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# An integrable Hamiltonian hierarchy, a high-dimensional loop algebra and associated integrable coupling system

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A subalgebra of loop algebra $\tilde{A}_2$ is established. Therefore, a new isospectral problem is designed. By making use of Tu's scheme, a new integrable system is obtained, which possesses bi-Hamiltonian structure. As its reductions, a formalism similar to the well-known Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy and a generalized standard form of the Schrödinger equation are presented. In addition, in order for a kind of expanding integrable system to be obtained, a proper algebraic transformation is supplied to change loop algebra $\tilde{A}_2$ into loop algebra $\tilde{A}_1$. Furthermore, a high-dimensional loop algebra is constructed, which is different from any previous one. An integrable coupling of the system obtained is given. Finally, the Hamiltonian form of a binary symmetric constrained flow of the system obtained is presented.

**Keywords:** loop algebra, integrable system, Hamiltonian structure, constrained flow
**PACC:** 0340K, 0220, 0365G

## 1. Introduction

Researching new integrable systems is an interesting and important topic in soliton theory. Many integrable hierarchies were obtained by the use of $2\times2$ isospectral problems.[1−7] Few of them were given by $3\times3$ isospectral problems. In this paper, a new subalgebra of loop algebra $\tilde{A}_2$ is established, which has two powers of spectral parameter margin. An isospectral problem is thus designed. By utilizing the Tu scheme, a new integrable hierarchy is obtained, possessing a bi-Hamiltonian structure. As its reduction case, a coupled nonlinear Schrödinger equation containing three equations is presented. A formalism of integrable system, which has a similar form to the well-known hierarchy proposed by Ablowitz, Kaup, Newell and Segur, we briefly call it AKNS hierarchy (but actually not the original AKNS hierarchy), and a generalized standard Schrödinger equation are given. In addition, in order to obtain an expanding integrable system, a proper algebraic transformation is assumed to turn a $3\times3$ loop algebra into $2\times2$ loop algebra. Furthermore, a high-dimensional loop algebra, which is different from any one in Refs. [8–16], is constructed. We find that the loop algebra is miracle since it is constructed only in the frame of loop algebra, not from Lie algebra. It follows that a kind of integrable coupling system, which is of course an expanding integrable system of the hierarchy obtained, is engendered. Finally, by establishing a binary symmetry constraint, the Hamiltonian form of the constrained flows is presented.

## 2. A new loop algebra and an integrable system

Consider the following matrices

$$e_1=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix},\quad e_2=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},$$

$$e_3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\quad e_4=\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix}.$$

We find that

$$[e_1,e_2]=-2e_4,[e_1,e_3]=0,[e_1,e_4]=-2e_2,$$
$$[e_2,e_3]=e_4,[e_2,e_4]=2e_1,$$
$$[e_3,e_4]=-e_2.$$

If we let $a=\sum_{i=1}^{4}a_ie_i,\ b=\sum_{i=1}^{4}b_ie_i,\ c=\sum_{i=1}^{4}c_ie_i$, we have
$[[a,b],c]+[[b,c],a]+[[c,a],b]=0$, that is to say, Jacobi identity holds. Hence $\{e_1,e_2,e_3\}$ constitutes a Lie algebra. Let

$$\begin{aligned}e_1(n)=&e_1\lambda^{2n}, e_2(n)=e_2\lambda^{2n+1},\\ e_3(n)=&e_3\lambda^{2n}, e_4(n)=e_4\lambda^{2n+1}.\end{aligned}\tag{1}$$

Then we have

$$\begin{aligned}&[e_1(m),e_2(n)]=-2e_4(m+n),\\ &[e_1(m),e_4(n)]=-2e_2(m+n),\\ &[e_2(m),e_3(n)]=e_4(m+n),\\ &[e_2(m),e_4(n)]=2e_1(m+n+1),\\ &[e_3(m),e_4(n)]=-e_2(m+n),\\ &[e_1(m),e_3(n)]=0,\\ &\deg e_1(n)=\deg e_3(n)=2n,\\ &\deg e_2(n)=\deg e_4(n)=2n+1.\end{aligned}\tag{2}$$

Therefore, $\{e_1(n),e_2(n),e_3(n)\}$ is a loop algebra, note it as $\overline{A}_2$. Set

$$\varphi_x=U\psi,\lambda_t=0,\quad \varphi=(\varphi_1,\varphi_2,\varphi_3)^{\mathrm{T}},$$

$$U=\begin{pmatrix}\lambda^2+q & 0 & (r-s)\lambda\\ 0 & -\lambda^2 & 0\\ (r+s)\lambda & 0 & -q\end{pmatrix}.\tag{3}$$

