1\alpha^2B \\ &+ 12pb_1\alpha^4AC + b_1\alpha B\beta' y + 6sb_1C\alpha^2b_2 \\ &+ sa_1B\alpha^2b_2 + b_1\alpha B\gamma' z + hb_1B\beta^2 = 0, \end{aligned} \tag{6k}$$

$$\begin{aligned} &4b_2\alpha B\omega + 4b_2\alpha B\gamma' z + 72pb_2\alpha^4AC \\ &+ 4sa_0b_2\alpha^2B + 6sb_2^2C\alpha^2 + 4hb_2B\beta^2 \\ &+ 2sb_1^2B\alpha^2 + 16pb_2\alpha^4B^2 + 4b_2\alpha B\beta' y \\ &+ 4gb_2B\gamma^2 = 0. \end{aligned} \tag{6l}$$

With the aid of Maple, we find

1.

$$\begin{aligned} &a_1 = b_1 = a_2 = 0, b_2 = -\frac{12Ap(t)\alpha^2}{s(t)}, \\ &a_0 = -\frac{4p(t)\alpha^4B + \alpha\beta'(t)y + \alpha\gamma'(t)z + \alpha\omega + h(t)\beta^2(t) + g(t)\gamma^2(t)}{s(t)\alpha^2}. \end{aligned} \tag{7}$$

2.

$$\begin{aligned} &a_1 = b_1 = b_2 = 0, a_2 = -\frac{12Cp(t)\alpha^2}{s(t)}, \\ &a_0 = -\frac{4p(t)\alpha^4B + \alpha\beta'(t)y + \alpha\gamma'(t)z + \alpha\omega + h(t)\beta^2(t) + g(t)\gamma^2(t)}{s(t)\alpha^2}. \end{aligned} \tag{8}$$

3.

$$a_1 = b_1 = 0, a_2 = -\frac{12Cp(t)\alpha^2}{s(t)}, b_2 = -\frac{12Ap(t)\alpha^2}{s(t)},$$
$$a_0 = -\frac{4p(t)\alpha^4 B + \alpha\beta'(t)y + \alpha\gamma'(t)z + \alpha\omega + h(t)\beta^2(t) + g(t)\gamma^2(t)}{s(t)\alpha^2}. \tag{9}$$

Substituting Eqs. (7)–(9) into Eq. (5) and using the special solutions of Eq. (1), we obtain the following doubly periodic solutions of Eq. (4)

$$u_1 = -\frac{-4p(t)\alpha^4(1+m^2) + \alpha\beta'(t)y + \alpha\gamma'(t)z + \alpha\omega + h(t)\beta^2(t) + g(t)\gamma^2(t)}{s(t)\alpha^2} - \frac{12\alpha^2 p(t)(\varepsilon + \eta m^2 \mathrm{sn}^4\xi)}{s(t)\mathrm{sn}^2\xi}. \tag{10}$$

$$u_2 = -\frac{-4p(t)\alpha^4(1+m^2) + \alpha\beta'(t)y + \alpha\gamma'(t)z + \alpha\omega + h(t)\beta^2(t) + g(t)\gamma^2(t)}{s(t)\alpha^2} - \frac{12\alpha^2 p(t)(\varepsilon \mathrm{dn}^4\xi + \eta m^2 \mathrm{cn}^4\xi)}{s(t)\mathrm{cn}^2\xi\mathrm{dn}^2\xi}. \tag{11}$$

$$u_3 = -\frac{4p(t)\alpha^4(2m^2-1) + \alpha\beta'(t)y + \alpha\gamma'(t)z + \alpha\omega + h(t)\beta^2(t) + g(t)\gamma^2(t)}{s(t)\alpha^2} - \frac{12\alpha^2 p(t)[\varepsilon(1-m^2) - \eta m^2 \mathrm{cn}^4\xi]}{s(t)\mathrm{cn}^2\xi}. \tag{12}$$