In terms of Eqs. (1) and (2), let

$$V=\begin{pmatrix}a & 0 & (b-c)\lambda\\ 0 & 0 & 0\\ (b+c)\lambda & 0 & -a\end{pmatrix},$$

where

$$a=\sum_{m=0}^{\infty}a_m\lambda^{-2m}, b=\sum_{m=0}^{\infty}b_m\lambda^{-2m},$$
$$c=\sum_{m=0}^{\infty}c_m\lambda^{-2m}.$$

Solving the stationary zero-curvature equation

$$V_x=[U,V]\tag{4}$$

leads to the recursive relations:

$$\begin{aligned}b_{mx}=&-c_{m+1}-2qc_m+2sa_m,\\ b_{m+1}=&-c_{mx}-2qb_m+2ra_m,\\ a_{mx}=&2rc_{m+1}-2sb_{m+1}\\ =&-2rb_{mx}-4qrc_m+2sc_{mx}+4qsb_m,\\ b_0=&c_0=0, a_0=\alpha, a_1=\alpha(s^2-r^2),\\ b_1=&2\alpha r, c_1=2\alpha s,\\ a_2=&4\alpha(rs_x-r_xs)+3\alpha(s^2-r^2)^2+8\alpha q(r^2-s^2),\\ b_2=&-2\alpha s_x-4\alpha qr+2\alpha r(s^2-r^2),\\ c_2=&-2\alpha r_x-4\alpha qs+2\alpha s(s^2-r^2),\\ b_3=&2\alpha r_{xx}+8\alpha qs_x+4\alpha q_xs-2\alpha s_x(s^2-r^2)\\ &-2\alpha s(s^2-r^2)_x+8\alpha q^2r+8\alpha r(rs_x-r_xs)\\ &+6\alpha r(s^2-r^2)^2+20\alpha qr(r^2-s^2),\\ c_3=&2\alpha s_{xx}+8\alpha qr_x+4\alpha q_xr-2\alpha r_x(s^2-r^2)\\ &-2\alpha r(s^2-r^2)_x+8\alpha q^2s+8\alpha s(rs_x-r_xs)\\ &+6\alpha s(s^2-r^2)^2+20\alpha qs(r^2-s^2).\end{aligned}\tag{5}$$

Note that

$$\begin{aligned}V_+^{(n)}=&\sum_{m=0}^{n}(a_me_1(n-m)\\ &+b_me_2(n-m)+c_me_4(n-m)),\\ V_-^{(n)}=&\lambda^{2n}V-V_+^{(n)},\end{aligned}$$

then Eq. (4) can be written as

$$-V_{+x}^{(n)}+[U,V_+^{(n)}]=V_{-x}^{(n)}-[U,V_-^{(n)}].\tag{6}$$

It is easy to verify that the terms on the left-hand side of Eq. (6) are of degree $\geq 0$, while the terms on the right-hand side are of degree $\leq 1$. Therefore, the terms of both sides are of degree 0, 1.

Thus,

$$\begin{aligned}-V_{+x}^{(n)}+[U,V_+^{(n)}]=&(2sb_{n+1}-2rc_{n+1})e_1(0)\\ &+c_{n+1}e_2(0)+b_{n+1}e_4(0).\end{aligned}$$

Taking $V^{(n)}=V_+^{(n)}$, then the zero-curvature equation

$$U_t-V_x^{(n)}+[U,V^{(n)}]=0\tag{7}$$

determines the Lax integrable system as follows:

$$u_{t_n}=\begin{pmatrix}q\\ r\\ s\end{pmatrix}_{t_n}=\begin{pmatrix}-2sb_{n+1}+2rc_{n+1}\\ -c_{n+1}\\ -b_{n+1}\end{pmatrix}=\begin{pmatrix}a_{nx}\\ -c_{n+1}\\ -b_{n+1}\end{pmatrix}=\begin{pmatrix}-\frac{\partial}{2} & 0 & 0\\ 0 & 0 & \frac{1}{2}\\ 0 & -\frac{1}{2} & 0\end{pmatrix}\begin{pmatrix}2a_n\\ 2b_{n+1}\\ -2c_{n+1}\end{pmatrix}=JG_n,\tag{8}$$

where $J$ is a Hamiltonian operator.

From Eq. (5), we have

$$G_n = \begin{pmatrix} -2\partial^{-1}q\partial & -2\partial^{-1}r\partial & 2\partial^{-1}s\partial \\ 0 & -2q-4r\partial^{-1}r\partial+8r\partial^{-1}q\partial & -\partial-8r\partial^{-1}qr+4r\partial^{-1}s\partial \\ 0 & -\partial+8s\partial^{-1}qs-4s\partial^{-1}r\partial & -2q-8s\partial^{-1}qr+4s\partial^{-1}s\partial \end{pmatrix} G_{n-1} = LG_{n-1}.$$

Hence, Eq. (8) can be written as

$$u_{t_n} = \begin{pmatrix} q \\ r \\ s \end{pmatrix}_{t_n} = JL^n \begin{pmatrix} 2\alpha \\ 4\alpha r \\ -4\alpha s \end{pmatrix}. \tag{9}$$

Some reduction cases of Eq. (9) are as follows.

**Case 1** Taking $n$=2 in Eq. (9) gives rise to a coupled nonlinear Schrödinger equation with three equations

$$\begin{aligned} r_{t_2} =& -2\alpha s_{xx} - 8\alpha q r_x - 4\alpha q_x r + 2\alpha r_x(s^2-r^2) \\ &+ 2\alpha r(s^2-r^2)_x - 8\alpha q^2 s - 8\alpha s(rs_x - r_x s) \\ &- 6\alpha(s^2-r^2)^2 - 20\alpha qs(r^2-s^2), \\ s_{t_2} =& -2\alpha r_{xx} - 8\alpha q s_x - 4\alpha q_x s + 2\alpha s_x(s^2-r^2) \\ &+ 2\alpha(s^2-r^2)_x - 8\alpha q^2 r - 8\alpha r(rs_x - r_x s) \\ &- 6\alpha r(s^2-r^2)^2 - 20\alpha qr(r^2-s^2), \\ q_{t_2} =& 4\alpha(rs_{xx} - r_{xx}s) + 12\alpha(s^2-r^2)(ss_x - rr_x) \\ &+ 8\alpha q_x(r^2-s^2) + 16\alpha q(rr_x - ss_x). \end{aligned}$$

**Case 2** Taking $q$=0 in Eq. (9) yields the formalism of the AKNS hierarchy as follows:

$$\tilde{u}_{t_n} = \begin{pmatrix} r \\ s \end{pmatrix}_{t_n} = \tilde{J}\tilde{L} \begin{pmatrix} 4\alpha r \\ -4\alpha s \end{pmatrix}, \tag{10}$$

where

$$\tilde{J} = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix},$$

$$\tilde{L} = \begin{pmatrix} -4r\partial^{-1}r\partial & -\partial+4r\partial^{-1}s\partial \\ -\partial-4s\partial^{-1}r\partial & 4s\partial^{-1}s\partial \end{pmatrix}.$$

**Remark 1** Here $\tilde{L}$ is different from $L$ in AKNS hierarchy,[1] hence the system (10)is not AKNS system.

**Case 3** Taking $n$=2 in Eq. (10) yields the following nonlinear generalized Schrödinger equation:

$$\begin{aligned} r_{t_2} =& -2\alpha s_{xx} + 2\alpha r_x(s^2-r^2) + 2\alpha r(s^2-r^2)_x \\ &- 8\alpha s(rs_x - r_x s) - 6\alpha s(s^2-r^2)^2, \\ s_{t_2} =& -2\alpha r_{xx} + 2\alpha s_x(s^2-r^2) + 2\alpha s(s^2-r^2)_x \\ &- 8\alpha r(rs_x - r_x s) - 6\alpha r(s^2-r^2)^2. \end{aligned} \tag{11}$$

Let $Q = r+s = r+\mathrm{i}s_1$, $t_2 = t$, $\alpha = -\mathrm{i}$, where $r$ and $s_1$ are all real functions. In terms of Eq. (11), we obtain a generalized standard form of Schrödinger equation

$$\mathrm{i}Q_t + 2Q_{xx} - 2Q|Q|_x^2 + 6Q|Q|^4 + 2Q_x|Q|^2 - 4Q^2\overline{Q}_x = 0.$$

A direct calculation reads

$$\begin{aligned} &\left\langle V, \frac{\partial U}{\partial q} \right\rangle = 2a, \left\langle V, \frac{\partial U}{\partial r} \right\rangle = 2b\lambda^2, \\ &\left\langle V, \frac{\partial U}{\partial s} \right\rangle = -2c\lambda^2, \\ &\left\langle V, \frac{\partial U}{\partial \lambda} \right\rangle = 2(a+rb-sc)\lambda. \end{aligned}$$