$$u_4 = -\frac{4p(t)\alpha^4(2-m^2) + \alpha\beta'(t)y + \alpha\gamma'(t)z + \alpha\omega + h(t)\beta^2(t) + g(t)\gamma^2(t)}{s(t)\alpha^2} + \frac{12\alpha^2 p(t)[\varepsilon(1-m^2) + \eta \mathrm{dn}^4\xi]}{s(t)\mathrm{dn}^2\xi}. \tag{13}$$

$$u_5 = -\frac{4p(t)\alpha^4(2-m^2) + \alpha\beta'(t)y + \alpha\gamma'(t)z + \alpha\omega + h(t)\beta^2(t) + g(t)\gamma^2(t)}{s(t)\alpha^2} - \frac{12\alpha^2 p(t)[\varepsilon(1-m^2)\mathrm{sn}^4\xi + \eta \mathrm{cn}^4\xi]}{s(t)\mathrm{sn}^2\xi\mathrm{cn}^2\xi}. \tag{14}$$

$$u_6 = -\frac{4p(t)\alpha^4(2m^2-1) + \alpha\beta'(t)y + \alpha\gamma'(t)z + \alpha\omega + h(t)\beta^2(t) + g(t)\gamma^2(t)}{s(t)\alpha^2} - \frac{12\alpha^2 p(t)[\varepsilon \mathrm{dn}^4\xi - \eta m^2(1-m^2)\mathrm{sn}^4\xi]}{s(t)\mathrm{sn}^2\xi\mathrm{dn}^2\xi}. \tag{15}$$

$$u_7 = -\frac{2p(t)\alpha^4(m^2-2) + \alpha\beta'(t)y + \alpha\gamma'(t)z + \alpha\omega + h(t)\beta^2(t) + g(t)\gamma^2(t)}{s(t)\alpha^2} - \frac{3\alpha^2 p(t)[\varepsilon m^2 \mathrm{sn}^4\xi + \eta(1 \pm \mathrm{dn}\xi)^4]}{s(t)\mathrm{sn}^2\xi(1 \pm \mathrm{dn}\xi)^2}. \tag{16}$$

$$u_8 = -\frac{2p(t)\alpha^4(1-2m^2) + \alpha\beta'(t)y + \alpha\gamma'(t)z + \alpha\omega + h(t)\beta^2(t) + g(t)\gamma^2(t)}{s(t)\alpha^2} - \frac{3\alpha^2 p(t)[\varepsilon \mathrm{sn}^4\xi + \eta(1 \pm \mathrm{cn}\xi)^4]}{s(t)\mathrm{sn}^2\xi(1 \pm \mathrm{cn}\xi)^2}. \tag{17}$$

$$u_9 = -\frac{2p(t)\alpha^4(1-2m^2) + \alpha\beta'(t)y + \alpha\gamma'(t)z + \alpha\omega + h(t)\beta^2(t) + g(t)\gamma^2(t)}{s(t)\alpha^2} - \frac{3\alpha^2 p(t)[\varepsilon \mathrm{cn}^4\xi + \eta(\sqrt{1-m^2}\mathrm{sn}\xi \pm \mathrm{dn}\xi)^4]}{s(t)\mathrm{cn}^2\xi(\sqrt{1-m^2}\mathrm{sn}\xi \pm \mathrm{dn}\xi)^2}. \tag{18}$$

$$u_{10} = -\frac{2p(t)\alpha^4(1+m^2) + \alpha\beta'(t)y + \alpha\gamma'(t)z + \alpha\omega + h(t)\beta^2(t) + g(t)\gamma^2(t)}{s(t)\alpha^2} - \frac{3\alpha^2(m^2-1)p(t)[\varepsilon \mathrm{dn}^4\xi + \eta(1 \pm m\mathrm{sn}\xi)^4]}{s(t)\mathrm{dn}^2\xi(1 \pm m\mathrm{sn}\xi)^2}. \tag{19}$$

$$u_{11} = -\frac{2p(t)\alpha^4(1+m^2)+\alpha\beta'(t)y+\alpha\gamma'(t)z+\alpha\omega+h(t)\beta^2(t)+g(t)\gamma^2(t)}{s(t)\alpha^2} - \frac{3\alpha^2(1-m^2)p(t)[\varepsilon\mathrm{cn}^4\xi+\eta(1\pm\mathrm{sn}\xi)^4]}{s(t)\mathrm{cn}^2\xi(1\pm\mathrm{sn}\xi)^2}. \tag{20}$$