Substituting the above formulae into trace identity yields

$$\frac{\delta}{\delta u} 2(a+rb-sc)\lambda = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma} \begin{pmatrix} 2\alpha \\ 2b\lambda^2 \\ -2c\lambda^2 \end{pmatrix}. \tag{12}$$

Comparing the coefficients of $\lambda^{-2n-1}$ gives

$$\frac{\delta}{\delta u} 2(a_{n+1} + rb_{n+1} - sc_{n+1}) = (-2n+\gamma)G_n. \tag{13}$$

Taking $n$=0, inserting the initial values of Eq. (5) into Eq. (13) leads to $\gamma$=0. Therefore,

$$\frac{\delta H_n}{\delta u} = G_n, \quad H_n = \frac{sc_{n+1} - rb_{n+1} - a_{n+1}}{n}$$

are Hamiltonian functions. Thus, we obtain the Hamiltonian structure of the system (9) as follows:

$$u_{t_n} = J\frac{\delta H_n}{\delta u} = K\frac{\delta H_{n-1}}{\delta u}. \tag{14}$$

It is easy to verify that $J_1 = c_1 J + c_2 K$ is also a Hamiltonian operator, where $c_1$ and $c_2$ are constants. Hence, Eq. (14) is a bi-Hamiltonian structure of the system (9). Furthermore, we can test that $JL = L^*J$. Thus, the system (9) is integrable in the Liouville sense.

## 3. Integrable coupling of the system (9)

Regarding the concept of the integrable couplings, one can refer to Ref. [17]. Using the method proposed in Refs. [8–16] cannot obtain integrable couplings of the system (9) since this system is given with basis on loop algebra $\tilde{A}_2$, not basis on loop algebra $\tilde{A}_1$. Therefore, we must take a proper algebraic transformation to turn the loop algebra (1), (2) into its equivalent loop algebra like the form of $\tilde{A}_1$. For this purpose, we take a Lie algebra $A_1$:

$$\tilde{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tilde{e}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\tilde{e}_3 = \tilde{e}_1, \tilde{e}_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$[\tilde{e}_1, \tilde{e}_2] = 2\tilde{e}_4, [\tilde{e}_1, \tilde{e}_4] = 2\tilde{e}_2,$$

$$[\tilde{e}_2, \tilde{e}_4] = -2\tilde{e}_1, [\tilde{e}_1, \tilde{e}_3] = 0. \quad (15)$$

**Remark 2** We take $\tilde{e}_3 = \tilde{e}_1$ in Eq. (15) so that Eq. (15) may correspond to the relations in Eqs. (2). Let us make the scalar transformation: $e_i \to \alpha_i \tilde{e}_i$, $i$=1,2,3,4, where $\alpha_i$ are to be determined. Solving the following algebraic equations

$$\frac{-2\alpha_4}{\alpha_1\alpha_2} = 2, \frac{-2\alpha_2}{\alpha_1\alpha_4} = 2,$$

$$\frac{\alpha_4}{\alpha_2\alpha_3} = -2, \frac{2\alpha_1}{\alpha_2\alpha_4} = -2, \frac{-\alpha_2}{\alpha_3\alpha_4} = 2,$$

yields a set of solutions: $\alpha_1 = -1, \alpha_2 = \alpha_4 = 1$, $\alpha_3 = -\frac{1}{2}$.

Thus, we have a subalgebra of Lie algebra $A_1$:

$$\overline{e}_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \overline{e}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\overline{e}_3 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \overline{e}_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$[\overline{e}_1, \overline{e}_2] = -2\overline{e}_4, [\overline{e}_1, \overline{e}_3] = 0, [\overline{e}_1, \overline{e}_4] = -2\overline{e}_2,$$

$$[\overline{e}_2, \overline{e}_3] = \overline{e}_4, [\overline{e}_2, \overline{e}_4] = 2\overline{e}_1, [\overline{e}_3, \overline{e}_4] = -\overline{e}_2. \quad (16)$$

Obviously, Eq. (16) has the same communicative relations as the loop algebra $\overline{A}_2$. Therefore, by making use of loop algebra (16), one also deduces the system (9). For simplicity, we omit the bars in Eq. (16). Now we expand Eq. (16) into the following high-dimensional loop algebra $\tilde{G}$, which is different from any one in Refs. [8–16]:

$$e_1(n) = \begin{pmatrix} -\lambda^{2n} & 0 & 0 \\ 0 & \lambda^{2n} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_2(n) = \begin{pmatrix} 0 & \lambda^{2n+1} & 0 \\ \lambda^{2n+1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_3(n) = \frac{1}{2}\begin{pmatrix} -\lambda^{2n} & 0 & 0 \\ 0 & \lambda^{2n} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_4(n) = \begin{pmatrix} 0 & \lambda^{2n+1} & 0 \\ -\lambda^{2n+1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_5(n) = \begin{pmatrix} 0 & 0 & \lambda^{2n} \\ 0 & 0 & \lambda^{2n-1} \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_6(n) = \begin{pmatrix} 0 & 0 & \lambda^{2n} \\ 0 & 0 & -\lambda^{2n-1} \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_7(n) = \begin{pmatrix} 0 & 0 & \lambda^{2n} \\ 0 & 0 & \lambda^{2n+1} \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_8(n) = \begin{pmatrix} 0 & 0 & \lambda^{2n} \\ 0 & 0 & -\lambda^{2n+1} \\ 0 & 0 & 0 \end{pmatrix},$$

$$[e_1(m), e_2(n)] = -2e_4(m+n),$$
$$[e_1(m), e_3(n)] = 0,$$
$$[e_1(m), e_4(n)] = -2e_2(m+n),$$
$$[e_2(m), e_3(n)] = e_4(m+n),$$
$$[e_2(m), e_4(n)] = 2e_1(m+n+1),$$
$$[e_3(m), e_4(n)] = -e_2(m+n),$$
$$[e_1(m), e_5(n)] = -e_6(m+n),$$
$$[e_2(m), e_5(n)] = e_7(m+n),$$
$$[e_3(m), e_5(n)] = \frac{-1}{2}e_6(m+n),$$
$$[e_4(m), e_5(n)] = -e_7(m+n),$$
$$[e_1(m), e_6(n)] = -e_5(m+n),$$
$$[e_2(m), e_6(n)] = -e_8(m+n),$$
$$[e_3(m), e_6(n)] = \frac{-1}{2}e_5(m+n),$$
$$[e_4(m), e_6(n)] = -e_7(m+n),$$
$$[e_1(m), e_7(n)] = -e_8(m+n),$$

$$
\begin{aligned}
&[e_2(m), e_7(n)] = e_5(m+n+1),\\
&[e_3(m), e_7(n)] = \frac{-1}{2}e_8(m+n),\\
&[e_4(m), e_7(n)] = e_6(m+n+1),\\
&[e_1(m), e_8(n)] = -e_7(m+n),\\
&[e_2(m), e_8(n)] = -e_6(m+n+1),\\
&[e_3(m), e_8(n)] = -\frac{1}{2}e_7(m+n),\\
&[e_4(m), e_8(n)] = -e_5(m+n+1),\\
&[e_5(m), e_6(n)] = [e_5(m), e_7(n)]\\
=&[e_5(m), e_8(n)] = [e_6(m), e_7(n)]\\
=&[e_6(m), e_8(n)] = [e_7(m), e_8(n)] = 0,\\
&\deg e_1(n) = \deg e_3(n)\\
=&\deg e_5(n) = \deg e_6(n) = 2n,\\
&\deg e_2(n) = \deg e_4(n)\\
=&\deg e_7(n) = \deg e_8(n) = 2n+1, \qquad (17)
\end{aligned}
$$

Let

$$
\begin{aligned}
\tilde{G}_1 =&\text{span}\{e_1(n), e_2(n), e_4(n)\}\\
\tilde{G}_2 =&\text{span}\{e_5(n), e_6(n), e_7(n), e_8(n)\},
\end{aligned}
$$

then

$$\tilde{G} = \tilde{G}_1 \dot{+} \tilde{G}_2, \qquad (18)$$

$\tilde{G}_1$ is isomorphic to $\overline{A}_1$.

In what follows, we use Eqs. (17) and (18) to deduce integrable coupling of the system (9), which is also a kind of expanding integrable system of Eq. (9). Set

$$
\begin{aligned}
\psi_x =&U\psi, \lambda_t = 0,\\
U =&e_3(1) + u_1e_1(0) + +u_2e_2(0) + u_3e_4(0)\\
&+ u_4e_5(0) + u_5e_6(0) + u_6e_7(0) + u_7e_8(0). \qquad (19)
\end{aligned}
$$

Let

$$
\begin{aligned}
V = \sum_{m\geq 0} &(a_me_1(-m) + b_me_2(-m)\\
&+ c_me_4(-m) + d_me_5(-m) + f_me_6(-m)\\
&+ g_me_7(-m) + h_me_8(-m)).
\end{aligned}
$$