$$u_{12} = -\frac{2p(t)\alpha^4(1+m^2)+\alpha\beta'(t)y+\alpha\gamma'(t)z+\alpha\omega+h(t)\beta^2(t)+g(t)\gamma^2(t)}{s(t)\alpha^2} + \frac{3\alpha^2p(t)[\varepsilon(1-m^2)^2+\eta(m\mathrm{cn}\xi\pm\mathrm{dn}\xi)^4]}{s(t)(m\mathrm{cn}\xi\pm\mathrm{dn}\xi)^2}. \tag{21}$$

$$u_{13} = -\frac{2p(t)\alpha^4(1+m^2)+\alpha\beta'(t)y+\alpha\gamma'(t)z+\alpha\omega+h(t)\beta^2(t)+g(t)\gamma^2(t)}{s(t)\alpha^2} - \frac{3\alpha^2p(t)[\varepsilon(1-m^2)^2\mathrm{sn}^4\xi+\eta(\mathrm{dn}\xi\pm\mathrm{cn}\xi)^4]}{s(t)\mathrm{sn}^2\xi(\mathrm{dn}\xi\pm\mathrm{cn}\xi)^2}. \tag{22}$$

$$u_{14} = -\frac{2p(t)\alpha^4(m^2-2)+\alpha\beta'(t)y+\alpha\gamma'(t)z+\alpha\omega+h(t)\beta^2(t)+g(t)\gamma^2(t)}{s(t)\alpha^2(m^2-2)} - \frac{3\alpha^2p(t)[\varepsilon m^4\mathrm{cn}^4\xi+\eta(\sqrt{1-m^2}\pm\mathrm{dn}\xi)^4]}{s(t)\mathrm{cn}^2\xi(\sqrt{1-m^2}\pm\mathrm{dn}\xi)^2}. \tag{23}$$

If $m\to 1$,then $\mathrm{sn}\xi\to\tanh\xi, \mathrm{cn}\xi, \mathrm{dn}\xi\to\mathrm{sech}\xi$. Thus we can obtain the following multiple soliton solutions of Eq. (4)

$$u_{15} = -\frac{-8p(t)\alpha^4+\alpha\beta'(t)y+\alpha\gamma'(t)z+\alpha\omega+h(t)\beta^2(t)+g(t)\gamma^2(t)}{s(t)\alpha^2} - \frac{12\alpha^2p(t)(\varepsilon+\eta\tanh^4\xi)}{s(t)\tanh^2\xi}. \tag{24}$$

$$u_{16} = -\frac{-2p(t)\alpha^4+\alpha\beta'(t)y+\alpha\gamma'(t)z+\alpha\omega+h(t)\beta^2(t)+g(t)\gamma^2(t)}{s(t)\alpha^2} - \frac{3\alpha^2p(t)[\varepsilon\tanh^4\xi+\eta(1\pm\mathrm{sech}\xi)^4]}{s(t)\tanh^2\xi(1\pm\mathrm{sech}\xi)^2}. \tag{25}$$

Where $\xi=\alpha x+\beta(t)y+\gamma(t)z+\omega t$, ε and η are arbitrary elements of {0,1}.

Remark 1 Eqs. (24) and (25) are only part of special cases of Eqs. (10)–(23). The others are omitted.

Remark 2 If $m\to 0$,then $\mathrm{sn}\xi\to\sin\xi$, $\mathrm{cn}\xi\to\cos\xi$, $\mathrm{dn}\xi\to 1$. We can obtain triangular periodic solutions of Eq. (4). (omitted)

Remark 3 The above solutions are essentially those solutions on a curve in (3+1)-dimension. They can be regarded as the generalized travelling wave solutions.

4. Conclusion

We have presented a generalized Jacobi elliptic function method and used it to solve the (3+1)-dimensional KP equation with variable coefficients. In fact, we have obtained not only the exact solutions of KP equation, but also the exact solutions of KdV equation as $h(t)=g(t)=0$. This method can be applied to many other nonlinear PDEs.

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