Solving the auxiliary equation similar to Eq. (4) yields

$$
\begin{aligned}
a_{mx} =&2u_2c_{m+1} - 2u_3b_{m+1},\\
b_{mx} =&-c_{m+1} - 2u_1c_m + 2u_3a_m,\\
c_{mx} =&-b_{m+1} - 2u_1b_m + 2u_2a_m,\\
d_{mx} =&-\frac{1}{2}f_{m+1} - u_1f_m + u_2g_{m+1}\\
&- u_3h_{m+1} + u_5a_m - u_6b_{m+1} + u_7c_{m+1},\\
f_{mx} =&-\frac{1}{2}d_{m+1} - u_1d_m - u_2h_{m+1}\\
&+ u_3g_{m+1} + u_4a_m - u_6c_{m+1} + u_7b_{m+1},\\
g_{mx} =&-\frac{1}{2}h_{m+1} - u_1hm + u_2d_m - u_3d_m\\
&- u_3f_m - u_4b_m + u_4c_m + u_5c_m + u_7a_m,\\
h_{mx} =&-\frac{1}{2}g_{m+1} - u_1g_m - u_2f_m + u_5b_m + u_6a_m,\\
b_0 =&c_0 = d_0 = f_0 = g_0 = h_0 = 0, a_0 = \alpha,\\
a_1 =&\alpha(u_3^2 - u_2^2), b_1 = 2\alpha u_2, c_1 = 2\alpha u_3,\\
g_1 =&2\alpha u_6, f_1 = 2\alpha u_5, d_1 = 2\alpha u_4, h_1 = 2\alpha u_7. \qquad (20)
\end{aligned}
$$

Similar to Eq. (6), here we have

$$
\begin{aligned}
&- V_{+x}^{(n)} + [U, V_+^{(n)}]\\
=&(2u_3b_{n+1} - 2u_2c_{n+1})e_1(0)\\
&+ c_{n+1}e_2(0) + b_{n+1}e_4(0)\\
&+ \left(\frac{1}{2}f_{n+1} - u_2g_{n+1} + u_3h_{n+1}\right.\\
&\left.+ u_6b_{n+1} - u_7c_{n+1}\right)e_5(0)\\
&+ \left(\frac{1}{2}d_{n+1} + u_2h_{n+1} - u_3g_{n+1}\right.\\
&\left.+ u_6c_{n+1} - u_7b_{n+1}\right)e_6(0)\\
&+ \frac{1}{2}h_{n+1}e_7(0) + \frac{1}{2}g_{n+1}e_8(0).
\end{aligned}
$$

Set $V^{(n)} = V_+^{(n)}$, then the zero-curvature equation

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0 \qquad (21)$$

determines the Lax integrable system as follows:

$$
u_{t_n} = \begin{pmatrix} u_1\\ u_2\\ u_3\\ u_4\\ u_5\\ u_6\\ u_7 \end{pmatrix}_{t_n} = \begin{pmatrix} 2u_2c_{n+1} - 2u_3b_{n+1}\\ -c_{n+1}\\ -b_{n+1}\\ -\frac{1}{2}f_{n+1} + u_2g_{n+1} - u_3h_{n+1} - u_6b_{n+1} + u_7c_{n+1}\\ -\frac{1}{2}d_{n+1} - u_2h_{n+1} + u_3g_{n+1} - u_6c_{n+1} + u_7b_{n+1}\\ -\frac{1}{2}h_{n+1}\\ -\frac{1}{2}g_{n+1} \end{pmatrix}
$$

$$= \begin{pmatrix} -\frac{\partial}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{u_6}{2} & -\frac{u_7}{2} & 0 & -\frac{1}{2} & -u_2 & -u_3 \\ 0 & \frac{u_7}{2} & \frac{u_6}{2} & -\frac{1}{2} & 0 & -u_3 & -u_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 2a_n \\ 2b_{n+1} \\ -2c_{n+1} \\ d_{n+1} \\ f_{n+1} \\ -g_{n+1} \\ h_{n+1} \end{pmatrix} = J \begin{pmatrix} 2a_n \\ 2b_{n+1} \\ -2c_{n+1} \\ d_{n+1} \\ f_{n+1} \\ -g_{n+1} \\ h_{n+1} \end{pmatrix}. \tag{22}$$

From Eq. (20), we obtain

$$\begin{pmatrix} 2a_n \\ 2b_{n+1} \\ -2c_{n+1} \\ d_{n+1} \\ f_{n+1} \\ -g_{n+1} \\ h_{n+1} \end{pmatrix} = \begin{pmatrix} -2\partial^{-1}u_1\partial & -2\partial^{-1}u_2\partial & -2\partial^{-1}u_3\partial \\ 0 & -2u_1 - 4u_2\partial^{-1}u_2\partial + 8u_2\partial^{-1}u_1u_3 & -\partial - 8u_2\partial^{-1}u_1u_2 + 4u_2\partial^{-1}u_3\partial \\ 0 & -\partial - 4u_3\partial^{-1}u_2\partial + 8u_3\partial^{-1}u_1u_3 & -2u_1 + 4u_3\partial^{-1}u_3\partial - 8u_3\partial^{-1}u_1u_2 \\ -4u_4\partial^{-1}u_1\partial & 4u_6\partial - 4u_1u_7 + 4u_2u_4 & 4u_1u_6 - 2u_7\partial - 4u_2u_4 \\ & +4u_3u_5 - 4u_4\partial^{-1}u_2\partial & -4u_2u_5 + 4u_4\partial^{-1}u_3\partial \\ -4u_5\partial^{-1}u_1\partial & 4u_2u_5 - 4u_5\partial^{-1}u_2\partial^{-1} & 4u_5\partial^{-1}u_3\partial + 2u_6\partial \\ & +4u_1u_6 + 4u_3u_4 - 2u_7\partial & -4u_1u_7 - 4u_3u_4 - 4u_3u_5 \\ -4u_6\partial^{-1}u_1\partial & 2u_5 - 4u_6\partial^{-1}u_2\partial & 4u_6\partial^{-1}u_3\partial \\ -4u_7\partial^{-1}u_1\partial & -2u_4 - 4u_7\partial^{-1}u_2\partial & 2u_4 + 2u_5 + 4u_7\partial^{-1}u_3\partial \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2u_1 - 4u_2^2 + 4u_2u_3 & -2\partial - 4u_3^2 - 4u_2u_3 & 4u_2\partial u_1u_3 & 4u_1u_2 - 4u_3\partial \\ -2\partial + 4u_2u_3 + 4u_4^2 & -2u_1 - 4u_2^2 + 4u_3^2 & -4u_1u_2 + 4u_3\partial & -4u_2\partial + 4u_1u_3 \\ 0 & -2u_2 & -2u_1 & -2\partial \\ 2u_2 - 2u_3 & -2u_3 & -2\partial & -2u_1 \end{pmatrix}$$

$$\begin{pmatrix} 2a_{n-1} \\ 2b_n \\ -2c_n \\ d_n \\ f_n \\ -g_n \\ h_n \end{pmatrix} = L \begin{pmatrix} 2a_{n-1} \\ 2b_n \\ -2c_n \\ d_n \\ f_n \\ -g_n \\ h_n \end{pmatrix}.$$

Therefore, Eq. (22) can be written as

$$u_{t_n} = JL^n \begin{pmatrix} 2\alpha \\ 4\alpha u_2 \\ -4\alpha u_3 \\ 2\alpha u_4 \\ 2\alpha u_5 \\ -2\alpha u_6 \\ 2\alpha u_7 \end{pmatrix}. \tag{23}$$

Comparing the structures of $J$ and $L$ in Eq. (23) with those in Eq. (9) and according to the definition of integrable couplings, we conclude that Eq. (23) is an integrable coupling of the system (9). Let $u_1 = q$, $u_2 = r$, $u_3 = s$, $u_4 = u_5 = u_6 = u_7 = 0$, then Eq. (23) reduces to Eq. (9). Therefore, the system (23) is also a type of expanding integrable system of Eq. (9).

**Remark 3** Here we have constructed a new high-dimensional loop algebra $\tilde{G}$ in order to obtain integrable couplings of the system (9). Two open problems still remain. As we know from the above, the system (23) is Lax integrable. How can we prove the system (23) is Liouville integrable? How can we obtain the Hamiltonian structures of the system (23) and the infinite conserved laws? These problems will be discussed or solved in future by the author or together with other researchers who are interested in them.

# 4. Binary symmetry constrained flows

Consider the conjugate isospectral problem of the system (3):

$$\psi_x = U^*\psi, \lambda_t = 0, \psi = (\psi_1, \psi_2, \psi_3)^{\mathrm{T}},$$

$$U^* = \begin{pmatrix} -\lambda^2 - q & 0 & -(r+s)\lambda \\ 0 & \lambda^2 & 0 \\ (s-r)\lambda & 0 & q \end{pmatrix}. \tag{24}$$

We restrict $\varphi_i$ $\psi_i$ ($i$=1,2,3) and their derivatives in Eqs. (3) and (24) tending to zero when $|x| \to \infty$. A direct calculation shows that

$$\frac{\delta\lambda}{\delta u} = \frac{1}{E}\begin{pmatrix} \varphi_1\psi_1 - \varphi_3\psi_3 \\ \lambda(\varphi_1\psi_3 - \varphi_3\psi_1) \\ \lambda(\varphi_1\psi_3 - \varphi_3\psi_1) \end{pmatrix},$$

$$E = -\int_{-\infty}^{+\infty} [2\lambda\varphi_1\psi_1 - 2\lambda\varphi_2\psi_2 + (r+s)\varphi_1\psi_3 + (r-s)\varphi_3\psi_1]\mathrm{d}x.$$

For $N$ distinct $\lambda_j$ ($j = 1, 2, \cdots, N$) and some fixed integer $k$ ($k = 0, 1, 2, \cdots$), bi-symmetry constraint can in general be expressed as[19]

$$J\frac{\delta H_k}{\delta u} = J\beta_k \sum_{j=1}^{N} E_j \frac{\delta\lambda_j}{\delta u}, \tag{25}$$

where $\beta_k$ is a constant to be determined, $E_j$ are normal constants.

As far as Eq. (14) goes, Eq. (25) can reduce to

$$\begin{pmatrix} 2a_k \\ 2b_{k+1} \\ -2c_{k+1} \end{pmatrix} = \beta_k \begin{pmatrix} \langle \Phi_1, \Psi_1\rangle - \langle \Phi_3, \Psi_3\rangle \\ \langle \Lambda\Phi_1, \Psi_3\rangle + \langle \Lambda\Phi_3, \Psi_1\rangle \\ \langle \Lambda\Phi_1, \Psi_3\rangle - \langle \Lambda\Phi_3, \Psi_1\rangle \end{pmatrix}, \tag{26}$$

where $\Phi_i = (\varphi_{i1}, \cdots, \varphi_{iN})^{\mathrm{T}}$, $\Psi_i = (\psi_{i1}, \cdots, \psi_{iN})^{\mathrm{T}}$, $i$=1,2,3. When $k$=0, $\beta_1 = 2\alpha$, Eq. (26) reduces to the following constraint

$$2r = \langle \Lambda\Phi_1, \Psi_3\rangle + \langle \Lambda\Phi_3, \Psi_1\rangle,$$

$$2s = \langle \Lambda\Phi_3, \Psi_1\rangle - \langle \Lambda\Phi_1, \Psi_3\rangle. \tag{27}$$

Here $q$ is arbitrary, without loss of generality, we set it to 1. Inserting Eq. (27) into Eqs. (3) and (24) yields the space symmetry constrained flows as follows:

$$\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix}_x = U|_A \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix},$$

$$\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix}_x = U^*|_A \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix}, \tag{28}$$

where the symbol $|_A$ denotes substitution of Eqs. (27) into corresponding isospectral problems.

Letting

$$H_0 = \langle \Lambda^2\Phi_1 + \Phi_1, \Psi_1\rangle + \langle \Lambda\Phi_1, \Psi_3\rangle\langle \Lambda\Phi_3, \Psi_1\rangle - \langle \Lambda^2\Phi_2, \Psi_2\rangle - \langle \Phi_3, \Psi_3\rangle,$$

then we obtain a Hamiltonian form of the constrained flows (28) as follows:

$$P_{ix} = \frac{\partial H_0}{\partial Q_i},$$

$$Q_{ix} = -\frac{\partial H_0}{\partial P_i}, i = 1, 2, 3, \tag{29}$$

where $P_i = (\varphi_{i1}, \cdots, \psi_{iN})^{\mathrm{T}}$, $Q_i = (\psi_{i1}, \cdots, \psi_{iN})^{\mathrm{T}}$.

**Remark 4** Here we only deduce the Hamiltonian form of the $x$-constrained flows when $k$=0. By similar discussion, we can obtain the Hamiltonian form of the temporal constrained flows. When $k$=2,4,$\cdots$, we must introduce Jacobi–Ostrogradsky coordinates to obtain the Hamiltonian forms of Eq. (28) by making use of the approach in Refs. [18–20]. When $k = 1, 3, \cdots$, we only obtain nonregular constrained flows from Eq. (25).[21] Using the method proposed in Ref. [21], we can construct the Hamiltonian structures. Since such calculations are more complicated, we shall discuss them in another paper.

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